Finitism Revisited: Takeuti's philosophy of mathematics

Andrew Arana, Université de Lorraine, Archives Poincaré

Fudan Logic Seminar

29 April 2026

Galileo (1564–1642) observed that there is a correspondence between the natural numbers and the square numbers: we can associate 0 to 0, 1 to 1, 2 to 4, 3 to 9, and in general n to n^2.

This correspondence suggests that there are as many square numbers as there are natural numbers.

But there are natural numbers that are not square numbers, so it seems obvious that there are more natural numbers than square numbers.

Galileo's response to this paradox was that the infinite transcends our understanding, which is necessarily finite.

Descartes, *Principiae philosophiae* (1644)

Thus we will never be involved in tiresome arguments about the infinite. For since we are finite, it would be absurd for us to determine anything concerning the infinite; for this would be to attempt to limit it and grasp it. So we shall not bother to reply to those who ask if half an infinite line would itself be infinite, or whether an infinite number is odd or even, and so on. It seems that nobody has any business to think about such matters unless he regards his own mind as infinite.

Hilbert basis theorem. If a ring R is noetherian, then the polynomial ring $R[x_1, \ldots, x_n]$ is also noetherian, for any n.

Later analysis would show that Hilbert's proof relied on transfinite induction.

Of Hilbert's proof of this theorem, his colleague Paul Gordan said, "This is not mathematics, this is theology!".

Leopold Kronecker (1823–1891)

God created the integers, all else is the work of man.

David Hilbert, "Über das Unendliche" (1925)

Nowhere is the infinite realized: it is neither present in nature nor admissible as a foundation in our rational thought.

According to Hilbert, forbidding the infinite from mathematics would be "to deprive the boxer of his gloves".

Hilbert, "Über das Unendliche" (1925)

No one shall be able to drive us from the paradise that Cantor created for us. It is necessary to make inferences everywhere as reliable as they are in ordinary elementary number theory, which no one questions and in which contradictions and paradoxes arise only through our carelessness.

Thus our reasoning in elementary arithmetic, concerning the finite, is Hilbert's gold standard of mathematical security.

He wanted to show that we can reason with the infinite as securely as we do with the finite.

A finitary proof of the consistency of infinitary arithmetical reasoning would demonstrate on the most reliable grounds the security of this infinitary reasoning, according to Hilbert.

We could then be said to know the conclusions of proofs making use of the infinite, as an ideal tool that aids our intuitive thought.

The search for such a consistency proof became known as Hilbert's program.

Hilbert and Bernays, *Grundlagen der Mathematik* (1934)

Regarding this goal [of proving consistency], I would like to emphasize that an opinion, which had emerged intermittently—namely that some more recent results of Gödel would imply the infeasibility of my proof theory—has turned out to be erroneous. Indeed, that result shows only that—for more advanced consistency proofs—the finitist standpoint has to be exploited in a manner that is sharper [*schärferen*] than the one required for the treatment of the elementary formulations.

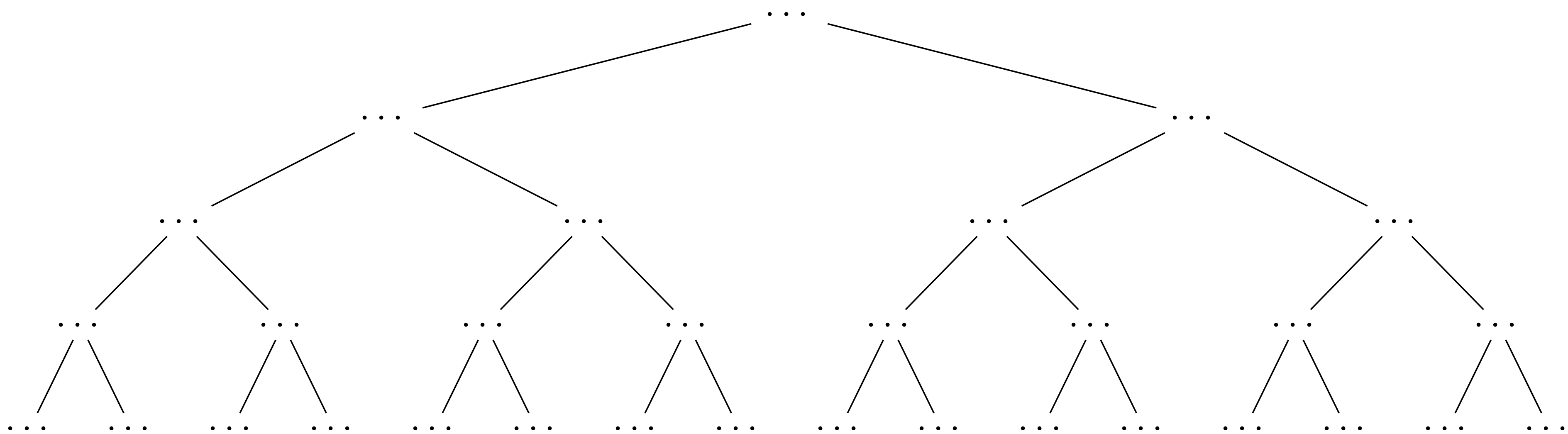
Let's outline Gentzen's proof of the consistency of PA.

He designs a system of ordinals and an ordering of these ordinals that are each concrete and thus finitistically acceptable.

This ordering has type ϵ_0.

Proofs in PA are assigned these ordinals according to the rules of inference used.

$$\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}}$$



Gentzen gives a procedure for reducing proofs so that each proof of inconsistency gets reduced to another proof of inconsistency with a smaller ordinal.

If there is a proof of inconsistency, this procedure generates an infinitely decreasing sequence of such ordinals.

By the well-ordering of the ordering of type ϵ_0, such a sequence is impossible.

Thus there is no proof of inconsistency in PA.

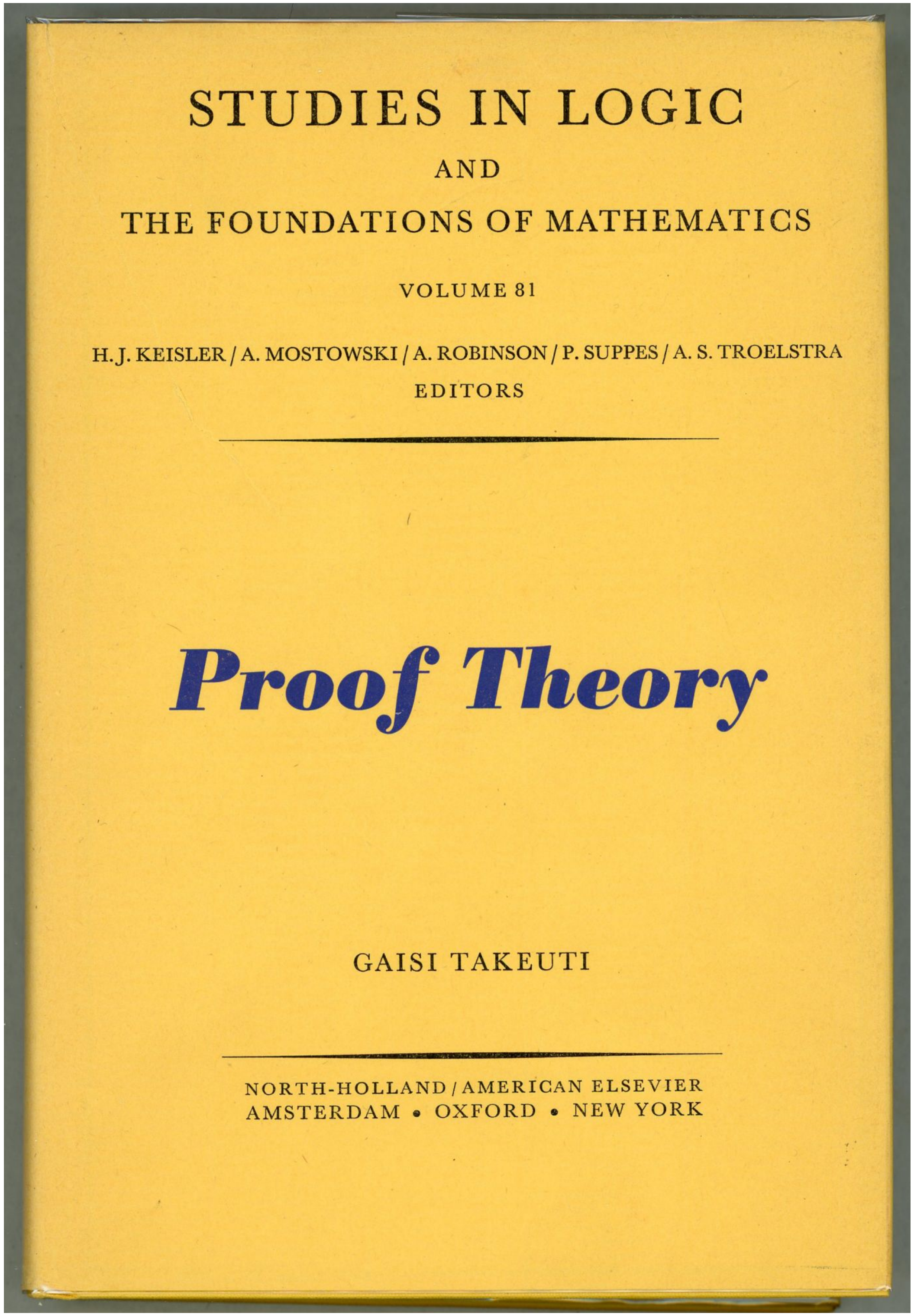
Tarski, "Contribution to the discussion of P. Bernays *Zur Beurteilung der Situation in der beweistheoretischen Forschung*" (1954)

Gentzen's proof of the consistency of arithmetic is undoubtedly a very interesting metamathematical result, which may prove very stimulating and fruitful. I cannot say, however, that the consistency of arithmetic is now much more evident to me (at any rate, perhaps, to use the terminology of the differential calculus more evident than by an epsilon) than it was before the proof was given.

Takeuti, “Consistency Proofs and Ordinals” (1975)

Anyway since I am a logician and am very familiar with the magic of quantifiers Gentzen’s consistency proof, which consists of the elimination of quantifiers and an accessibility proof for the ordinals less than ϵ_0, is greatly reassuring. It does add to my confidence in the consistency and truth of Peano arithmetic.

Gaisi Takeuti (1926–2007) was born in Kanazawa, Japan; doctorate in mathematics at the University of Tokyo (1956); researcher at Institute for Advanced Study in Princeton under Gödel; then professor of mathematics at the University of Illinois, Urbana-Champaign until retirement in 1992.


STUDIES IN LOGIC
AND
THE FOUNDATIONS OF MATHEMATICS
VOLUME 81
H. J. KEISLER / A. MOSTOWSKI / A. ROBINSON / P. SUPPES / A. S. TROELSTRA
EDITORS
Proof Theory
GAISI TAKEUTI
NORTH-HOLLAND / AMERICAN ELSEVIER
AMSTERDAM • OXFORD • NEW YORK

We believe that our standpoint is a natural extension of Hilbert's finitist standpoint, similar to that introduced by Gentzen, and so we call it the Hilbert-Gentzen finitist standpoint.

Now a Gentzen-style consistency proof is carried out as follows:

(1) Construct a suitable standard ordering, in the strictly finitist standpoint.

(2) Convince oneself, in the Hilbert-Gentzen standpoint, that it is indeed a well-ordering.

(3) Otherwise use only strictly finitist means in the consistency proof.

We now present a consistency proof of this kind for **PA**.

To understand why Takeuti thought transfinite induction to ϵ_0 was still finitary, we first need to understand better what Hilbert meant by finitary.

Hilbert and Bernays, *Grundlagen der Mathematik* (1934)

Our treatment of the basics of number theory and algebra was meant to demonstrate how to apply and implement direct contentual inference that takes place in thought experiments on intuitively conceived objects and is free of axiomatic assumptions. Let us call this kind of inference "finitist" inference for short, and likewise the methodological attitude underlying this kind of inference as the "finitist" attitude or the "finitist" standpoint.... With each use of the word "finitist", we convey the idea that the relevant consideration, assertion, or definition is confined to objects that are conceivable in principle, and processes that can be effectively executed in principle, and thus it remains within the scope of a concrete treatment.

Hilbert, “Über das Unendliche” (1925)

Contentual logical inference is indispensable. . . [A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Hilbert, "Über das Unendliche" (1925)

Contentual logical inference is indispensable. . . [A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Hilbert, “Über das Unendliche” (1925)

Contentual logical inference is indispensable. . . [A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Hilbert, “Über das Unendliche” (1925)

Contentual logical inference is indispensable. . . [A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Kant, *Critique of pure reason* (1781/87)

All human cognition begins with intuitions, goes from there to concepts, and ends with ideas.

> So fängt denn alle menschliche Erkenntnis mit Anschauungen an, geht von da zu Begriffen und endigt mit Ideen.
>
> Kant, Kritik der reinen Vernunft,
> Elementarlehre 2. T. 2. Abt.

Einleitung.

Die Geometrie bedarf — ebenso wie die Arithmetik — zu ihrem folgerichtigen Aufbau nur weniger und einfacher Grundsätze. Diese Grund-

Kant, *Critique of Pure Reason*

In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and at which all thought as a means is directed as an end, is intuition. This, however, takes place only insofar as the object is given to us; but this in turn, is possible only if it affects the mind in a certain way. The capacity (receptivity) to acquire representations through the way in which we are affected by objects is called sensibility. Objects are therefore given to us by means of sensibility, and it alone affords us intuitions. (A19/B33)

Kant, *Critique of Pure Reason*

In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and at which all thought as a means is directed as an end, is intuition. **This, however, takes place only insofar as the object is given to us**; but this in turn, is possible only if it affects the mind in a certain way. The capacity (**receptivity**) to acquire representations through the way in which we are affected by objects is called sensibility. **Objects are therefore given to us by means of sensibility, and it alone affords us intuitions.** (A19/B33)

According to Kant, then, intuition is a mode of relation between mind and world.

This relation is receptive and accordingly passive.

The receptivity of intuition has consequences for our knowledge of the infinite, for we are incapable of receiving anything infinite completely.

Heidegger, *Kantbuch* (1929)
Finite knowledge is non-creative intuition. . . the finitude of intuition lies in its receptivity.

Indeed, this observation is key to Kant's mathematical antinomies.

As a result, he relegated the infinite to the ideal, as a useful tool for organizing our thought but not real and not knowable.

He saw this as consequential for metaphysics rather than for mathematics, because the mathematics he knew was not concerned with the infinite.

Our inability to know the infinite would be a problem for accommodating the new mathematics of the infinite of the 19th century within a Kantian epistemology.

We want to investigate how Takeuti's Hilbert-Gentzen finitist standpoint finitist in the Kant-Hilbert sense.

Key to that sense is that finitary knowledge is intuitive knowledge.

While in Kant's secularized Christian epistemology, intuition is passive and receptive, Takeuti comes from a different tradition in which intuition is understood actively.

Our view is that Takeuti's active conception of intuition, arising from his Japanese context, explains why he judged Gentzen-style consistency proofs to be finitist.

Kitaro Nishida (1870–1945), born near Kanazawa, Japan, is the founder of the Kyoto school of Japanese philosophy. He is the foremost Japanese philosopher of the twentieth century.

The Philosopher's Path in Kyoto, which Nishida walked each day to Kyoto University.

Nishida contrasted the individual finite human mind 人 with 神, the infinite mind or God.

Nishida, 善の研究 (*An Inquiry Into The Good*, 1911)

Because there is Buddha, there are sentient beings. Because there are sentient beings, there is Buddha. Because there is a creator God, there is a created world. Conversely, because there is a created world, there is God.

Nishida was a practitioner of Kegon Buddhism, the Japanese name for Huayan Buddhism (華嚴宗).

Its central text is the Avataṃsaka Sūtra (*Huayan Jing* 華嚴經)).

The school is best known for the doctrine of *interpenetration*— the idea that all phenomena (dharmas) are mutually dependent, interwoven, and perfectly fused without obstruction.

Avatamsaka-sutra (4th century)

As is the mind, so is the Buddha;
As the Buddha, so living beings:
Know that Buddha and mind
Are in essence inexhaustible.

If people know the actions of mind
Create all the worlds,
They will see the Buddha
And understand Buddha's true nature.

In Nishida's view, active intuition is the fundamental mode of experience of the world, and it does not suppose a separation between subject and object.

John Maraldo, "Nishida Kitarō", on active intuition (2019)

For Nishida, the artist takes in or intuits the world and transforms or enacts it, both of which are but two moments in a single unfolding—not only of the world but of the artist as well. Both artist and work are formed mutually and are reflected in one another.

Nishida, 善の研究 (*An Inquiry Into The Good*, 1911)

We reach the quintessence of good conduct only when subject and object merge, self and things forget each other, and all that exists is the activity of the sole reality of the universe. At that point we can say that things move the self or that the self moves things, *that Sesshū painted nature or that nature painted itself through Sesshū*. There is no fundamental distinction between things and the self, for just as the objective world is a reflection of the self, so is the self a reflection of the objective world.

Nishida, *From the Acting to the Seeing* (1927)

Active intuition is seeing everything that is and everything that is at work as reflections of what mirrors itself within itself by itself becoming nothing. I want to conceive, at the root of all things, a seeing without a seer.

Nishida, "Logic and Life" (1936)

There may be objections to conceiving the mechanistic world on the basis of the formative act of active intuition. But when we say we possess tools, while things become our body, the body in turn becomes a thing. That the I becomes a thing means that the I is losing the I. Ultimately this means that the I, after a fashion, disappears.

Nishida, “Logic and Life” (1936)

Our self must be active. The existence of human beings lies in acting. To act means to make things with tools. . . . But there is no seeing when faced with things lacking any sort of connection with our movement. . . . Accordingly, making is seeing and seeing is making.

Nishida, “Absolutely Contradictory Self-Identity” (1939)

The world of reality is necessarily a world of interaction among objects. It is possible to think of reality in the form of a mutual relationship among objects, as what comes about as a result of the interaction between object and object. Now the action of an object indicates the self-negation of the object, the disappearance of the object as such. . . . The world continually advances from the *made* to the *maker*.

Avatamsaka-sutra, "The Incalculable"

If untold buddha-lands are reduced to atoms,
In one atom are untold lands,
And as in one,
So in each.

The buddha-lands on a hairtip are infinite,
The buddha-lands in an atom are infinite;
They can go to all these buddha-lands
And see the infinite buddhas.

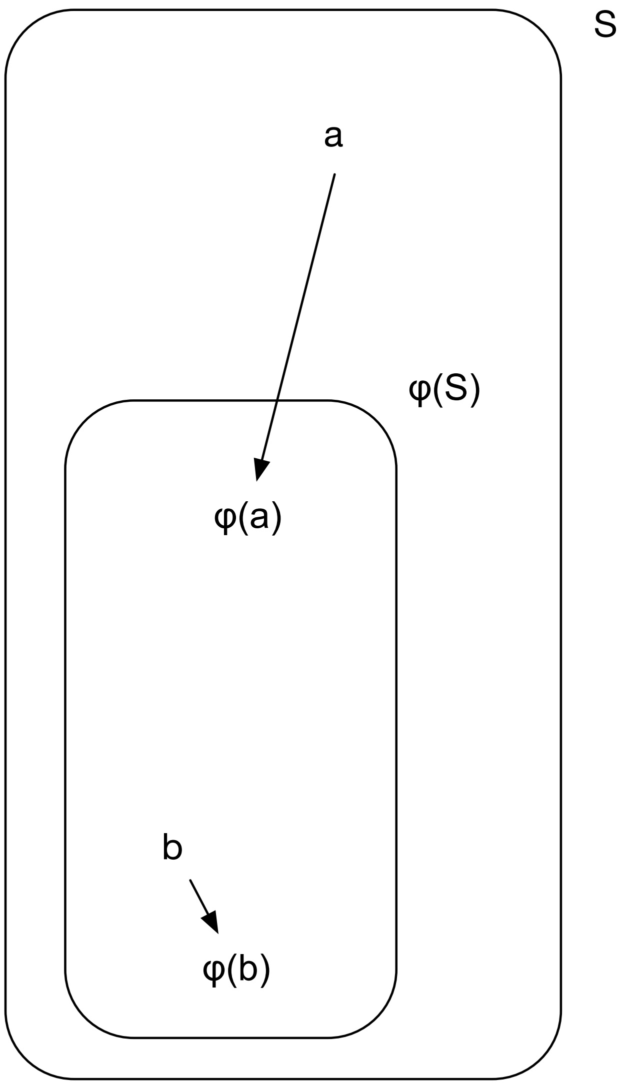
The next component of Nishida's view is that of self-reflection or self-awareness, what Nishida called *jikaku* 自覚.

Here Nishida started from his Buddhist understanding of the infinite, but also drew on the work of Richard Dedekind and Josiah Royce.

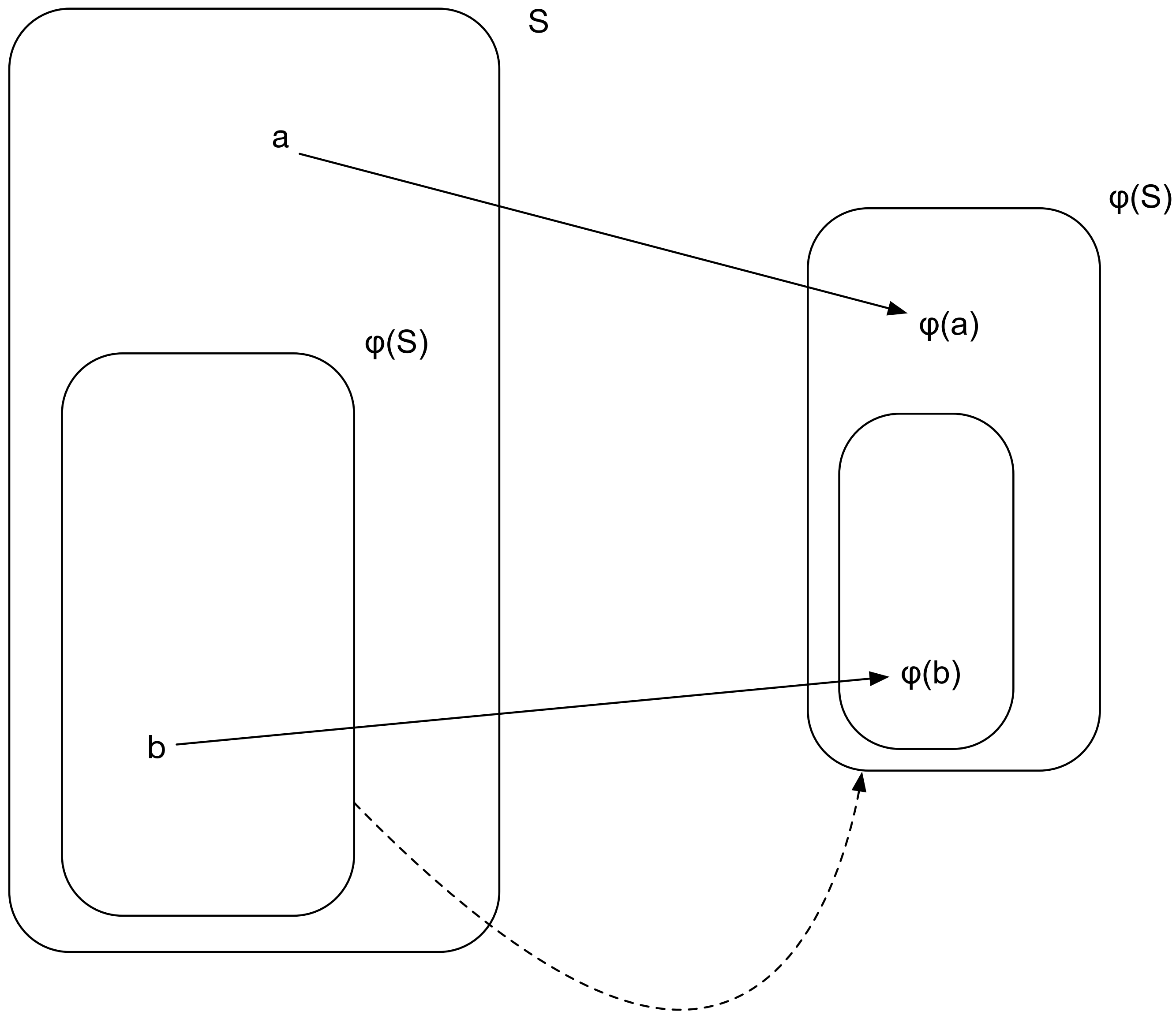
In his book *Was sind und was sollen die Zahlen?* (1888), Richard Dedekind gave a new definition of the infinite.

A set S is Dedekind infinite if there exists a function $\varphi : S \to S$ that is injective but not surjective.

$\varphi : S \to S$ injective but not surjective



$\varphi : S \to S$ injective but not surjective



Josiah Royce, "The Concept of the Infinite" (1902)

An object that contained, as part of itself, a perfect picture of itself—in other words, a self-representative object or system, of the type here in question—would of necessity prove to be an object whose complexity of structure no finite series of details could exhaust; for it would contain a picture of itself. within which there was to be found a picture of this picture, and a picture of this second picture, and so on without end.


24
PORTIONS/PORTIES
La vache qui rit
RICHE EN
CALCIUM
RIJK AAN
400g e
Tendrement bon • Heerlijk zacht

Following Dedekind and Royce, Nishida talks of drawing of a map of its own location, including the map being drawn.

Nishida calls this capacity self-awareness [*jikaku* 自覚] and holds that the object constructed in this way is "the true sense of the infinite" ("Understanding in logic and in mathematics", 1912).

By means of self-awareness [*jikaku*], we can make ourselves the object of our thought, in an active intuition; and then make the content of that intuition into a further object, in; and so on.

In this way, we can have an active intuition of the infinite, and thus we finite beings can have intuitive knowledge of the infinite.

Takeuti was familiar with the works of his teacher and friend Joichi Suetsuna (1898 – 1970), a number theorist at the University of Tokyo (and student of Teiji Takagi, who was a student of Hilbert's at Göttingen).

Suetsuna also wrote on philosophy of mathematics and argued for a type of finitism.

He had an active correspondence with Helmut Hasse, Hermann Weyl's successor at Göttingen.

Suetsuna was deeply influenced by his friend Nishida.

After his encounters with Nishida he became a Kegon Buddhist monk.

Suetsuna,「数理と論理」(“Mathematics and logic”, 1947)

It is indeed our intuition that enables us to comprehend the continuum. However, this intuition, I think, is fundamentally an active intuition intertwined with our bodily movements. We grasp the continuum through our active intuition, grounded in the premise of transitioning from one point to another. This encompasses physical actions such as moving our hands or shifting our gaze from one position to another, or even assuming such movements.

In his 1944 paper "Yugen no Tachiba to Kyokugen Gainen" (有限の立場と極限概念, "The Finitist Standpoint and the Notion of Limit"), Suetsuna argues that active intuition can ground knowledge of transfinite ordinals up to ϵ_0.

Takeuti, 「形式主義の立場からII」 ("From the point of view of formalism II", 1956)
I call the intuition for grasping a function or a set, in general, an active intentional intuition. Indeed, when attempting to grasp a function or a set, this active intention comes into play. What would be thought of by this active intention, the object of this active intention, or the intention itself, would be called a function or a set. . . Here, the word "active intentional intuition" is a word coined under Suetsuna's influence (Suetsuna scolded me that "intentional" is not quite appropriate).

Following Nishida, Takeuti makes no separation between the act by a subject of constructing an object such as a set, and the object so constructed.

Then both this act and the object of this act are unified in an active intuition.

Having adopted Nishida's active intuition, following Suetsuna, he next adopts Nishida's self-reflection (*jikaku*).

Takeuti, 「数学について」 ("About mathematics", 1972)

Now, for example, consider a finite mind. If the function of a finite mind is completely finite, then our mathematics remains finite. However, our finite mind is the so-called potential infinity. In other words, we can indicate infinitary objects like $0, 1, 2, 3, \ldots$ [...] Why? This is because our mind has the ability of self-reflection, that is, the ability to observe what we are doing and to know what we are doing.

He says infinite mind does the same for constructing the transfinite ordinals.

$$\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} = \omega^{\epsilon_0}$$

In virtue of the self-representing nature of ϵ_0, Takeuti thought we, as finite minds, could know it: an application of Nishida's *jikaku*.

He sought other ordinals of this self-representing form for even more powerful mathematical systems ("Takeuti's conjecture").

Finite mind actively participates in the construction / constitution of ϵ_0 and other transfinite sets. The finite and the infinite are woven together, reflecting each other.

The intimate relationship between the finite and the infinite we have seen in Nishida and Takeuti's thought is more permeable than it is in the Kantian tradition.

On this view, humans from their finite standpoint can have knowledge of infinite sets, not by establishing that infinite sets as ideal elements are secure instruments of reasoning as Hilbert would have it, but by knowing those infinite sets themselves, as we know ourselves and our creative activity, by an active intuition making use of self-awareness [*jikaku*].

Gentzen-style proofs of consistency are thus intuitive, even for finite minds.