# 目 录

# 会议手册

# 温馨提示

热烈欢迎您参加“2025 数学哲学国际学术研讨会”。为了方便您在会议期间的工作和生活，请注意以下事项：

**1. 会议时间和地点：**

时间：2025 年 7 月 25-27 日（25 日报到，27 日中午离会）

地点：报到及住宿地点为萃英大酒店，会议地点为兰大城关校区逸夫科学馆 201 会议室。

**2. 住宿安排：**

本次会议预定的酒店为萃英大酒店，位于兰州大学城关校区西门向北约 90 米处。酒店前台电话 0931-8631999，入住时请携带本人的身份证。

**3. 其他事项：**

（1）会议期间正值暑期，兰州白天气温较高，但昼夜温差仍较明显，建议您根据气温变化适时增减衣物，既要注意防暑，也避免夜间受凉；

（2）请妥善保管私人物品，如需任何帮助，请及时联系会务人员；

（3）会务组联系方式：

| | |
|---|---|
| 陈佳 | 15205150891 |
| 赵航纬 | 18005792538 |
| 郭子夏 | 19567256530 |
| 田秉昌 | 18919120827 |
| 王健权 | 13893100755 |

# 交通指南

**兰州中川机场至兰州大学**（距离约 71 公里）

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|---|---|
| **方案一：** | 乘坐机场巴士一号线到兰州大学站下车，费用约 30 元 |
| **方案二：** | 乘坐出租车，路程约 1 小时，费用约 200 元 |
| **方案三：** | 从高铁中川机场东站（近 T3 航站楼）乘城际铁路到兰州西站或兰州站，再转公交、地铁或打车。城际铁路约 50 分钟，费用 20 元，后续需换乘公交、地铁或出租车（参见兰州西站和兰州站方案） |

**兰州西站至兰州大学**（距离约 12 公里）

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|---|---|
| **方案一：** | 乘坐地铁 1 号线至兰州大学站下车（D 口近萃英大酒店和兰州大学），共 8 站约 20 分钟路程，费用 4 元 |
| **方案二：** | 乘坐出租车，路程约 30 分钟，费用约 40 元 |

**兰州站至兰州大学**（距离约 2 公里）

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|---|---|
| **方案一：** | 东出站口乘坐 1、10、16、110、131 路公交车至兰州大学站下车，费用 2 元 |
| **方案二：** | 乘坐出租车，费用约 10 元 |

N
酒 店
会 场
兰大西门
餐 厅
齐云楼
飞云楼
格致楼
计算机中心
逸夫科学馆
萃英大酒店
贵勤楼
正门
校友广场
网球场
篮球场
钟灵园
观云楼
丹桂苑（餐饮）
研究生公寓8#
研究生公寓9#
研究生公寓6#
研究生公寓7#
研究生公寓10#
研究生公寓12#
研究生公寓11#
留学生公寓
南一门

# 日程安排

## 7 月 25 日（周五）  25th Jul. (Friday)

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|---|---|
| 9:00-21:00 | 签到（萃英酒店大堂）<br>Check In (at the Hall of Cui-Ying Hotel) |
| 18:30-20:00 | 晚餐（丹桂苑二楼）<br>Dinner (at Dan-Gui-Yuan) |

## 7 月 26 日（周六）  26th Jul. (Saturday)

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| 09:00-09:30 | 开幕式（逸夫科学馆 201 会议室）<br>Opening Ceremony (at Room 201 in Yifu Science Center) |
| 09:30-11:40 | 会议（逸夫科学馆 201, 202 会议室）<br>Presentations (at Rooms 201, 202 in Yifu Science Center) |
| 11:50-14:10 | 午餐及午休<br>Lunch and Rest |
| 14:10-17:50 | 会议（逸夫科学馆 201, 202 会议室）<br>Presentations (at Rooms 201, 202 in Yifu Science Center) |
| 18:10-19:40 | 晚餐<br>Dinner (at Dan-Gui-Yuan) |

## 7 月 27 日（周日）  27th Jul. (Sunday)

| | |
|---|---|
| 09:00-11:10 | 会议（逸夫科学馆 201 会议室）<br>Presentations (at Rooms 201 in Yifu Science Center) |
| 11:10-11:40 | 闭幕式（逸夫科学馆 201 会议室）<br>Closing Ceremony (at Room 201 in Yifu Science Center) |
| 11:50-13:00 | 午餐、离会<br>Lunch, Check Out |

# 会议议程

## 07 / 25　全天

9:00-21:00 萃英大酒店大堂

**签到及办理入住**

## 07 / 26　上午

9:00-9:30 逸夫科学馆 201

**开幕式**

**主持：** **陈　佳** | 兰州大学

**致辞：** **邓汉华** | 兰州大学社会科学处

**杨睿之** | 复旦大学

**合影**

9:30-10:20 逸夫科学馆 201

**主旨报告**（50 分钟）

**主持：姚博凯** | 北京大学

**Joel D. Hamkins |** University of Notre Dame

*How the continuum hypothesis might have been a fundamental axiom*

10:20-10:40

**茶歇**（20 分钟）

10:40-11:40

**常规报告**（每人 30 分钟）

| **第一分会场**<br>逸夫科学馆 201 | **第二分会场**<br>逸夫科学馆 202 |
|---|---|

**主持：高贝贝** | 华南师范大学

**胡扬** | 华南师范大学
*The Norm for Logical Proof: Speech Act, Natural Deduction, and the Connecting Force*

**曹航杰** | 华南师范大学
*Symmetry Groups of Boolean Self-referential Systems*

**主持：陈佳** | 兰州大学

**罗广龙** | 南开大学
*Polymorphism and the Ontology of numbers*

**于宝山** | 西北师范大学
布劳威尔和胡塞尔对序数和基数的构造及其时间意识模型比较

## 11:50-14:10

**午餐及午休**

用餐地点：丹桂苑 2 楼（岷山厅、陇山厅、关山厅、寿鹿厅）

# 07 / 26　下午

## 14:10-15:40

**常规报告**（每人 30 分钟）

### 第一分会场

逸夫科学馆 201

**主持：梁晓龙** | 山西大学

**鞠大恒** | 复旦大学（线上）
数学的内部应用与结构主义解释——以组合学为例

**陈明坤** | 清华大学
基于分体拓扑学对物理组合问题的系列解

**李浩宁** | 北京师范大学
AI for Science：两种科学、两种 AI 和两种数学观

### 第二分会场

逸夫科学馆 202

**主持：陈龙** | 北京师范大学

**周星哲** | 北京师范大学
几何实在论的两次升华

**郭婵婵** | 延安大学
*On Clifford's Concept of Space: Dialogues with Riemann and Helmholtz*

**王月儿** | 山西大学
“统一几何”的空间哲学——以外尔的空间观为例

## 15:40-16:00

**茶歇**（20 分钟）

## 16:00-16:50 逸夫科学馆 201

**主旨报告**（50 分钟）

**主持：陈龙** | 北京师范大学

**Alexander Paseau** | University of Oxford（线上）
*Metaphysical Problems for Plenitudinous Platonism*

## 16:50-17:50

**常规报告**（每人 30 分钟）

### 第一分会场
逸夫科学馆 201

**主持：薄谋** | 兰州大学

**张文馨** | 北京师范大学（线上）
理念追求与经验根基之纠葛——重审古希腊数学之知识形态

**彭桢** | 中国科学技术大学
跨学科共同体的制度化构建：从 ZAMM 到 GAMM 的战间期德国应用数学革命（1921-1933）

### 第二分会场
逸夫科学馆 202

**主持：姚宁远** | 复旦大学

**龙羽江** | 北京师范大学
*Reanalysis of Penrose's New Argument and a Modified Version of it Formalised in DTK System*

**陈俊宏** | 复旦大学
*The Coding Conception of In*

## 18:10-19:40

**晚餐**

用餐地点：食堂故事本味餐厅

# 07 / 27　上午

## 9:00-9:50 逸夫科学馆 201

**主旨报告**（50 分钟）

**主持：邢滔滔** | 北京大学

**姚博凯** | 北京大学
*Set Theory with Plenitudinous Urelements*

## 9:50-10:10

**茶歇（20 分钟）**

## 10:10-11:10 逸夫科学馆 201

**常规报告（每人 30 分钟）**

**主持：杨睿之** | 复旦大学

**陈泽晟** | 独立学者（线上）

*A belated foundational role of set theory*

**刘力恺** | 北京大学

古典数学的公理化及其确定性

## 11:10-11:40 逸夫科学馆 201

**闭幕式**

**主持：** **陈龙** | 北京师范大学

**致辞：** **邢滔滔** | 北京大学

**薄谋** | 兰州大学

## 11:50-13:00

**午餐及离会**

用餐地点：丹桂苑 2 楼（陇山厅、关山厅、寿鹿厅）

# 参会人员名单

| 姓名 | 单位 |
|---|---|
| 薄谋 | 兰州大学 |
| 曹航杰 | 华南师范大学 |
| 陈佳 | 兰州大学 |
| 陈俊宏 | 复旦大学 |
| 陈龙 | 北京师范大学 |
| 陈明坤 | 清华大学 |
| 陈泽晟 | 独立学者 |
| 崔佳楠 | 晋中学院 |
| 高贝贝 | 华南师范大学 |
| 郭婵婵 | 延安大学 |
| 郭子夏 | 兰州大学 |
| Hamkins, Joel David | University of Notre Dame |
| 胡扬 | 华南师范大学 |
| 黄蕾 | 河南财经政法大学 |
| 鞠大恒 | 复旦大学 |
| 李浩宁 | 北京师范大学 |
| 梁晓龙 | 山西大学 |

| | |
|---|---|
| 刘佶鑫 | 四川大学 |
| 刘力恺 | 北京大学 |
| 龙羽江 | 北京师范大学 |
| 罗广龙 | 南开大学 |
| Paseau, Alexander | University of Oxford |
| 彭桢 | 中国科学技术大学 |
| 王健权 | 兰州大学 |
| 王月儿 | 山西大学 |
| 汝强 | 中国矿业大学 |
| 史健健 | 西北师范大学 |
| 田秉昌 | 兰州大学 |
| 邢滔滔 | 北京大学 |
| 熊明 | 华南师范大学 |
| 杨睿之 | 复旦大学 |
| 姚博凯 | 北京大学 |
| 姚宁远 | 复旦大学 |
| 于宝山 | 西北师范大学 |
| 张文馨 | 北京师范大学 |
| 赵航纬 | 兰州大学 |
| 周星哲 | 北京师范大学 |

# 论文、摘要集

# How the continuum hypothesis might have been a fundamental axiom

Joel David Hamkins, University of Notre Dame

Abstract: I shall describe a historical thought experiment showing how our attitude toward the continuum hypothesis could easily have been very different than it is. If our mathematical history had been just a little different, I claim, if certain mathematical discoveries had been made in a slightly different order, then we would naturally view the continuum hypothesis as a fundamental axiom of set theory, necessary for mathematics and indeed indispensable for calculus.

# Metaphysical Problems for Plenitudinous Platonism

A.C. Paseau, University of Oxford

Abstract: Plenitudinous Platonism is the thesis that there are as many types of mathematical object as possible. Because it takes mathematics to be the study of abstract objects, it is a form of platonism; and because it takes any coherently describable mathematical structure to exist, it is also a form of structuralism. An umbrella term, Plenitudinous Platonism also goes by the name of Full-Blooded Platonism or Egalitarian Platonism. My talk will raise some problems for the view. As my title indicates, I will focus on the metaphysical side of things.

# Set Theory with Plenitudinous Urelements

Bokai Yao, Peking University

Abstract: The Axiom of Plenitude asserts that every set is equinumerous with a set of urelements. We offer philosophical motivations for treating Plenitude as a natural axiom and examine its interaction with other foundational principles in ZF set theory with urelements. Assuming that cardinality is definable, Plenitude unifies the Collection and Reflection principles (which are otherwise conjectured to be non-equivalent) and implies both if every set is equinumerous with its cardinality.

# The Norm for Logical Proof:

## Speech Act, Natural Deduction, and the Connecting Force

HU Yang

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The norm for proof indicates the *compelling* character of a given proof: “to follow a proof is to be in some sense *compelled* to its conclusion.” (Restall, 2024) The question to be addressed here is where this kind of *proof norm*, namely that the premises of a given proof are *forced to* be stepwise connected and to go forward its conclusion, comes from. Put it differently, how does such *forceful compulsion* arise?

There seem to be two reasons to investigate this question.

From a practical point of view, the accounts of mathematical or logical proof have social-cultural impact and even give rise to genuine legal disputes. In 1980’s, the UK Ministry of Defence’s Royal Signals and Radar Establishment (RSRE) intended to use varieties of formal proof methods to verify the correctness of a commercially designed microprocessor for safety-related systems. This was then an important UK hardcore verification program called “VIPER”. However, due to the widespread controversy on how a mathematical proof can verify the physical chips, RSRE was charged by a licensed production company of the microprocessor that RSRE claimed in the brochure that “the chip was proven faultless. Even though the lawsuit never came to court, this was the first case in which the meaning of mathematical proof became a legal case.”(Heintz, 2003)

From a theoretical point of view, it is quite reasonable to say that the forceful compulsion of proof comes from mathematical or logical rigor. The proof is usually said to be rigorously rule-governed. However, as Tanswell (2024) observes, a *practical turn* has occurred recent decades in the philosophical inquiry about mathematics. It turns out to thus be highlighted that mathematicians or logicians’ interest, value or activity play an important role in a well-rounded understanding of the rigor. Obviously, given the rapid development of various automatic proof assistants like Lean, Coq, and Z3 (etc.), alongside AI tools, how to untangle the distinct role of *human provers* in contribution to the proof norm in question seems to be a philosophically engaging question.

Our approach to the proof norm focuses on the role of speech act in constructing a formal proof. We shall first argue that the origin of (formal) proof norm has to do with speech acts pervasively performed therein.

It has been widely held that speech act theory focuses on the use of natural language, while logic and mathematics typically rely on formal languages whose rigor is supposed to be exclusive of pragmatic factors as much as possible (Ganesalingam, 2013). This common view has been challenged.

Ruffino et al. (2020, 2021) argue that mathematical texts are shot through with speech acts, with “blocks” like theorems, lemmas, propositions, corollaries, and definitions reflecting the author’s “assertions” or “declarations” [following Searle & Vanderveken’s (1985) classification of speech acts]. Through these speech acts, mathematical content is organized into hierarchical mathematical theories. Schmidt & Venturi (2023) examine the speech acts in mathematical axioms and postulates, viewing them as “hybrid acts” of declarations, assertions, and directives. They suggest that analyzing the hybrid nature of axioms and postulates not only clarifies the differing approaches of Frege and Hilbert to axiomatic systems but also offers a new perspective for fully understanding mathematical practice. While Ruffino et al. (2021) and Schmidt & Venturi (2023) focus on static mathematical texts from the prover’s perspective, Tanswell (2023, 2024) and Weber (2023) use empirical studies of mathematical research papers and set theory textbooks to highlight the role of various speech acts in the process of prove and the interaction between provers and proof-readers. As Restall (2022) notes, proofs involve more than just assertions, revealing the complexity of proof structures: “There are different steps where objects are given names, and speech acts, other than asserting, are involved as well. Proofs have complex structure. A crucial constraint, though, is that a proof is not simply a statement of the premises and the conclusion – at least, not in most cases. To prove some conclusion C from some premises $p_1, \ldots, p_n$, you must somehow trace the connection from $p_1, \ldots, p_n$ to C.” Clearly, Restall emphasizes the “connectivity” of speech acts in proofs, which leads to the second argument of our approach.

We shall argue that the proof norm has two resources both of which are closely related with the performance of speech acts in proof. The first consist in the conventionally illocational forces ascribed to varieties of speech acts. Restall (2024) recognizes both polar question (performed with interrogative force) and justification request (preformed with imperative force) revolving around the assertions of a given proof as the impetus for the rule-governed dynamics of proof. Pagin (2024) gives a detailed explanation of the force of speech acts performed as parts of a natural deduction proof like assumption, hypothetical and quantified assertion. According to his index application model, the force of a given speech act *s* is an ascription of the content of *s* to an actual index *i* for it to have a truth value at *i*. Particularly, Pagin thinks of the endorsement of a proof as bearing the inferential force which ascribing the content of both the premises and conclusion to an index to yield a truth value sequence. Pezlar (2023) takes the absurdity involved in the rules like EFQ, ¬I,¬E and RAA to be an impossible command bearing the imperative force. This command is interpreted as "stopping derivation". Since the absurdity is considered as an impossible command, it means that such a command cannot be followed, the derivation is thus compelled to go forward. This is a special case showing the compelling character of a proof.

The second resource for proof norm is what I intend to further develop: the conversationally illocational force ascribed to the explicit or implicit speech acts in the inference rules, which is commonly presumed between the prover and proof-readers.(Stalnaker, 2014 & Corredor, 2017) I call it "connecting force".

We here focus on one of logical proofs: natural deduction. Taking for instance ∀-introduction rule, ∃-elimination rule and some specific proofs in natural deduction, we shall analyze such a kind of connecting force in two dimensions. The first dimension is how different types of speech acts bearing the connecting force enhance *the internally procedural norm of a given proof*. The second dimension is how these speech acts foster *the externally interactive norm between provers and proof-readers*. By establishing a"connection coordinate", the role of speech acts in the proof norm can be concretely illustrated.

It should be noted that speech acts in proof don't imply that proof content cannot be attainable without illocutionary force. Instead, we can clearly identify and separate this force to preserve the content's "purity". Frege emphasized illocutionary force in his logical system to clarify its role and avoid mixing it with content. Later, some hid this force, creating an illusion that it either doesn't exist or is everywhere. By recognizing the force and its role in proofs, we prevent this misconception.

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9. Ruffino, M., Mauro, L.S., and Venturi, G. (2021). Speech acts in mathematics. Synthese, 198(10): 10063–107.
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14. Tanswell, F.S. (2024). *Mathematical Rigor and Informal Proof*. Cambridge: Cambridge University Press.
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# Symmetry Groups of Boolean Self-referential Systems

Hangjie Cao Ming Hsiung

**Abstract.**

The no-no paradox is a semantic paradox arising from symmetry, consisting of two self-referential statements that mutually declare each other false. Sorensen regards it as paradoxical because classical logic assigns different truth values to these two statements, which conflicts with their symmetry. Sorensen also generalizes the original no-no paradox to many other paradoxes of the same type. The paradoxical nature of the paradoxes of such type stems from the violation of a symmetry principle: in a self-referential statement system, statements with specific symmetry should be assigned the same truth value.

Our work in this paper is mainly based on a symmetry characterization for no-no type paradoxes proposed by [3]. Informally, the symmetry of a no-no type paradox can be characterized by a permutation group: all permutations within this group (and only these permutations) keep this paradox invariant when applying these permutations to interchange the indices of its constituent statements. For example, the no-no paradox remains invariant under the identity mapping $e$ and the transposition $(12)$ (swapping the two statements). The cyclic group $C_2$, generated by these two permutations, is defined as the "symmetry group" of the no-no paradox [3, p. 1926].

The notation of a symmetry group can be further generalized to Boolean systems. A Boolean system is a finite set of self-referential statements, where each statement is a Boolean combination of statements within the system. The no-no type paradoxes we focus on belong to the class of Boolean systems. Meanwhile, Boolean systems that are paradoxical in the more classical sense—those generating contradiction under classical truth valuations—are called Boolean paradoxes.

A natural question arises: which permutation groups characterize the symmetry of no-no type paradoxes? To this question, [3] posits the following conjecture: for every permutation group , there exists a no-no type paradox whose symmetry group

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is just . In [1], it is demonstrated that the alternating group $A_4$ cannot be the symmetry group of any Boolean system, and thus, it cannot be the symmetry group of any no-no type paradox. In our current work, we extend this result by proving that for all $n \geq 4$, the alternating group $A_n$ cannot be the symmetry group of any Boolean system.

We use binary sequences to represent possible assignments of statements in a Boolean system, and the semantic properties of the system are determined by a transition function defined between these sequences. By introducing permutations that act on binary sequences, we characterize the symmetry of a Boolean system as the equivariance of its transition function under permutation actions. To be specific, a Boolean system has symmetry with respect to ($\sigma$-symmetry) if its transition function is equivariant under the action of $\sigma$. Furthermore, a Boolean system has symmetry with respect to all permutations in a given group $G$ if its transition function is equivariant under the action of all permutations in $G$. The main non-representation result adopts the existing methods for studying symmetry of Boolean functions. As established in [4, p. 384] (cf. [2, p. 571]), a representation theorem states that a permutation group is representable by a Boolean function (i.e. it is the symmetry group of this Boolean function) if and only if it is the maximal one among all the groups with the same orbit partition on binary sequences. The sufficiency argument proceeds as follows: given two distinct groups sharing identical orbit partitions, any Boolean function preserving the symmetry of one group necessarily acquire extraneous symmetries from the other. As a consequence, functions intended to represent the smaller group invariably exhibit excessive symmetry, rendering the non-representability of the smaller group.

The proof of our main results requires, beyond these foundations, a further analysis of the specific symmetry requirements for transition functions. Crucially, we demonstrate that in addition to orbit partitions, the invariance preorder relations among stabilizers also play a fundamental role in determining a transition function constructions with specified symmetries. Our central theorem establishes that when two distinct groups induce identical structures in both aspects—that is, they share the same orbit partition on the binary sequences set and identical invariance preorder relation among stabilizers —then any transition function maintaining the symmetry of the smaller group must acquire extraneous symmetries from the larger group. This consequently demonstrates the non-representability of the smaller group.

As noted in [1], the alternating group $A_4$ and the symmetric group $S_4$ induce identical structures in both these two aspects. This paper generalizes this to any $A_n$ and $S_n$ with $n \geq 4$ , thereby proving that $A_n$ is non-representable in the sense described earlier. Therefore, no alternating group $A_n$ can be the symmetry group of a Boolean system.

A corollary of the non-representability proof is the preorder relation among sequences induced by the action of $S_n$. This enables us to delineate all possible constructions of transition functions endowed with maximal symmetry. We further analyze the sense in which transition functions reflect the paradoxical property of Boolean systems: one is the classical Boolean paradox, and the other is the no-no type paradox, which is of particular interest to us. Based on these, we provide construction procedures for both types of paradoxes with $S_n$ as their symmetry group respectively.

By applying group-theoretic tools to the structural analysis of logical systems, we provide a novel perspective for the study of semantic pathology. Building on existing work, this paper further clarifies the limitations of symmetry in paradoxes and the broader class of self-referential systems they represent: specifically, no alternating group $A_n$ for $n \geq 4$ can serve as the symmetry group of such systems. Additionally, we provide constructive procedures for several types of paradoxes under specific symmetry constraints, revealing the intrinsic connection between broader symmetry and semantic paradoxes.

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# Polymorphism and the Ontology of Number

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**Abstract**

In English, (1) “Jupiter has four moons” and (2) “The number of Jupiter’s moons is four” appear to have the same truth conditions, the inference from (1) to (2) seems to be trivial. But this also raises several puzzles, one of prominent ones is the so called FREGE’S OTHER PUZZLE (FOP): Whereas ”four” in (1) appears to be an adjective, functioning as a modifier, ”four” in (2) appears to be a name, functioning as a singular term, so how can a single expression (e.g.,”four”) possess different semantic functions while the inference from (1) to (2) seems to be trivial? From Frege on, several authors have attempted to resolve FOP without success. In this paper, following Snyder’s Polymorphic approach, we will argue that Polymorphism is the best solution to FOP. We will accompany apart from Snyder et al. on the philosophical consequences of Polymorphism: while Snyder et al. conclude that the polymorphic nature of number words supports Referentialism and Realism about numbers, we will conclude that the polymorphic analysis of number expressions results in at most Referentialism but not Realism.

**Keywords:** Frege’s Other Puzzle, Polymorphism, Numericals, Degrees, Type-Shifting

## 1 Introduction

In English, the following pair of sentences appear to have the same truth conditions:

(1) Jupiter has four moons.
(2) The Number of Jupiter’s moons is four.

The inference from (1) to (2) seems to be trivial. The trivial character of the inference has also been employed by many philosophers(Wright,Hale, 2001, Schiffer, 2005, Thomasson, 2009) as an easy argument to establish the existence of numbers. But this has raised several puzzles, one is the ontological puzzle: how can we obtain something from nothing? i.e., how can we get abstract objects so easily by understanding (1) and (2) while the ontological status of abstract objects is still hotly debated by philosophers? Another is called by Hofweber (2016) as FREGE'S OTHER PUZZLE (FOP):

> On the one hand, "four" occurs as an adjective [in (1)], which is to say that it occurs grmammatically in sentences in a position that is commonly occupied by adjectives... similar to "green" in [(3)]
>
> (3) Jupiter has green moons.
>
> On the other hand, "four" occurs as a singular term [in (2)], which is to say that it occurs in a position that is commonly occupied by a paradigmatic cases of singular terms,...[so that] "four" [in (2)] seems to be just like "Wagner" in
>
> (4) The composer of Tannhaeuser is Wagner.
>
> And both of these statements seem to be identity statements, ones with which we claim that what four singular terms stand for is identical.
>
> But that number words can occur both as singular terms and as adjectives is puzzling. Usually adjectives cannot occur in a position occupied by a singular term, and the other round, without resulting in ungrmammaticality and nonsense. To give just one example, it would be ungrmmatical to replace "four" with "the number of moons of Jupiter" in [(5)]:
>
> (5) Jupiter has the number of Jupter's moons moons.
>
> This ungrammaticality results even though "four" and "the number of moons of Jupiter" are both apparently singular terms standing for the same object in [(5)]. So, how can it be that number words can occur both as singular terms and as adjectives, while other adjectives cannot occur as singular terms, and other singular terms cannot occur as adjectives?
>
> Even though Frege raised this question more than 100 years ago, I dare say that no satisfactory answer has ever been given to it.(Hofweber, 2016, XXX)

Compared with the ontological puzzle, FOP in this form is a puzzle about semantics and syntax, it asks how a single expression of certain category can have different semantic functions in different syntactic positions while other expressions of the same category cannot without leading to ungrammaticality. It should also be noted that FOP arises not only in the pair (1) and (2), but also arises in other situations where the number word "four" possesses very different functions, as witnessed in the following sentences:[1]

(6)a. Jupiter's moons are four (in number). (predicative)
b. Those are Jupiter's four moons. (attributive)
c.Jupiter has four moons. (quantificational)
d. Mary drank four ounces of water. (meausrement)

[1]Here we use Snyder's examples in Snyder (2021) with minor changes

e. Jupiter's moons number four (in total). (verbal complement)
f.The number of Jupiter's moons is four. (specificational)
g. Mary is contestant number four. (ordinal)
h. Four is an even number. (numeral)
i. The number four is even. (predicative numeral)

In the above sentences, the labels "predicative", "attributive", etc. are employed to indicate how "four" is used in the accompanying examples. Those different usages of "four" are semantically significant in the following sense: on the one hand, just as Hofweber observed, when "four" is in different syntactic positions, it will probably obtain different semantic functions. For example, when "four" occurs in a predicative position (e.g., (6a)), it functions more like "green" in (7) below, where "green" has the semantic type of $\langle e, t \rangle$ :

(7) Jupiter's moons are green.

But when it is used as a numeral (as in (6h), semantically it is more like "green" in the following sentence, where it has the semantic type $e$:

(8) Green is a color.

On the other hand, just as "green" in (7) and (8) are systematically connected, the meaning of the number expression "four" in the above sentences are also systematically related: they are all the potential application of the number words in different situations: counting, measuring, and ordering.

So a successful resolution of FOP has to cope with these differences as well as connections. To be more specific, a good semantics for number words has to meet two desiderata: i) it should compositionally provide a semantics for all uses of number words, and ii) it should provide an account for the systematic connection of different meanings of number words. In this paper,following Snyder et. al.(2017, 2021, 2022, 2024) we will argue that the only kind of semantic theory capable of meeting both desiderata is Polymorphism, i.e., number expressions are polysemic, and they accomplish a systematic connection via the well-attested type-shifting principles. But we depart from Snyder et.al. on the philosophical consequences of Polymorphism and type-shifting: while Snyder et. al. argue that Polymorphism supports Referentialism and thus Realism about numbers, we will argue below that Polymorphism at most supports Referentialism but not realism. To be more specific, we will argue that words can function as singular terms in some cases and therefore are capable of referring, but that the most appropriate candidacy for this reference is a concrete property, *degree*, rather than *numbers*.

The paper is structured as follows: In section 2, we will review the literature on the different proposals to resolve FOP and argue that most of them are empirically unsupported. In section 3, we will present Snyder's (2021) Polymorphism and vindicate that it is one of most successful semantic theories to the number expression "four" in sentences like (6a)-(6g) listed above when we take "four" primarily as an adjective. We suggest then in section 4, when taking "four" as an adjective, its semantic value had better be a nominalized property correlate, *degree*. We will also develop an empirically more adequate but confined semantics for degrees, and conclude that when explaining for the differences as well as connections of meanings of "four" in (6a)-(6g), a nominalist friendly semantic theory of *degree* is enough, thereby debunking that Snyder's claim that Referentialism entails Realism. Then we will expand this semantic theory to the numerals cases such as (6h)-(6i), we will conclude that degree as an object is also capable of accounting for the fact that number words can be used in arithmetic cases without obviating the fact that number expressions are polymorphic and they are connected via a type-shifting principle. In section 5, we will consider two potential objections to our polymorphic approaches and argue that they are not worrisome as they appear. Section 6 is the conclusion.

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# 布劳威尔和胡塞尔对序数和基数的构造及其时间意识模型比较

于宝山　西北师范大学哲学学院

## 1.1 引言

对于我们如何认识数学对象的问题，胡塞尔和布劳威尔认为，数学对象的存在需要通过数学直观的意识构造，数学知识的客观性通过主观性获得认识论上的有效性。同时，数学直觉主义虽然致力于具体的数学对象的心智构造，但缺少充分论证的哲学基础；而胡塞尔的超越论现象学虽然致力于严格的意识哲学体系的构建，但缺少具体构造的数学对象。[1]因此，二者可以在数学认识中进行相互补充，对比解释，从而提供一种现象学的数学直觉主义的数学认识论解释。

胡塞尔的超越论现象学与布劳威尔的数学直觉主义之间的思想关系最早由其学生：贝克尔（Oscar Becker）、外尔（Hermann Weyl）、海廷（Arend Heyting）在各自的数学哲学中进行过引述和讨论，但都没有进行过系统的比较和阐释。[2]按照海廷的观点，数学直觉主义的主要特征在于（1）数学不仅是形式的，它也具有直观内容；(2)数学对象是通过心智直接感知的，但数学知识并不依赖于经验。[3]当前关于数学直觉主义与现象学的关联研究中，Richard Tieszen 批评了通常论述中将数学直观误解为神秘认识的观点，讨论了笛卡尔、康德、胡塞尔哥德尔关于直观的认识论观点，主张直观作为数学知识的必要条件。他认为胡塞尔与布劳威尔都认为数学本质上是一种无语言的思维活动，强调数学认识活动中心理行为与反思的关系。[4]Mark van Atten 通过现象学的分析来解读数学直觉主义，证明了选择序列作为数学对象的合法性，[5]并且通过胡塞尔的滞留-前摄的内时间意识结构，

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[1] 在本章的讨论中，为避免混淆，不会使用胡塞尔现象学中的直观（Intuition）替代布劳威尔数学直觉主义中的直觉（Intuition），二者都是时间性和意识构造（Constitution）的意义上使用。进一步地，我们也只在数学对象与形式系统的关系角度使用数学建构 (Construction)一词。

[2] 贝克尔对数学形式主义和直觉主义进行了系统的现象学解释，Cf. Becker, Oskar. *Mathematische Existenz, Untersuchungen zur Logik und Ontologie mathematischer Phänomene*. 2nd ed., Max Niemeyer Verlag, 1973 [1927].

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解释了布劳威尔将迭代行为作为直观对象的可能性。

在 Richard Tieszen 与 Mark van Atten 的研究基础上，本文首先分析布劳威尔基于空的二一性的时间直观对序数构造，然后比较这种时间意识结构与胡塞尔的时间意识结构的异同。在相似的时间意识结构的前提下，进一步讨论二者对基数与序数的不同构造方案和时间性的关系。最后分析，胡塞尔基于意向性之链和“如此等等”的构造模式对潜无限的构造，以及布劳威尔对“如此等等”的迭代模式的批评，讨论在此基础上的时间连续统与选择序列解释中的前摄问题。

## 1.2 胡塞尔与布劳威尔的关系背景：阿姆斯特丹讲座

1928 年 4 月 22 日至 29 日，胡塞尔应荷兰哲学家与语言学家 Herman J. Pos 的邀请，在阿姆斯特丹举行了两场题为《现象学的心理学与超越论现象学》的讲座。在邀请信中，Herman J. Pos 提及了他代表布劳威尔对胡塞尔的关注和邀请，并提议胡塞尔演讲他的数学哲学，但是胡塞尔拒绝了这个提议：

> 我很想见到布劳威尔先生，但我肯定会让他失望。因为目前的数学哲学与我现在的思想有些脱节，这需要我花费很长时间重新熟悉，尽管我过去曾经研究过这一领域。我更愿意讲“现象学心理学与超越现象学。”[1]

胡塞尔和布劳威尔最终在阿姆斯特丹会面。布劳威尔满怀激动地写道：“此时，胡塞尔就在我身边，强烈地吸引着我。”[2]胡塞尔在写给海德格尔的书信中对布劳威尔和他的助手也进行了高度称赞：

> 在阿姆斯特丹，与布劳威尔的长时间对话是最有趣的部分，他给我留下了非常深刻的印象，一个完全独创的、彻底诚实的、真正的、非常现代的人。他一直都有自己的哲学助手，一位非常聪明的女士，完全熟悉我的著作，也包括《观念》，她确实是我在阿姆斯特丹演讲后的讨论中唯一明智的人。[3]

胡塞尔和布劳威尔两人于 1928 年在阿姆斯特丹亲自会面并进行讨论时，其

[1] Brief. IV, S. 339-442.
[2] Van Dalen, Dirk. *Companion to The Selected Correspondence of L.E.J. Brouwer*. Springer, 2011, P. 1515.
[3] Brief. IV, S. 156.胡塞尔致海德格尔1928年5月9日。

具体对话内容已经无从得知，但彼时各自的学术体系都已经成熟[1]。在1928年5月19日致英伽登的信中，胡塞尔提到布劳威尔计划到弗莱堡拜访他。此外，在5月12日致曼科的信中，胡塞尔谈及布劳威尔曾向他询问阿姆斯特丹大学一个空缺的哲学教职的任命事宜，并提到贝克尔。胡塞尔则推荐曼科和贝克尔作为该职位的候选人。同时，胡塞尔催促海德格尔筹划出版已经由施泰茵修订的关于内时间意识现象学的早期讲座稿与研究稿。他在其中提及了布劳威尔，并和海德格尔讨论了《内时间意识的现象学讲座》中时间意识和内在时间意识的区分问题。但我们无法确定，胡塞尔对于时间讲稿的出版动机是否与布劳威尔的会谈有关。

这些书信构成了胡塞尔与布劳威尔会面的历史见证，但是这次历史性的会面对胡塞尔的哲学发展或布劳威尔的数学发展并没有产生某种深刻的联结。与魏尔斯特拉斯、黎曼、康托尔和希尔伯特等人对胡塞尔的影响不同，布劳威尔的著作在胡塞尔的哲学中几乎没有得到任何直接引用。同样的，布劳威尔本人也并没有直接论述过现象学的思想资源。当布劳威尔作为学生（大约在1904年）开始发展直觉主义时，胡塞尔已经出版了两部关于数学和逻辑哲学的主要著作：1891年的《算术哲学》和1900—1901年的《逻辑研究》，但布劳威尔似乎并不了解这两部作品。在他1904—1907年期间准备论文的九本笔记本中也根本没有提到胡塞尔。[2]因此海德格尔认为“直觉主义在本质上受到现象学的影响”的观点是错误的。[3]另一方面，胡塞尔拥有一份布劳威尔的论文《形式主义和直觉主义》，该论文收藏的时间是在胡塞尔发表《形式逻辑与超越论逻辑》之后，[4]他在1912年的一份手稿中提及到了布劳威尔与外尔：

> 一个人在对数学的判断中，**其全部意义完全取决于这些[基础]概念，是否应该遵循希尔伯特、布劳威尔或其他人**？我们能否如此确定，尽管这正是今天的共识，但古典数学和同样的物理学没有得到更好的建议？但我们不会在那里做得更好。它从未完成，而是它自身正在形成，因此问题重演，不可

[1] 在1928年3月10日和14日，布劳威尔在维也纳举办了两场讲座，哥德尔和维特根斯坦都在其中。在另一场讲座之后，布劳威尔和维特根斯坦在一起讨论了整整一天的数学基础问题，这促使维特根斯坦决定重新返回哲学。到了4月份，布劳威尔在阿姆斯特丹与胡塞尔进行了会面和交谈。相关信息可参见https://plato.stanford.edu/entries/brouwer/1928–1929。

[2] Van Atten, Mark. *Brouwer Meets Husserl: On the Phenomenology of Choice Sequences*. Springer, 2007. p.278.

[3] 海德格尔：《时间概念史导论》，欧东明译，商务印书馆，2014，第4-5页。

[4] 米雅·哈蒂莫：《胡塞尔的科学背景（1917—1938）：胡塞尔私人藏书室调研》，于宝山译，《中国现象学与哲学评论》2023年第2期，第351-411页。

能有一个确定的选择来确定我们的规范。[1]

## 1.3 胡塞尔与布劳威尔的时间意识模型比较

### 1.3.1 布劳威尔与胡塞尔论数学对象的全时性

布劳威尔的直觉主义与逻辑主义、形式主义的不同之处在于，数学不是依赖于外部公理或逻辑规则，而是一个基于时间直观的内在构造过程。在其就职演讲《直观主义和形式主义》中，将他的数学哲学立场表述为“我放弃康德的空间的先天形式，但更坚决地坚持时间的先天形式”。[2]这种数学直觉的内在时间形式虽然符合康德关于时间定义的先天形式，但与康德不同的是，布劳威尔更加关注时间意识发生和数学构造。布劳威尔进一步区分了“直观”的时间和“科学”的时间，他认为，数学认识源自对时间意识的内在感知，是一种自主的内在心智的构造活动。[3]通过时间意识，我们能够构造出基本的数学概念和对象。因此，直觉主义数学要求任何数学命题的“存在性”必须通过数学直觉的构造证明。

从静态现象学到发生现象学的转变中，胡塞尔关于数学对象与时间性的关系论述也发生了关键转变。在《逻辑研究》时期，他认为数学对象是无时间的（Unzeitlich），[4]而在《经验与判断》中，则从构造性的角度将数学对象定义为全时性（Allzeitlich）的。[5]一方面，作为形式对象，它们的概念中不包含任何感性内容，而是指涉纯粹地适用于一般对象的结构，保持纯粹的形式特性。另一方面，数学对象要成为意识对象，就必须与意识发生关系，始终在意识中被重新理解，与整个时间范围相一致。在 1918 年的贝尔瑙研究手稿中，胡塞尔指出，观念对象的超时间性（Überzeitlichkeit）并不意味着与时间性无关。与之相反，其本质在于它能够在任何一个时间点上本原的构造，在意识的统一性中被重复的构造且具有同一性。具有重新意向生活的所有构造性综合都建基于内时间意识的被动原

[1] 转译自 Van Atten, Mark. *Brouwer Meets Husserl*, pp.64-65.具体可参见：胡塞尔手稿 D15, S.7-23（Husserl. *Urassoziation und Zeitigung; Konstitution des Realen, Zeit, Raum, Kausalitä*t. Transcribed by Marly Biemel, corrected by Karl Schuhmann, 1932）.

[2] Brouwer, L. E. J. “Intuitionism and Formalism. ”*Bulletin of the American Mathematical Society*, vol. 20, 1913, p. 85.

[3] Brouwer, L. E. J. Over de grondslagen der wiskunde. *PhD diss*., Universiteit van Amsterdam, 1907. English translation in Brouwer, 1975, p. 61; Brouwer, L. E. J. “Brouwer’s Cambridge Lectures on Intuitionism.” *In D. van Dalen* (Ed.), Cambridge University Press, 1981, p. 92.

[4] Hua XIX/1, S. A124/$B_1$123.

[5] EU, § 64.

综合之上或者活的当下。[1]因此，数学的客观性必须通过主观性的构造才能解释其有效性：

> 数学家并不以反思的方式探寻主观上构造的客观性的意义和可能性之主观来源和最终问题。这件事是哲学家的任务。[2]

综上，胡塞尔和布劳威尔都同意，客观知识的有效性都必须直接或间接地回溯至有限性主体的直观或构造，在其自身经验中被直接给予。

**1.3.2 布劳威尔的二一性与胡塞尔的三一性时间意识结构**

数学直觉主义为理解数学对象提供了一种基于时间感知的构造理论。布劳威尔强调，时间的感知和记忆（保持）是数学直觉的核心，而这种直觉通过自我展开的方式生成了数学对象。布劳威尔使用“二一性”（two-ity）来描述时间意识中的基础结构，即意识如何感知当前时刻与刚刚过去时刻的关系。这个结构对数学构建至关重要，因为它为构建数学对象提供了基础。布劳威尔的基本观点是数学直觉植根于对时间流动的感知。这种感知中可以描述为一个生命时刻分裂为两个完全不同的单元，其中一个会依次让位于另一个。主体因此首先有了对第一单元的感知，随着感知的单元让位于另一个单元，主体又有了对新的第二个单元的感知，但同时又在记忆中持存着对第一个单元的感知。在这种感知模式下，二一性出现了。布劳威尔进一步指出，当我们意识到这种“二一性”作为一种形式，不考虑在各个阶段给出的感官内容时，我们就拥有了布劳威尔所说的“数学的基本直觉”。主体能够剥离这种二一性的所有质性和感觉材料，从而获得“空的二一性”（empty twoity）[3]，这种空的形式就是数学的基本直觉。

> 直觉主义数学是一种本质上无语言的心灵活动，其源起于对时间流动的感知。这种时间流动的感知可以描述为一个生命时刻的分裂，分裂成两个截然不同的事物，其中一个让位给另一个，但被记忆持留。如果这样产生的二一性被剥夺了所有的性质，它将进入所有二一性的共同底层的空的形式。正

[1] Hua XXXIII, S. 322；Hua I, S. 79.
[2] 埃德蒙德·胡塞尔：《逻辑学与认识论导论》，郑辟瑞译，商务印书馆，2016，第 204、453–454 页。
[3] Brouwer, L. E. J. “Historical Background, Principles and Methods of Intuitionism.” *South African Journal of Science*, vol. 49, 1952, pp. 139–146.

是这种共同底层，这种空的形式，构成了数学的基本直觉。[1]

在内部时间的流动中，人们可以意识到一个生命时刻分裂成两个不同的事物，其中一个让位给另一个但被记忆保留。人们可以辨识出一个之前和之后，总有一个“之间”。

> 这个基础，剥离了一切质性，剥离了任何变化的感知，是连续性与离散性的统一，是将若干实体通过一个“之间”连接在一起的可能性，这一基础不会被新的实体的加入所耗尽。[2]

布劳威尔指出，离散单位之间总是存在某种“之间”状态，连接我们在时间中感知两个事物的两个元素。这个“之间”不能通过新单元的插入而耗尽。这个“之间”永远不能被认为仅仅是单元的集合。[3]离散和连续是互补的，因为时间的离散性或分离性的要求是存在一个“之间”状态，而时间流动的意识只能通过对过去和现在的认知，或者过去和现在的区分来实现。

空的二一性及其可迭代操作是从时间移动体验的内部结构中抽象出来的。布劳威尔通过“二一性”这一概念，展示了从个体经验中抽象出数学概念的过程，这个空的“二一性”和它所组成的两个单元构成了基本的数学构造单位[4]。这一过程不仅适用于形成数字一和二的基本直觉，还可以将二一性的其中一个单元视为一个新的二一性，这一过程可以无限重复，形成所有限序数。这里需要注意的是，空的两一性的第一和第二单元，必须彼此不同。如果两个单元最终是相同的，那么显然就违反了布劳威尔规定的“一个事物让位于另一个事物”的直觉。但仅仅从一个单元开始获得自然数是不够的。一旦我们意识到感知中的一个单元转变为另外一个单元，我们便有了从中生成自然数的抽象二一性的基础。例如，如果我们将二一性的两个单元和相继关系规定为：$x_0=|(|)$，括号用来表示时间上的后继的记忆内容。如果我们总是用一个二一性替换最左边的字符串，这种原“二一性”便可以成为一个新的二一性，则我们得到：$x_1=|(|)=|(|(|))$，$x_2=|(|(|))=|(|(|(|)))$……$x_n=|(|(|(\cdots(|))\cdots))$（共滞留 n 次）。其中括号内的部

[1] Brouwer, L. E. J. *Brouwer's Cambridge Lectures on Intuitionism*. Cambridge University Press, 1981, pp. 4–5.
[2] Brouwer, L. E. J. Over de grondslagen der wiskunde. *PhD diss*., Universiteit van Amsterdam, 1907. English translation in Brouwer, 1975, p. 17.
[3] Brouwer, L. E. J. *Brouwer's Cambridge Lectures on Intuitionism*. Cambridge University Press, 1981, p. 40.
[4] Brouwer, L. E. J. “Points and Spaces. ”*Canadian Journal of Mathematics*, vol. 6, 1954, p. 2.

分则表示我们刚刚所持留下来的次级回忆。这个过程意味着第一个原感觉在绝对的过渡中流动着的转变为它的滞留，这个滞留又转变为对此滞留的滞留，如此等等。但同时随着第一个滞留而有一个新的“现在”、一个新的原感觉在此，它与第一个滞留以连续—瞬间的方式相联结，以至于这河流的第二时段是这个新的现在的原感觉并且是以前的现在的滞留，而第三个时段中又是一个带有第二个原感觉的滞留的原感觉并且是第一个原感觉的滞留的滞留，如此等等。[1]

> 内省中可以意识到，这一基本操作通过记忆的持续保持，逐步生成每一个自然数、自然数的无限序列、任意有限序列，以及先前获得的数学系统的无限延续序列，最终形成一个不断扩展的数学系统，对应于经典数学中“可分”系统的概念。[2]

所有数学概念的构造都始于空的二一性所包含的两个单元作为起点以及形成二一性的可迭代性。这些构造基于脱离语言的内省意识和反思行为。我们通过内省意识，将二一性的一个单元视为一个新的二一性的基本操作，通过记忆的不变的持留，连续生成无限进行的自然数序列以及无限序数ω。

> 作为心智，**它承担了体验现在和过去的感觉作为对象的功能**。通过对这种二一性现象的重述，该对象可以扩展到一个充满杂多感知的世界。[3]

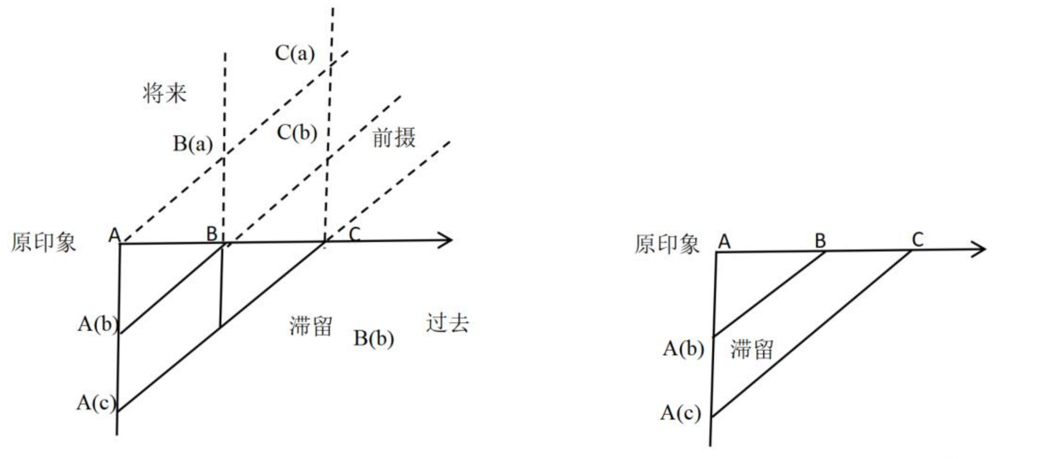
在胡塞尔看来，我们对时间的感知是一个统一的连续统，由原印象（primary impression）、滞留（retention）和前摄（protention）构成。在这个模型中，内在时间从原印象开始，这些原印象在意识中短暂“持留”，并且在这一保持过程中不断被变异：随着我在每一瞬间接收到新的原印象，之前的印象并没有完全从意识中消失，而是以变异的形式留存下来，它不再被视为当下的，而是被视为刚刚过去的。这种持留前一刻的原印象的意向性，我们称之为“滞留”，也就是布劳威尔在前面所论述的刚刚过去的被记忆的感知：主体又有了对新的第二个单元的感知，但同时又在记忆中持存着对第一个单元的感知。但是，在布劳威尔的空的“二

[1] Hua X, S. 435.进一步的论述还可参见肖德生：《胡塞尔在贝尔瑙手稿中对前摄的描述与分析》，《中山大学学报（社会科学版）》2011 年第 3 期，第 109—116 页。

[2] Brouwer, L. E. J. “Points and Spaces.” *Canadian Journal of Mathematics*, vol. 6, 1954, p. 2.

[3] Brouwer, L. E. J. “Consciousness, Philosophy, and Mathematics.” P*roceedings of the 10th International Congress of Philosophy*, Amsterdam, 1948, p. 1235.

一性”结构中，并没有出现前摄的要素。前摄在此作为滞留的对立面出现，其功能与滞留类似，但指向未来。[1]我们在这里改变布劳威尔的“二一性”中原印象—滞留的结构顺序┃(│）为（│）┃，这里（│）从滞留扭转为前摄。如果在时刻 $x_0$ 的前摄为（│)，那么在 $x_1$ 时刻则为：（│）│，此时 $x_0$ 时刻的前摄（│）充实为 $x_1$ 原印象│，此时 $x_1$ 时刻的前摄为（│)，以此类推。但是由于在布劳威尔的空的“二一性”结构中缺乏前摄要素，尽管前摄与滞留都是充实发生所必需的，[2]但二者的意向本质不同，因此我们用胡塞尔的字符而非布劳威尔的笔画串来刻画下图中时间意识的性结构。




胡塞尔的三一性时间意识结构　　布劳威尔的二一性时间意识结构（无前摄）

**图 1**

如果在时刻 B，我感知到 B 并且滞留流到 A，Ba，那么我还会预期 C，以此类推，在下一个瞬间 C，我将感知 C，滞留 B，Cb，并且保持 A 的二次滞留，Ca，同时预期 D，等等。[1]因此，胡塞尔与布劳威尔关于时间意识结构的共同之处在于：在后继阶段中，刚刚直观的内容以适当的方式持存并被记忆保留，这表明了记忆在构造过程中的重要作用。更具体地说，随着构造的开始并持续进行，刚刚直观的部分成为新鲜的回忆，虽然它们被保留并保持活跃，仍在处理构造的当前部分。实际上，如果不是这样，构造就无法作为一个统一的整体被感知。我们可以将这种保持过程的图示想象为图 1。水平轴表示连续的“现在”时刻或阶段的“流动现在”。水平轴下方的对角线表示所选择阶段中意识的一部分“持存”，这些保持随着它们逐渐沉入我们当下经验的较远部分而不断变化。在水平轴上，

[1] Hua X, S. 55.
[2] Hua XXIII, S. 46.

我们有一系列的后继者，而在每个阶段的垂直轴上，表示它们是如何在该阶段的一个意识中被统一或整合的。在水平轴上，我们有“多”，而在垂直轴上，我们有“一”，即综合。布劳威尔强调，在这个连续的、顺序的结构中，通过“之前”和“之后”的排列，我们的意识中如何将“之前”和“之后”或“一-二”统一在一起，从而使得我们在多中获得一。

从现象学角度看，布劳威尔的分析触及了数学概念构造的最深层次——所有对象在内时间中的构造层面：对时间直观的当下感知和对刚刚过去的直观的记忆的持留为布劳威尔的这种数学认识模式提供了必要的构造基础，也就是胡塞尔所谓的内时间意识结构的原印象和滞留，但是布劳威尔这里缺失了前摄环节。前摄涉及未来，而滞留及过去。[1]前摄通过对原印象充实的在先朝向而区别于滞留。胡塞尔认为，前摄之所以与滞留本质不同，是因为前瞻具有“努力”特征，前摄的努力特征是一种被动的指向性，这种意向性本质上属于前摄，它保证了时间的迭代进行。虽然滞留可以获得意向性，但它并不本质上具有意向性，[2]滞留使得时间的连续成为可能。胡塞尔认为意识的河流，在自身中朝向前行的方向是前摄，向着相反的方向则是滞留。在经验中，过去和未来之间存在一种不对称性。作为滞留的变异，在次级回忆中的过去是确定和固定的，我们没有改变它的自由。然而，前摄的意向和相关的未来是开放的，且更为不确定或模糊。我们将在下面分析布劳威尔和胡塞尔对序数和基数的不同进路的构造之后，最后讨论，由于选择序列在直观连续统构造的过程中对排中律的拒斥和其自由选择的自由度不可预测性和无法充实，布劳威尔在时间意识的论述中缺失了前摄的时间结构要素。

## 1.4 胡塞尔与布劳威尔关于基数、序数以及潜无限的构造性分析

在时间意识结构相似的前提下，胡塞尔和布劳威尔对数的概念分析和构造提出了相反的方案。胡塞尔认为数的基本概念是基数，序数在起源上依赖于基数。而布劳威尔则认为数的基本概念是序数，基数在起源上依赖于序数。我们将在下面分析这两种不同的构造方案及其差异的原因。

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[1] Hua X, S. 77; 373.
[2] Hua XI, S. 73;76-77.

### 1.4.1 胡塞尔对基数和潜无限的构造

胡塞尔认为数的基本概念是通过集合联结形成的基数（Anzahl），[1]而且任何数学哲学都必须从对数的概念的分析开始。

> 因此，在任何情况下，对基数（Anzahlen） 概念的分析都是算术哲学的重要前提。这是其首要前提，除非有证据表明，逻辑上的优先权属于序数概念，正如其他观点所主张的那样。如果能够在完全忽视序数的情况下对整数概念进行分析，那么这将是证明该观点不可接受的最有力证据。[2]

胡塞尔将数规定为多，他认为数是对“多少”的回答。因此，对数的概念的理解首先要从多的概念开始，而概念则又是具体现象的抽象的表象，所以数的表象也就奠基于多的现象。这些具体的多的现象构成一个集合，胡塞尔称这种“确定的全体或多”为集合表象。而使集合表象成立的可能性条件是什么呢？胡塞尔认为集合表象的特征在于其元素的完全任意性：一本书、一杯咖啡、一个人或者一只猫、一棵树、一轮夕阳，但这些不同类型的内容和现象的多之间存在一种使得元素联结成集合的共有结构，胡塞尔称之为“集合联结”：

> 每个表象客体，不管是物理或心理的、抽象的或具体的，也不管是通过感觉还是想象而被给予，都能够和任意其他的、无论多少的客体统一成一个全体，并因此也能够被计数。[3]

但是集合联结如何脱离于其自身的元素而被抽离？作为具体物的元素如何成为“某物”？胡塞尔在这里运用了注意力这个概念的功能。经过注意力的抽象之后，元素成了“任意的某物或任意的一”。集合联结的语义表达“和”与“一”作为两个形式结构单元共同构成数的概念内涵：一和一、.....，其中“……”代表不确定性。随着集合联结的进行，多的概念就会分解为不同的多的数：2、3、4。在数数的过程，我们得到集合 A 中有 3 个元素，基数就是 3。这时的 3 并不是指“第三个”，而是指总共有多少个元素——这就是基数的含义。在此基础

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[1] 为了方便比较，我们在这里将胡塞尔意义上的“个数”广泛地定义为基数。因为胡塞尔的“个数”意味着较小的可数的基数。

[2] Hua XII, S.14.

[3] Hua XII, S. 32.

上，胡塞尔认为基数（Anzahlen）指代集合，序数则指代序列，而序列是有序的集合。因此序数的概念实际上包含并预设了基数的概念。[1]但是从上面的分析可以看出，胡塞尔在早期的《算术哲学》中并没有将意识的时间结构与基数的起源或构成联系起来。

我们已经在第一章讨论过《算术哲学》中本真表象和符号表象的问题。胡塞尔认为通过符号表象，集合的概念不再局限于“偶然的”或有限的集合，而且突破了“所有认知本质上的必要限制”，进入了无限的领域。例如，在集合论中，我们可以用符号如“{x|x 是自然数}”来表示自然数集合，而无需列举出所有的自然数。[2]在此基础上，胡塞尔讨论了有限主体对无穷集合的认识及其符号表象的问题，并且区分了较大的有限集与无限集。有限集合的生成通常是通过有限步骤完成的，每一步都可以明确地确定并操作。当我们讨论自然数集合时，我们通过一个递增过程逐步生成集合中的元素。例如，假设我们需要处理一个集合{1, 2, 3, ... , 100}，这是一个有限集合，虽然我们不能在一瞬间“列出”每个元素，但我们可以逐一地构造每个元素的观念，从 1 开始，每次都加 1，生成新的数字（2, 3, 4, ...）其生成过程在达到 100 时就终结。这种逐步构造的认识方式适用于有限集合，因为它们有一个明确的边界，允许我们通过不断扩展来完成对集合的理解。但是当讨论无限集合时，我们不仅面临着无法通过逐步展开元素来完成认知的问题，还需要考虑无限集合的不可操作性。例如，考虑实数集合 R，它不仅包含所有的有理数（可通过有限步骤表示），还包含无理数。无限集合的生成不可能在任何一个特定的时刻完成，因此在无限集合的生成过程中，谈论“最后一个元素”或“最后一步”显然是荒谬的。虽然我们可以用符号化方法表示 R，但我们无法通过任何有限的操作或认知步骤来完全“列出”其中的所有元素。胡塞尔认为无限集合的本质是无限延续过程而非最终形成的集合。尽管我们无法在有限时间内完全列出所有自然数，但通过引入“如此等等”这一延续的过程，我们依然能够理解如何从一个有限步骤的递增过程延续到无限的。这一思路不仅适用于自然数集合，也可以扩展到其他无限集合的构建，例如直线上的无限点集。我们可以想象直线上的某些点序列，先通过某种已知的规则分布这些点。例如，假设我们在一条直线上分布了一些点，可以通过给定的规则决定每个点的位置。

[1] Hua XII, S. 12.
[2] Hua XII, S. 218.

这样，起初我们可能只能表示一个有限的点集。然而，借助于无限延续的观念，我们可以通过插入更多的点，使得这些点在直线的每两个相邻点之间逐渐填满，从而形成一个无限延续的点集。通过这一系列分析，我们可以清楚地看出，集合的概念，特别是无限集合的概念，不能仅仅依赖于有限集合的构建方式。虽然我们可以通过符号化表达有限集合，但在讨论无限集合时，我们必须引入一个扩展的概念，即通过无限延续的过程观念来构建集合。这一过程不仅是逐步扩展的，而是无限的，且其每个步骤都由明确的生成规则所决定。最终，这使得无限集合成为一个抽象的过程概念，而非一个简单的、通过逐一列举元素形成的集合。[1]

胡塞尔认为，尽管我们无法在有限时间内完全列出所有自然数，但通过引入“如此等等”这一持续的过程，我们依然能够构造潜无限。在《算术哲学》之后的《形式逻辑与超越论逻辑》的第 74 节“‘以此类推’之观念性，构造的无限性之观念性，及其主体相关项”，胡塞尔进一步给出了基数与无限集合的主观构造性起源。

> 我在这里想到了一个从未被逻辑学家强调过的“以此类推”（Und so weiter）的基本形式，这种迭代的“无限性”在主观上有其对应物，即“以此类推”。显然，这是一个观念化的过程，因为实际上没有人能够真正无限重复。然而，它在逻辑中无处不发挥着意义决定的作用。我们可以不断回到一个观念的意义单元，进而回到任何一个观念的统一体[…]数学是无限构造的领域，是一个观念性存在的国度，不仅包含“有限”的意义，也包含构造的无限性。显然，主观构造起源的问题在这里以隐藏的构造方法重复出现，这种方法应该被揭示并作为规范重建。[2]

他认为在逻辑学与数学构造过程中，“以此类推”的使用，表明我们能够一再重复地返回某一观念或过程。这一重复的行为，不仅是形式上的机械操作，更是一种观念化的延续过程。人们在构建无限集合时，实际上并没有通过逐一列举所有元素的方式，而是通过符号化手段，我们能够在观念上构造出一个无限延续的集合。每一个新的元素并不是通过物理操作得来的，而是通过符号规则不断迭代递推出来的。从某个符号初始值“a”开始，定义一个简单的操作“加 1”，通

[1] Hua XII, S. 218-221.
[2] Hua XVII, S. 196.

过每次加 1 得到下一个符号值“a+1”，并继续这一操作：a → a+1 → a+2 → a+3 → ... 这一过程是迭代的，一种通过自身定义的方式来构造或展开的过程，每一步都依赖于前一步的结果，并无限地递推下去。这个过程隐含着“以此类推”的思想，即每次加 1 的“以此类推”的迭代递推操作，通过这种符号迭代的方式，我们能够观念化地构建出一个无限的数列，符号化地表示一个潜在的“无限”过程。

胡塞尔将无限的概念作为通过“先天的可构造性”或“可计算性”构造出来的逻辑上明证的对象，且在“逻辑上是闭合的”。因为我们将“一”理解为最普遍意义上的抽象单元，集合从而被理解为相互独立单元的迭代关系组合。[1]这种集合概念的形成是通过不断加一（如 2 + 1 等于 3，3 + 1 等于 4 等）的系统化迭代过程。

> 此外，计数是一种概念方法，在这种方法中，通过向已知数的独立单元添加新单元来生成数字，这一过程不仅具有一个“开放”的视域，而且被设想为一种**逻辑明证性：“对于一个任意数，总是可以无条件地（加 1）……”**。由此产生了一个无限的基本概念，即无限性作为一个闭合的、预先完成的逻辑思维领域，它由先天的可构造性、“可计算性”来界定，成为“可构造的观念”的视域——这是算术的、科学的工作领域。[2]

> 我可以随即思考一个生成序列：1、2、3，依此类推。这是一种在观念上可实践的意向和实现（Verwirklichung）。在此过程中，每个生成物都作为一个过渡环节—意向在此过程中得以充实，但又通过它作为最初的意向对象而指向下一个环节。在其中通过现时化（Aktualisierung）而得到充实，但这又仅仅只是一个过渡（环节），以此类推[…]这意味着：我在进行生成活动，并由此构成了一条实践的意向性之链，一条由意向与充实组成的链条，意向性贯通于每一个充实的环节。[3]

胡塞尔在这里通过意向性之链进行潜无限的构造。意向性之链（Verketten der

[1] Hua XXIX, S. 204.
[2] Hua VII, S.152.
[3] Lohmar and Carlo Ierna. “Husserl’s Manuscript A I 35.”*Husserl and Analytic Philosophy*, edited by Guillermo E. Rosado Haddock, De Gruyter, 2016, pp. 289-291.

Intentionalität）是由意向与充实组成的链条，意向性贯通于每一个充实的环节，它穿过当前环节而指向下一环节，并在后者得到充实时并再次穿过，将前序意向性传递至后续环节，将离散的计数行为统摄为连贯的意向性链条，使得每个数字既作为“已生成序列的终点”（如“5”终结了“1-4”的意向性环节），又作为“未完成序列的起点”（如“5”指向“6”的可能性）。从一开始，意向性就作为统一体而延展，使得每个充实既是终点又是过渡点。胡塞尔认为实际计数行为始终受限于具体计数对象（如“3 个苹果”），但通过“依此类推”的规则，意向性可构造潜无限的序列。这种无限性并非经验外推，而是通过形式化（如皮亚诺公理中的“后继函数”）被构想为观念对象。由此，数学计数成为一种纯粹的概念操作和逻辑推演。基于对“一”和“集合”之间的重复递归和迭代，就可以引入“潜无限”的概念。因为我们总是可以在一个已有的数后面再加一个“1”，每加一个“1”就是在前面的基础上进行的再次构造。每个集合形式（如“包含 n 个元素的集合”）均可被观念化为“更大集合”的基础，从而过渡到更高阶的数（如 n+1）。[1]这种重复和同一性确保了无限概念的逻辑明证性。如果每次的迭代都是相同的结构（比如总是加 1），那么每一步的结果都在某种程度上是“自我重复”的无限，不再是一个没有规则、无法控制的扩展，而是依赖于某种结构来保持其可理解性和“有限性”——它通过固定的步骤和规则，在无限的过程中建立一种某种程度上封闭的，成为“可构造的观念”的视域，具有了胡塞尔所说的逻辑明证性。

通过符号表象获得的无限性，虽然超越了有限的认识主体，这种无限性本质上却仍然源于人类体验中的时间维度。这种形式上的无限既不是实际存在的，也不具有高斯所谓的神的计算的特征，而是一种时间意义上的无限。因此，无限集合只能作为“潜无限集合”存在。根据胡塞尔对无限集合的表象递推迭代方式，无穷数列的表象方式是由符号表象和在时间行为中无限叠加的“以此类推”将无穷表现为“潜无限”。对于有限主体的时间性构造和认识而言，我们只能通过一种不断扩展、潜在的方式来“接近”无穷，而无法将其作为一个完整的对象构造出来，无限集合的概念不仅仅是一个集合本身，而是一个通过过程不断生成的集合。尽管没有终点，但每一步的生成都有明确的定义和规则，每一个新的元素都由前面的规则推导出来，通过符号和逻辑规则，可以不断地生成新的元素，生成

[1] Lohmar and Carlo Ierna. “Husserl’s Manuscript A I 35.”*Husserl and Analytic Philosophy*, edited by Guillermo E. Rosado Haddock, De Gruyter, 2016, pp. 289-291.

潜无限集合。胡塞尔正是通过这一点阐述了无穷集合概念的动态性质，即它是一种开放的、未完成的构造，而非一个完全的、实际上可以把握的集合。朝向实无限的构造意向是“荒谬的”，无限集合仅能作为潜无穷的集合活动被设想，而不能够作为实无限的概念化存在，这不仅仅是人类的认知局限，而是一种逻辑上的不可能性，甚至“上帝”也无法做到这一点。

综上所述，胡塞尔通过对有限性和无穷性的现象学考察，提出了一种基于潜无穷的现象学的超越论构造方式：通过符号化的“以此类推”的“a”到“a+1”的迭代，建立了一个潜无限的过程，否定了传统神学中关于“神的算术”以及无限智力的假设。这一思考过程不仅回应了高斯“神在计算”的批判性讨论，[1]也引出了对无限问题进行超越论现象学构造的意识行为的迭代问题。

### 1.4.2 布劳威尔对序数和潜无限的构造

布劳威尔认为数的基本概念是建立在二一性基础上的序数（Ordinalzahl）。从时间移动的体验结构中抽象出来的空的二一性及其嵌套迭代结构是序数概念的一部分。

> 如果由此产生的二一性被剥去一切性质，那么剩下的就是所有二一性共同基础的空形式。正是这个共同基础，这个空形式，构成了数学的基本直观。[2]

在布劳威尔对时间意识和序数生成的分析中，二一性的两个部分不是相互独立的，其中一个部分嵌入另一个部分，并且不断分化迭代，这种嵌入式的迭代确定了顺序关系。首先，空的二一性扩展为|(|)，括号用来表示时间上的滞留内容，我们定义为序数 2，通过进一步的抽象行为，人们从空的二一元中得到数字 1。因此，从发生学的角度来看，序数 2 比序数 1 更为基础。随着原印象不断地下坠回退为滞留，二一元性的后半部分分裂成一个新的二一性，旧的二一性成为其中的一个部分。如果我们总是用一个二一性替换最右边的划线，则得到|(|(|))，这个新的二一性形式现在对应于数字 3。因为二一性的其中一个元素可以被视为一个

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[1] 关于胡塞尔对高斯的批评，我们将在本文的 8.4 节中对《算术哲学》中的潜无限和实无限问题的进一步论述中展开。

[2] Brouwer. “Historical Background, Principles and Methods of Intuitionism.” *South African Journal of Science*, *vol.* 49, 1952, p. 139.

新的二一性，只要单位的一个元素可以被认为是一个新的二一性，这个过程可以无限次重复，从而产生最小的无限序数ω（即 1,2,3，之后的第一个数字）。[1]布劳威尔试图试从康托尔的第二数类（所有可数无限序数的类）和更高数类中构造出意义时，他意识到这是行不通的，于是拒绝了更高数类，只留下所有有限序数和未完成或开放的可数无限序数集合。

从上面的分析可以看出，胡塞尔没有将意识的时间结构与序数的起源或构造联系起来。胡塞尔与布劳威尔关于基数和序数构造的分歧根源在于数字背后的心理学基本概念不是集合的概念，而是序列的概念。[2]因为集合的概念依赖于序列的概念，所以基数的概念依赖于序数的概念。序数帮助我们在数数时定义每个元素的顺序，而基数则总结了数数的总结果——即元素的数量。数数时的“第一、第二、第三”正是序数的作用，而最终的“3”则是基数的结果，表示集合中有多少个元素。在胡塞尔的分析过程中，我们可以看出，基数是通过数数的过程得来的。基数 3 的数数过程实际上是通过序数的递增（第一、第二、第三）来完成的。通过这个递增的序数过程，我们得到了基数 3，表示集合中有 3 个元素。对于有限集合，数数的过程是有限的，而对于无限集合，尽管数数的过程不能完全完成，但我们依然通过序数递增的方式来描述它的大小（基数）。因此，我们在分析之后得出结论：布劳威尔对序数的描述比胡塞尔在算术哲学中的描述更加准确，因为对于基数 n 的理解，必须是从结构中的单位顺序中抽象出来 。[3]

布劳威尔基于直觉主义的序数构造，进一步考虑了胡塞尔指出的这种“如此等等”的基本模式是否可以满足康托尔的超限数：

> 在这里，康托尔提到了某些无法被思考的东西，即无法被数学构造的东西；因为通过“如此等等”方式构造的整体只有在“如此等等”指向相同对象的序类型 ω 时才能被思考，而这个“等等”既不指向序类型 ω，也不指向相同的对象。在这里，康托尔失去了与数学坚实基础的联系。[4]

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[1] Brouwer. *Collected works I. Philosophy and Foundations of Mathematics*, ed. A. Heyting. North-Holland, Amsterdam, 1975, pp.127-128.
[2] 关于胡塞尔与布劳威尔对基数和序数的进一步分析参见 Van Atten, Mark. *Brouwer Meets Husserl*, pp.118-121.

[3] Tieszen, R. *Mathematical Intuition: Phenomenology and Mathematical Knowledge*. Kluwer, 1989, p. 105.
[4] Brouwer. *Collected works I*. Philosophy and Foundations of Mathematics, ed. A. Heyting. North-Holland, Amsterdam, 1975, p. 81.

但是与胡塞尔认为“如此等等”的基本结构是一种类似于 n+1 的迭代模式不同，布劳威尔认为通过“如此等等”的方式来构造数学对象，是不充分的。因为，首先“如此等等”在数学中是模糊的，缺乏精确的定义，而且没有明确规定的构造规则，不同的人可能会有不同的理解，导致构造出的对象可能并不相同。而序类型 ω 是指自然数的标准顺序类型，一个严格定义的无限序列。因此“如此等等”的模糊模式也无法严格地指向序类型 ω，更加无法保证构造出的对象是相同的。

其次，因为“0 和 1 之间的所有实数的集合”这个表述并没有指称任何对象。我们只能有效地构造出可数的对象集，无论是通过递归还是通过一系列自由选择。同样的对于超限基数的原则：康托尔将所有可数无穷序数的集合（即序数类型为 ω 的序数）的大小实际上是 $\aleph_0$（可数无限基数），并得出第一个不可数基数 $\aleph_1 > \aleph_0$ 的结论。布劳威尔认为使用全称表达“所有的…”并没有给定 $\aleph n$ 的任何指称对象，康托尔关于超限的这个命题是没有意义的，我们可以通过递归过程或自由选择构造出任何集合。[1]布劳威尔由此得出结论，基于对“如此等等”这个基本结构的模式理解，康托尔提到的是无法通过有限的、明确的步骤来构造的数学对象，这些超越人类思维能力的数学概念因而无法被真正“思考”或理解。基于此，只有在时间性中构造的潜无限在数学中才是合法和有效的。

## 1.5 序数与基数构造中的时间性问题

但是进一步的问题是，胡塞尔与布劳威尔在对基数与序数的构造中，数与时间性的关系是什么？Hill 认为胡塞尔不是弗雷格主义意义上的心理主义者，但他是布劳威尔主义者的心理主义理论仍然吸引着一些人，并有待于反驳。[2]她的观点依据是布劳威尔的数学起源于对时间运动的感知，但是胡塞尔在《算术哲学》一书中拒绝了基于时间直观的数的理论。她认为，布劳威尔的数学起源于对时间运动的感知，胡塞尔在《算术哲学》一书中拒绝了基于时间直观的数字理论，这两种观点形成了鲜明对比。但是胡塞尔所批评的是时间进入数字概念内容的观点，这并不是布劳威尔的观点。[3]序数的顺序当然是建立在时间固有的顺序上的，但

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[1] Brouwer. *Collected works I*. Philosophy and Foundations of Mathematics, ed. A. Heyting. North-Holland, Amsterdam, 1975, pp. 135-136.

[2] Hill, Claire Ortiz. “Husserl on Axiomatization and Arithmetic.” Edited by Mirja Hartimo, *Phenomenology and Mathematics*, Springer, 2010, pp. 64–65.

[3] 对 Hill 的这种观点的批评可参见 Van Atten, Mark. *Essays on Gödel’s Reception of Leibniz, Husserl, and*

时间本身并没有进入序数概念的内容。基数也是如此，它是根据序数和从排序中抽象出来的概念来定义的。

因此，我们看到，时间只是我们概念的心理前提，并且以两种方式……但我们发现，时间上的同时性和连续性都不会以任何方式进入多的(逻辑)内容;同样地，也不会进入数的表象。[1]

我们将通过分析胡塞尔在基数分析构造中集合联结中的时间意识与布劳威尔在序数构造中的二一性的时间意识结构的关系反驳这种观点,并进一步反驳通过时间性对数学认识进行心理学解释的错误。

胡塞尔在《算术哲学》中拒绝了康德通过时间性（通过纯粹直观）对数的解释，也批评了亚里士多德将数与时间并置在一起："时间是运动的数目"。[2]时间性确实是数字的"心理前提条件"，但并不是构造的根本。在数学对象的时间性问题中,胡塞尔认为,虽然在数的概念的起源中,时间只是我们概念的心理前提,而且时间上的同时性和连续性都不会以任何方式进入多的(逻辑)内容;同样地,也不会进入数的表象。集合联结虽然不是一种物理关系，但也并非等同于一种意识的统一形式或者时间的同时性和相继性。就意识统一形式而言，因为集合联结具有一种自发性，其本质不是拥有所有的意识内容，而是在于能够自发地关注到任何内容，因此二者并不相同；其次集合联结也并非等同于时间的同时，比如音乐的表象是相继的，而并非同时的；其次集合联结也不在于时间的相继性。因为时间相继的表象内容最终还需要一个整体的综合行为。因此，不论是时间的相继性还是同时性，仅是数和多数的心理学发生前提，而与其内涵意义无关。[3]在对数概念的集合行为的分析中,胡塞尔已经将发生心理学中的时间因素完全排除干净。

布劳威尔在这个角度会同意胡塞尔的观点,因为感知时间的连续性内容并不意味着将内容感知为时间的连续性。对他来说，序数概念是根据迭代的空二一性定义的,这些都是纯粹形式(范畴)对象。序数的顺序确实基于时间的内在顺序,但时间本身并没有进入序数概念的内容。基数也是如此，它们是根据序数和抽象

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*Brouwer*. Springer, 2015, pp.263-267.

[1] Hua XII, S. 33.

[2] Hua XXI, S. 32.

[3] Hua XII, S. 33.

化排序定义的。即使任何数学对象的构造都需要感知时间的流动，但这本身并不意味着时间进入了数学概念的内容（数的概念的就是一个例子）空的二一性及其可迭代性是从时间运动感知内部结构中抽象出来。从数学对象的构造角度，二人都同意数学对象的主观构造过程并没有使得这些对象在时间中被个体化，而且不关注个体的心理体验。他们认为，形式基于感性：任何形式对象的构造都必须以某种感性质料为起点，但随后质料被抽象掉了。纯粹范畴对象可以基于任何感性一物质内容构造，重要的只是形式。这意味着数学对象并非任意，它们的构造受到范畴形成法则的约束。[1]这与布劳威尔的观点相似，他认为空的二一性是主体在时间移动中剥离了所有的性质而抽象出来的空的二一性，这种空的二一性是整个数学构造的基础。

[1] Hua XVII, § 62.

## Abstract

Intuitionism provides a constructive theory for understanding mathematical objects grounded in temporal perception. Brouwer emphasized that the perception and retention (memory) of time form the core of mathematical intuition, which generates mathematical objects through a process of self-unfolding. Brouwer employed the concept of "two-ity" to describe the fundamental structure within temporal consciousness, specifically how consciousness perceives the relation between the present moment and the immediately preceding moment. This structure is crucial for mathematical construction, as it lays the groundwork for creating mathematical entities.

Brouwer’s essential viewpoint is that mathematical intuition is deeply rooted in the perception of the flow of time. This perception can be described as the splitting of a lived moment into two distinct units, where one unit sequentially gives way to another. Thus, the subject initially perceives the first unit; as this unit gives way to the subsequent one, the subject experiences perception of a new, second unit while simultaneously retaining in memory the perception of the first. Within this perceptual mode, "two-ity" emerges.

Brouwer further pointed out that when we become aware of this "two-ity" purely as a form, disregarding the sensory content present at each stage, we achieve what he termed "the fundamental intuition of mathematics." The subject can strip this two-ity of all qualitative and sensory materials, thereby obtaining what Brouwer called "empty two-ity." This empty form constitutes the fundamental mathematical intuition.Brouwer further pointed out that when we become aware of this "two-ity" purely as a form, disregarding the sensory content present at each stage, we achieve what he termed "the fundamental intuition of mathematics." The subject can strip this

two-ity of all qualitative and sensory materials, thereby obtaining what Brouwer called "empty two-ity." This empty form constitutes the fundamental mathematical intuition. All constructions of mathematical concepts begin with the two units contained in empty two-ity as a starting point and the iterability of forming duality. These constructions are based on introspective consciousness and reflective acts that are independent of language. Through introspective awareness, we treat one unit of two-ity as the fundamental operation for a new two-ity, continuously generating an infinite sequence of natural numbers and the infinite ordinal $\omega$ through the invariant retention provided by memory.

# 数学的内部应用与结构主义解释——以组合学为例

鞠大恒

**摘　要：**数学在自然科学中的可应用性一直是数学哲学中的重要话题之一。对此，一种流行的解释是所谓的"结构主义解释"，诉诸结构关系，或其细化"映射解释"，诉诸保持结构的映射。近来，在《Internal Applications and Puzzles of the Applicability of Mathematics》一文中，Marshall 考虑了数学的内部应用，即数学的一个分支在另一个分支中的应用，并对它也采用了上述的解释。本文以组合学为例，具体考察数学的内部应用，分析它们都是凭借怎样的结构关系而发生的。其结果将成为对结构主义解释的细化和对映射解释的订正，并对数学的外部应用的解释提供启发。

**关键词：**可应用性;结构主义解释;映射解释;组合学

## 1　引言

自 Wigner 的《The unreasonable effectiveness of mathematics in the natural sciences》（[1]）一文以来，数学在自然科学中的可应用性（applicability）一直是数学哲学中的重要话题之一。对此，一种流行的解释是所谓的"结构主义解释（structuralist account）"（[2]）或"映射解释（mapping account）"（[3]）：

……数学可应用于世界，单纯是因为数学宇宙的一些部分与物理领域的一个部分具有一些结构相似性（structural similarity）。（[4]）

……在世界与所讨论的数学结构之间存在一些保持结构的映射（structure-preserving mapping），这基本上就是事情的结束。（[4]）

近来，在《Internal Applications and Puzzles of the Applicability of Mathematics》（[5]）一文中，Marshall 考虑了数学的内部应用（internal applications），即数学的一个分支在另一个分支中的应用，并对它也采用了上述的解释：

要被应用于某些语境中，数学的一个分支不需要关于那个语境中包含的对象或对那些对象施加因果影响。一般被要求的是，在数学的该分支和给定的语境的主题之间存在特定的结构关系（structural relations）。这些结构关系的存在被它们之间的同态或其他保持结构的映射的存在所见证。涉及的对象是具体的还是抽象的，与相关的结构关系是否成立没什么关系。（[5]）

引人注意的是，他在脚注中写道：

我并不坚持数学的每一应用都是凭借保持结构的映射而发生的。（[5]）

不过，这并不矛盾，因为保持结构的映射只是结构关系的一种。毋宁说，应当区分两种解释，一是数学的可应用性是因为存在结构关系，二是数学的可应用性是因为存在保持结构的映射，后者是前者的一种细化，一个人（例如 Marshall）可以坚持前者而不坚持后者。我们认为，“结构主义解释”和“映射解释”理应分别指称这两种解释，而不是作为同一种解释的两个名字。

在本文中，我们将推进 Marshall 的工作，具体考察数学的内部应用，分析它们都是凭借怎样的结构关系而发生的。其结果将成为对结构主义解释的细化和对映射解释的订正，并对数学的外部应用的解释提供启发。

特别地，我们将集中注意力于组合学，考察组合学中的数学的其他分支的应用（一般称为“某某方法”）。其理由有三：一是我们对组合学了解相对较多；二是组合学中的内部应用的例子较多；三是组合学中的内部应用的专著较多，例如 Alon 的《Discrete Mathematics: Methods and Challenges》（[6]），Alon 和 Spencer 的《The Probabilistic Method》（[7]），Babai 和 Frankl 的《Linear Algebra Methods in Combinatorics》（[8]），Guth 的《Polynomial Methods in Combinatorics》（[9]）等，它们将成为本文的主要参考文献。

## 2 概率方法

概率方法是组合学中的内部应用的经典例子。其工作原理是：为了证明存在具有某种性质的组合对象，定义一个以组合对象为样本的概率空间，证明具有该性质的概率大于零。（[7]）显然，用结构主义解释的话来说，这里存在的结构关系就是保持结构的映射，从组合系统到概率空间。

下例是该方法的开端，来自 Erdós1947 年的论文（[10]）。

例 1（[7]） 拉姆塞数$R(k,l)$是最小的正整数$n$，满足任意对$n$个节点的完全图$K_n$的边做红蓝二染色，要么存在红色的$K_k$，要么存在蓝色的$K_l$。若$\binom{n}{k}2^{1-\binom{k}{2}} < 1$，则$R(k,k) > n$。

证明：对$K_n$的每条边随机地做红蓝二染色，对每一个$K_n$（的节点集）的$k$元子集$R$（共$\binom{n}{k}$个），令$A_R$为$K_n[R]$是单色的这一事件，显然$\Pr[A_R] = \frac{2}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}$。令$A$为存在一个单色的$K_k$这一事件，则$\Pr[A] = \Pr[\bigcup_R A_R] \le \sum_R \Pr[A_R] = \binom{n}{k}2^{1-\binom{k}{2}}$。若$\binom{n}{k}2^{1-\binom{k}{2}} < 1$，则$\Pr[A] < 1$，$\Pr[\bar{A}] > 0$，即存在一种二染色不包含单色的$K_k$。故$R(k,k) > n$。

但是，正如 Alon 和 Spencer 所指出的：

当然，人们可以断言，概率在上面给出的证明中不是本质的（essential）。一个同等简单的证明可以用计数来描述；我们仅仅核实$K_n$的二染色的总数量大于包含单色的$K_k$的那些的数量。（[7]）

具体地：

不应用概率方法的证明：考虑$K_n$的所有二染色（共$2^n$种），对每一个$K_n$（的节点集）的$k$元子集$R$（共$\binom{n}{k}$个），使$K_n[R]$单色的二染色有$2^{1+n-\binom{k}{2}}$种，故包含单色的$K_k$的二染色至多有$\binom{n}{k}2^{1+n-\binom{k}{2}}$种。若$\binom{n}{k}2^{1+n-\binom{k}{2}} < 2^n$，则至少存在一种二染色不包含单色的$K_k$。故$R(k,k) > n$。

让我们观察一下应用概率方法的证明为什么可以转换成不应用概率方法的证明。在所涉及的概率空间$(\Omega, \mathcal{F}, \Pr)$中，样本空间$\Omega$是$K_n$的所有二染色的集合，这是一个组合对象；事件域$\mathcal{F}$是$\Omega$的所有子集的集合，这也是一个组合对象，概率函数Pr把每个$A \in \mathcal{F}$映到$\Pr[A] = \frac{|A|}{|\Omega|}$，这是一个组合量（甚至计数量）。因此，这个概率空间只是披着羊皮的组合系统，没有更丰富的（richer）结构，只需要按部就班地脱下羊皮，就能得到不应用概率方法的证明。

这一观察显然是一般的。让我们再看一个例子。这个例子是所谓的依赖随机选择引理（dependent random choice lemma），它是一个非常有用的引理，被认为是“著名的概率方法的一个例子”（[11]）。

例 2（[11]）对正整数$a, r, m$，有$n$个节点和$\frac{nd}{2}$条边的（简单，下同）图$G$，若存在正整数$t$使得

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a,$$

则$G$（的节点集）包含一个至少$a$元的子集$U$，满足$U$中任意$r$个节点都有至少$m$个公共邻居。

证明：记一集节点$U$的公共邻居集为$N(U)$。随机地可重复地选取$t$个节点构成集合$T$。

令随机变量$X = |N(T)|$，

$$\mathbb{E}[X] = \sum_v \mathbb{E}[1_{v \in N(T)}] = \sum_v \Pr[v \in N(T)] = \sum_v \left(\frac{|N(v)|}{n}\right)^t \geq \frac{d^t}{n^{t-1}}。$$

令随机变量$Y$为$N(T)$的大小$= r$但公共邻居数$< m$的子集数，

$$\mathbb{E}[Y] = \sum_{|S|=r \wedge |N(S)|<m} \mathbb{E}[1_{S \subseteq N(T)}] = \sum_{|S|=r \wedge |N(S)|<m} \Pr[S \subseteq N(T)]$$

$$= \sum_{|S|=r \wedge |N(S)|<m} \left(\frac{|N(S)|}{n}\right)^t < \binom{n}{r}\left(\frac{m}{n}\right)^t。$$

于是$\mathbb{E}[X - Y] \geq \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a$，故存在一种取法$T$满足$X - Y \geq a$。

现在，从$N(T)$的每个大小$= r$但公共邻居数$< m$的子集$S$中选取一个节点，从$N(T)$中删除这些节点，所得的集合$U$至少包括$a$个节点，且其中任意$r$个节点都有至少$m$个公共邻居。

根据上面的观察，这一证明可以按部就班地转换成如下不应用概率方法的证明：

不应用概率方法的证明：可重复地选取$t$个节点构成集合$T$，共有$n^t$种取法$T_1, \cdots, T_{n^t}$。

记$|N(T_i)| = x_i$，

$$\sum_i x_i = \sum_v |\{i | v \in N(T_i)\}| = \sum_v |N(v)|^t \geq nd^t。$$

记$N(T_i)$的大小$= r$但公共邻居数$< m$的子集数为$y_i$，

$$\sum_i y_i = \sum_{|S|=r \wedge |N(S)|<m} |\{i | S \subseteq N(T_i)\}| = \sum_{|S|=r \wedge |N(S)|<m} |N(S)|^t < \binom{n}{r} m^t。$$

于是$\sum_i (x_i - y_i) \geq nd^t - \binom{n}{r} m^t \geq n^t a$，故存在一种取法$T_i$满足$x_i - y_i \geq a$。

现在，……

面对这一“非本质”的指控，Alon 和 Spencer 回应道：

理论上，这的确是实际情况。然而，在实践中，概率是本质的。要用计数论证替换本书中出现的许多工具的应用将是没有希望的，包括二阶矩方法、Lov á sz 局部引理和通过鞅的集中（concentration via martingales）……（[7]）

不过，对于我们的目的来说，我们没有必要考察组合学中的每一种概率方法，辨析它们是否是本质的，更没有必要主张它们都是非本质的。我们所满足于的收获是：存在一大类数学的内部应用是非本质的，这一非本质的可应用性是因为存在保持结构的映射，但仅仅保持结构，没有丰富（enrich）结构。

## 3 线性代数方法

线性代数方法是组合学中的内部应用的另一经典例子。“只要知道线性无关的概念，就可以在代数和组合学之间建立出乎意料的联系。”（[8]）在本节中，我们考察它的一个分支——所谓的“维数论证（dimension argument）”（[6]）。

让我们通过一个典型例子来看一下该方法的工作原理。

例 3（[8]） 设$C_1, \cdots, C_m$是$n$元集合$\{1, \cdots, n\}$的$m$个不同的非空子集，满足对任意$i = j$，$|C_i|$是奇数，对任意$i \neq j$，$|C_i \cap C_j|$是偶数，则$m \leq n$。

证明 1：对每个$i$，将$C_i$表示为它的关联向量$v_i \in \mathrm{M}_{n\times 1}(\mathbb{F}_2)$：若$j \in C_i$，则$v_i$的第$j$个分量为$1$，否则为$0$。

现在，条件可以表示为：对任意$i$，$v_i \cdot v_i \neq 0$，对任意$i \neq j$，$v_i \cdot v_j = 0$。

这蕴含$v_1, \cdots, v_m$线性无关：若有$\lambda_1, \cdots, \lambda_m \in \mathbb{F}_2$满足$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0$，两边点乘$v_i$可得$\lambda_i = 0$。

又$\dim(\mathrm{M}_{n\times 1}(\mathbb{F}_2)) = n$，故$m \leq n$。

证明 2：对每个$i$，将$C_i$表示为它的关联向量$v_i \in \mathrm{M}_{n\times 1}(\mathbb{Q})$。

现在，条件可以表示为：对任意$i$，$v_i \cdot v_i$不是$2$的倍数，对任意$i \neq j$，$v_i \cdot v_j$是$2$的倍数。

这蕴含$v_1,\cdots,v_m$线性无关：若有$\lambda_1,\cdots,\lambda_m\in\mathbb{Q}$满足$\lambda_1 v_1+\cdots+\lambda_m v_m=0$，反设$\lambda_1,\cdots,\lambda_m$不全为0，不妨设$\lambda_1,\cdots,\lambda_m$都是整数且互质，两边点乘$v_i$可得$\lambda_i(v_i\cdot v_i)$是2的倍数，又$v_i\cdot v_i$不是2的倍数，则$\lambda_i$是2的倍数，矛盾！故$\lambda_1=\cdots=\lambda_m=0$。

又$\dim(\mathrm{M}_{n\times 1}(\mathbb{Q}))=n$，故$m\le n$。

在这个例子中，我们是怎样应用线性代数的呢？答案很简单：我们自然地将集合表示为矩阵，将集合的条件表示为矩阵的条件。用结构主义解释的话来说，我们将$\mathcal{P}(\{1,\cdots,n\})$保持结构地映到$\mathrm{M}_{n\times 1}(\mathbb{F}_2)$和$\mathrm{M}_{n\times 1}(\mathbb{Q})$。

但是，这远远不是全部的故事。在将集合表示为矩阵之后，我们对它们进行了加法、数乘等运算。这些运算是线性空间所带有的，单纯的集合无法进行。这与前一节的两个例子形成了鲜明的对比：在那里，所有的运算都是披着羊皮的计数。

有人可能会说：虽然对证明 2 来说事情是如此，但对证明 1 来说事情并非如此。事实上，$\mathrm{M}_{n\times 1}(\mathbb{F}_2)$上的运算也是披着羊皮的组合运算：向量的加法就是集合的对称差；数乘只有0乘和1乘两种情况，前者就是将任何集合映到空集，后者就是将任何集合映到自身。

但是，这只是一个特例。一方面，这点只对$\mathbb{F}_2$成立，对一般的$\mathbb{F}_p$就不成立；另一方面，证明 2 可以被推广为一个一般的证明，而证明 1 则不能。

例 4（[8]） 设$p$是一个素数，$C_1,\cdots,C_m$是$n$元集合$\{1,\cdots,n\}$的$m$个不同的非空子集，满足对任意$i=j$，$|C_i|$不是$p^k$的倍数，对任意$i\ne j$，$|C_i\cap C_j|$是$p^k$的倍数，则$m\le n$。

证明：对每个$i$，将$C_i$表示为它的关联向量$v_i\in\mathrm{M}_{n\times 1}(\mathbb{Q})$。

现在，条件可以表示为：对任意$i$，$v_i\cdot v_i$不是$p^k$的倍数，对任意$i\ne j$，$v_i\cdot v_j$是$p^k$的倍数。

这蕴含$v_1,\cdots,v_m$线性无关：若有$\lambda_1,\cdots,\lambda_m\in\mathbb{Q}$满足$\lambda_1 v_1+\cdots+\lambda_m v_m=0$，反设$\lambda_1,\cdots,\lambda_m$不全为0，不妨设$\lambda_1,\cdots,\lambda_m$都是整数且互质，两边点乘$v_i$可得$\lambda_i(v_i\cdot v_i)$是$p^k$的倍数，又$v_i\cdot v_i$不是$p^k$的倍数，则$\lambda_i$是$p$的倍数，矛盾！故$\lambda_1=\cdots=\lambda_m=0$。

又$\dim(\mathrm{M}_{n\times 1}(\mathbb{Q}))=n$，故$m\le n$。

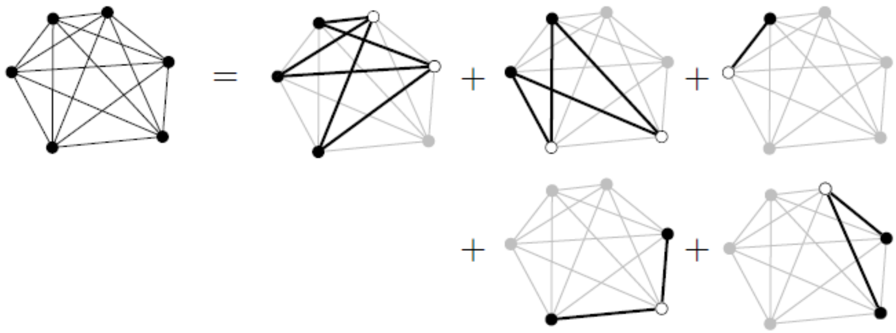
例 5（[8]） 存在$n$，存在$C_1,\cdots,C_m$是$n$元集合$\{1,\cdots,n\}$的$m$个不同的非空子集，满足：对任意$i=j$，$|C_i|$不是$2^2$的倍数，对任意$i\ne j$，$|C_i\cap C_j|$是$2^2$的倍数；但是，对每个$i$，将$C_i$表示为它的关联向量$v_i\in\mathrm{M}_{n\times 1}(\mathbb{F}_2)$，却有$v_1,\cdots,v_m$线性相关。

（换句话说，例 4 的证明中的$\mathrm{M}_{n\times 1}(\mathbb{Q})$一般不能被修改为$\mathrm{M}_{n\times 1}(\mathbb{F}_p)$。）

总而言之，在这个例子中，所涉及的线性空间相较于组合系统带有更丰富的结构，正是这些更丰富的结构使得我们完成了证明。对于这样的本质的应用来说，一般的“映射解释”是不够的：除了保持结构，更重要的是丰富结构！

让我们再看一个图论的例子。

例 6（[8][12]） 称一个图为完全二部图，若它的节点集可以被划分为两个部分，两个节点相邻当且仅当它们分属于两个部分。若$K_n$的边集被分解为$m$个完全二部图的边集的不交并，则$m \geq n-1$。



证明：记$K_n$的节点集为$\{1,\cdots,n\}$，$m$个完全二部图为$B_1,\cdots,B_m$，$B_i$的节点集的两个部分为$(X_i,Y_i)$。

对每个$i$，将$X_i$表示为它的关联向量$x_i \in \mathrm{M}_{n\times 1}(\mathbb{Q})$，将$Y_i$表示为它的关联向量$y_i \in \mathrm{M}_{n\times 1}(\mathbb{Q})$。每个以$\{1,\cdots,n\}$中的元素为节点的图$G$可以被表示为它的邻接矩阵$A_G \in \mathrm{M}_{n\times n}(\mathbb{Q})$：若$i,j$在$G$中相邻，则$A_G$的第$i$行第$j$列为1，否则为0。易见$A_{K_n} = J_n - I_n$（其中$J_n$为$n$阶全1矩阵，$I_n$为$n$阶单位矩阵），$A_{B_i} = x_i{y_i}^T + y_i{x_i}^T$。

现在，条件可以表示为：$A_{K_n} = \sum_i A_{B_i}$，亦即$\sum_i (x_i{y_i}^T + y_i{x_i}^T) = J_n - I_n$。

令$\sum_i x_i{y_i}^T = S$。一方面，显然$\mathrm{rank}(S) \leq \sum_i \mathrm{rank}(x_i{y_i}^T) = m$。另一方面，$S + S^T = J_n - I_n$，容易证明这蕴含$\mathrm{rank}(S) \geq n-1$（此略）。故$m \geq n-1$。

这个例子和上面的例子是类似的：我们自然地将组合对象表示为矩阵，将组合的条件表示为矩阵的条件，这样就可以利用矩阵的运算。同样类似的是，证明中的$\mathrm{M}_{n\times 1}(\mathbb{Q})$不能被修改为$\mathrm{M}_{n\times 1}(\mathbb{F}_2)$（进而被替换为$\mathcal{P}(\{1,\cdots,n\})$），这是因为在$\mathrm{M}_{n\times n}(\mathbb{F}_2)$中$S + S^T = J_n - I_n$并不蕴含$\mathrm{rank}(S) \geq n-1$，一个反例如下：

$$S = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

这再次展示了“更丰富的结构”的重要性：少了它，这一应用将无法发生！

## 4 多项式方法

多项式方法是线性代数方法的另一个分支，它利用了一类特殊的线性空间——多项式空间。

例 7（[12]） 若$\mathbb{R}^n$中的点集$\{p_1,\cdots,p_m\}$满足其中任意两点的距离均为$\delta_1$或$\delta_2$，则$m \leq \frac{n^2+5n+4}{2}$。

证明：令$x=(x_1,\cdots,x_n)$，考虑$\mathbb{R}[x_1,\cdots,x_n]$上的多项式$P_i(x)=(\|x-p_i\|-{\delta_1}^2)(\|x-p_i\|-{\delta_2}^2)$，条件可以表示为：对任意$i\neq j$，$P_i(p_j)=0$。

这蕴含$P_1,\cdots,P_m$线性无关：若有$\lambda_1,\cdots,\lambda_m\in\mathbb{R}$满足$\lambda_1P_1+\cdots+\lambda_mP_m=0$，两边代入$p_i$可得$\lambda_i{\delta_1}^2{\delta_2}^2=0$，故$\lambda_i=0$。

又易见所有$P_i$都在诸$1,x_i,x_ix_j,x_i\|x\|,\|x\|^2$（共$\frac{n^2+5n+4}{2}$项）张成的线性空间V中，故$m\leq\dim(\mathrm{V})\leq\frac{n^2+5n+4}{2}$。

看起来，这个例子和上一节中的例子没有什么不同：我们在带有更丰富结构的线性空间中做维数论证，只不过这里的线性空间是多项式空间$\mathbb{R}[x_1,\cdots,x_n]$。但是，映射解释的支持者应当注意到了：我们并没有把$\mathbb{R}^n$中的点保持结构地映到$\mathbb{R}[x_1,\cdots,x_n]$中的多项式，而是将前者代入后者！换句话说，这里存在的结构关系并不是一个映射！

有人可能会说：将$\mathbb{R}^n$中的点$p$代入$\mathbb{R}[x_1,\cdots,x_n]$中的多项式，就是将$\mathbb{R}[x_1,\cdots,x_n]$中的多项式代入$\mathrm{Hom}(\mathbb{R}[x_1,\cdots,x_n],\mathbb{R})$中的赋值映射$E_p:P(x)\mapsto P(p)$。因此，映射解释的支持者可以将这个例子解释为：我们将$\mathbb{R}^n$中的点保持结构地映到$\mathrm{Hom}(\mathbb{R}[x_1,\cdots,x_n],\mathbb{R})$中的赋值映射。但是，这一解释有两个问题：第一，维数论证发生在$\mathbb{R}[x_1,\cdots,x_n]$中而不是$\mathrm{Hom}(\mathbb{R}[x_1,\cdots,x_n],\mathbb{R})$中，换句话说，我们利用的是$\mathbb{R}[x_1,\cdots,x_n]$ 而不是$\mathrm{Hom}(\mathbb{R}[x_1,\cdots,x_n],\mathbb{R})$；第二，$\mathrm{Hom}(\mathbb{R}[x_1,\cdots,x_n],\mathbb{R})$并不自然地构成一个$\mathbb{R}$-线性空间，上述的映射很难说是“保持结构”的。

总而言之，这个例子见证了存在保持结构的映射之外的结构关系，构成了映射解释的反例。结合前两节的讨论，我们将对数学的内部应用的结构主义解释总结如下：

- 数学的内部应用
  - 非本质的：仅仅保持结构的映射
  - 本质的：丰富结构
    - 映射
    - 非映射

为了展示这远非特例，我们再看一个多项式方法的例子。

定理 1（[9]）（参数计数引理（parameter counting lemma）） 记域$\mathbb{F}$上度数$\leq D$的$n$元多项式的空间为$\mathbb{F}[x_1,\cdots,x_n]_D$。若$S\subseteq\mathbb{F}^n$满足$|S|<\binom{D+n}{n}$，则存在一个非零多项式$P\in\mathbb{F}[x_1,\cdots,x_n]_D$，满足对任意$p\in S$有$P(p)=0$（称为在$S$上消没）。

定理 2（[9]）（消没引理（vanishing lemma））

(1) 若$P\in\mathbb{F}[x_1,\cdots,x_n]_D$，且$P$在$\mathbb{F}^n$上的$D+1$个点上消没，则$P$是零多项式。

(2) 若$P\in\mathbb{F}[x_1,\cdots,x_n]_D$，且$P$在直线$l\subseteq\mathbb{F}^n$上的$D+1$个点上消没，则$P$在$l$上消没。

(3) 若$P\in\mathbb{F}_q[x_1,\cdots,x_n]_{q-1}$，且$P$在${\mathbb{F}_q}^n$上消没，则$P$是零多项式。

这两个引理可以用来解决很多组合，特别是相交几何问题。其开端是 Dvir 解决有限域上的 Kakeya 问题（[13]）。

例 8（[9]） 集合$K\subseteq{\mathbb{F}_q}^n$称为 Kakeya 集，若对任意向量$a\in{\mathbb{F}_q}^n\backslash\{0\}$，存在向量$b$满足直线$\{at+b|t\in\mathbb{F}_q\}\subseteq K$，则$|K|\geq(10n)^{-n}q^n$。

证明：反设存在 Kakeya 集$K$满足$|K| < (10n)^{-n}q^n$。利用定理 1 可得，存在非零多项式$P \in \mathbb{F}_q[x_1, \cdots, x_n]$满足$P$在$K$上消没，且$\mathrm{Deg}(P) \triangleq D < q$。令$P_D$为$P$的诸$D$次项之和，$\mathrm{Deg}(P_D) = D$。

对任意向量$a \in \mathbb{F}_q{}^n \backslash \{0\}$，存在向量$b$满足直线$\{at + b | t \in \mathbb{F}_q\} \subseteq K$。考虑多项式$R(t) = P(at + b)$，则$R \in \mathbb{F}_q[t]_{q-1}$且在$\mathbb{F}_q$上消没，由定理 2，$R$是零多项式。特别地，$P_D(a) = 0$。于是$P_D \in \mathbb{F}_q[x_1, \cdots, x_n]_{q-1}$且在$\mathbb{F}_q{}^n$上消没，由定理 2，$P_D$是零多项式，矛盾！

在这个例子中，我们工作于多项式空间$\mathbb{F}_q[x_1, \cdots, x_n]_{q-1}$（及其子空间$\mathbb{F}_q[t]_{q-1}$）中。同样地，它和$\mathbb{F}_q{}^n$之间的结构关系不是一个映射。

关于这两个引理的其他应用，可参见《Polynomial Methods in Combinatorics》（[9]）一书。

最后，让我们来看一个复杂一点的例子。

定理 3（[14]）（组合零点定理（Combinatorial Nullstellensatz）） 若$P \in \mathbb{F}[x_1, \cdots, x_n]$满足$Deg(P) = \sum_{i=1}^n t_i$，其中$t_i$是自然数，且$\prod_{i=1}^n x_i{}^{t_i}$的系数非0，则对$S_1, \cdots, S_n \subseteq \mathbb{F}$满足$|S_i| > t_i$，存在$s_1 \in S_1, \cdots, s_n \in S_n$使得$P(s_1, \cdots, s_n) \neq 0$。

这个定理也可以用来解决很多组合问题，以下仅举一例。关于这个定理的其他应用，可参见《Combinatorial Nullstellensatz: With applications to graph colouring》（[15]）一书。

例 9（[15]） 称一个图是$n$-正则的，若它的每个节点的度数均为$n$。设$p$是一个素数，若图$G$的平均度数$\geq 2p - 2$，最大度数$\leq 2p - 1$，则它包含一个$p$-正则的子图。

证明：记$G$的节点为$1, \cdots, n$，边为$e_1, \cdots, e_m$。记$a_{i,e_j}$如下：若$i$与$e_j$关联，则$a_{i,e_j} = 1$，否则为 0。$G$的无孤立点的子图$G'$可以用一个$m$维向量$v = (s_1, \cdots, s_m)$来表示：若$e_j \in G'$，则$s_j = 1$，否则为0。

现在，要证的命题可以表示为：存在非零的零一向量$(s_1, \cdots, s_m) \in \mathbb{Q}^m$，满足对任意$i$，$\sum_{j=1}^m a_{i,e_j} s_j = 0, p$。这等价于，存在非零的零一向量$(s_1, \cdots, s_m) \in \mathbb{F}_p{}^m$，满足对任意$i$，$\sum_{j=1}^m a_{i,e_j} s_j = 0$。

考虑$\mathbb{F}_p[x_1, \cdots, x_m]$中的多项式$P(x_1, \cdots, x_m) = \prod_{i=1}^n (1 - \left(\sum_{j=1}^m a_{i,e_j} x_j\right)^{p-1}) - \prod_{j=1}^m (1 - x_j)$。由定理 3（令$t_1 = \cdots = t_m = 1$，$S_1 = \cdots = S_m = \{0,1\}$，易见满足条件），存在零一向量$(s_1, \cdots, s_m) \in \mathbb{F}_p{}^m$满足 $P(s_1, \cdots, s_m) \neq 0$。

下证$(s_1, \cdots, s_m)$满足要求。若$(s_1, \cdots, s_m) = 0$，则$P(s_1, \cdots, s_m) = 0$，故$(s_1, \cdots, s_m) \neq 0$。于是$\prod_{j=1}^m (1 - s_j) = 0$，进而$\prod_{i=1}^n (1 - \left(\sum_{j=1}^m a_{i,e_j} x_j\right)^{p-1}) \neq 0$，进而对任意$i$，$\left(\sum_{j=1}^m a_{i,e_j} x_j\right)^{p-1} \neq 1$，而这等价于$\sum_{j=1}^m a_{i,e_j} x_j = 0$。证毕。

正如 Bueno 和 Colyvan 所言，“有时数学的应用将是一个多阶段的过程”（[4]）。在这个例子中，我们经历了两个阶段：从$G$的无孤立点的子图的集合到

$\mathbb{F}_p{}^m$，再从$\mathbb{F}_p{}^m$到$\mathbb{F}_p[x_1, \cdots, x_m]$。其中，在第一个阶段中，结构关系是一个映射，在第二个阶段中，结构关系不是一个映射。这再一次展示了，对于结构主义解释来说，结构关系是否是一个映射并不是关键。

## 5 结语

本文的主要工作是通过对组合学中具体例子的考察，将对数学的内部应用的结构主义解释总结如下：

- 数学的内部应用
  - 非本质的：仅仅保持结构的映射
  - 本质的：丰富结构
    - 映射
    - 非映射

本文既可以看作当下流行的“数学实践哲学（philosophy of mathematical practice）”之一例，也可以看作对王浩“知识学（epistemography）”的一次践行。

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# The Internal Applications of Mathematics and the Structuralist Account

## —Taking Combinatorics as a Case Study

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Abstract: The applicability of mathematics in the natural sciences has long been a central topic in the philosophy of mathematics. A prevalent account for this phenomenon is the "structuralist account," which attributes applicability to structural relations, or its refined version, the "mapping account," which attributes applicability to structure-preserving mappings. Recently, in the paper *Internal Applications and Puzzles of the Applicability of Mathematics*, Marshall examined the internal applications of mathematics—where one branch of mathematics is applied to another—and extended the aforementioned accounts to such cases. In this paper, we advance Marshall's work by taking combinatorics as a case study, examining some internal applications of mathematics in detail and analyzing what kind of structural relations they occur by means of. Our findings will refine the structuralist account, revise the mapping account, and provide insights into accounting for the external applicability of mathematics.

# On a Mereotopological Series-style Answer to Special Composition Question

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**Abstract.** The Special Composition Question (SCQ) asks: "When do some things compose something?" The most popular answer is Universalism, while Nihilism has gained increasing attention in recent years. Restrictivism attempts to specify certain conditions under which composition occurs. The Series-Style Answer (SSA), an important approach within Restrictivism, claims that different types of objects have different composition criteria. However, SSA faces challenges such as the Circularity Problem, the Sorites Paradox, and especially the Transitivity Problem. This paper focuses on the Transitivity Problem, arguing that Carmichael's current strategy fails to fully resolve it. To address this, a mereotopological version of SSA is proposed, using exterior boundaries to characterize series-style composition relations, aiming to provide a stronger theoretical foundation for SSA.

## 1. Introduction

It is natural to suppose that ordinary material objects, e.g. tables, chairs, atoms, walls, and so on, exist, and they can compose further material objects. Van Inwagen reflects this commonsense supposition (Van Inwagen, 1995) by introducing the Special Composition Question (SCQ): "when does something compose other things"? This question investigates the conditions that, once which are satisfied, material compositions to compose further objects would happen. SCQ is a request to fill the following blank:

> **Special Composition Schema**: necessarily, for any xs, there is some y such that, xs compose y
> (when)________________________________________________
> .

The most popular answer to SCQ is Universalism, according to which, composition always happens, automatically. Universalism fills "at any time" in the blank and holds that for any things that no two of which have a common part, there is something that they compose. (See Lewis, 1986; Sider, 2001; Varzi, 2005; Bigelow, 2006. For objections to Universalism, see Markosian, 1998; Comesana, 2008) Thus, according to this view, there is a further object composed of the Eiffel Tower plus the moon plus Donald Trump's hair.

Meanwhile, the second popular answer, which grows in popularity, especially in recent years, is Nihilism, according to which, compositions never happen. Nihilism fills "at no time" in the blank and holds that two or more things never compose or add up to anything. (See Cameron, 2010; Sider, 2013; Cornell, 2017; Merricks, forthcoming. For objections to Universalism, see Shaffer, 2009; Rea, 1998; Lewis, 1991) In this view, ordinary objects like gloves, atoms, tables, etc. never exist.

Restrictivism, as the least popular answer, plans to fill some conditions in the blank. This view holds that sometimes, under certain conditions, two or more non-overlapping things compose something, and that sometimes they do not compose anything. (See, Markosian, 1998; Korman, 2015; Carmichael, 2015) This strategy can intuitively accept the existence of ordinary objects to which we are so familiar, while rejecting the existence of those composites, such as the Eiffel Tower plus the moon plus Donald Trump's hair, or trout-turkey, which are, after all, so exotic and strange, at least in our everyday language.

Among Restrictivism, Series-Style Answer (SSA) is a controversial but highly appealing project. According to its most typical and standard version, there are different composition *criteria* for different *types* of objects. In Van Inwagen's word, "multigrade *relations* for certain *relata*" (Van Inwagen, 1995: 63). I am composed of my cells, which are composed of particles. Particles stand in nuclear relation to compose cells, and cells stand in biochemical relation to compose me. And the story ends.
However, although this seems to be very intuitive, it still faces many rebuttals. For example, some philosophers argue that SSA suffers from Van Inwagen's Circularity problem, which says that any proposed SSA answer to SCQ is covertly mereological in nature, e.g. particles already *compose* cells in our case. (For this objection, see Van Inwagen, 1995; for the response to this objection, see Silva, 2013; Carmichael, 2015); in addition, some philosophers also argue that SSA involves Sorites Paradox, namely, there are no cut-off criteria for composition to happen. Then every answer in the series is impossible. (For this objection, see Unger, 1980; Varzi, 2007; for the response to this objection, see Korman, 2015; Carmichael, 2011), etc.

In this paper, speaking for SSA theorists, I do not force myself to solve all the objections. Rather, I only focus on one of them, the *Transitivity problem*. In section 1, I will present SSA and non-transitivity in mereology with plural logic. In section 2, I argue that Carmichael's strategy to solve this problem fails to solve the Transitivity problem between different kinds of composition, so we have reasons to improve it; in section 3, in order to save transitivity, I will raise a mereotopological version of SSA, using mereotopological exterior boundaries to characterize series-style compositions; in section 4 and 5, I will offer my positive justifications for this SSA and show why it is a better choice for SSA theorists.

Before I start my work, however, two things should be noted: 1) indeed, some SSAs can be built, based on "non-standard" versions of mereological composition, which can avoid the transitivity problem. For instance, some philosophers have proposed several understandings of "Fusion" "Sum" "Non-transitivity sum" to define compositions. (See Tarski, 1935; Lewis, 1991; Gruszczyński, 2013; Pietruszczak, 2014). But in this paper, limited by space and my purpose, I will suppose a standard understanding of composition, namely, the composition sum in classical mereology. 2) I do not promise that my solution can remove all objections to SSA. My purpose is only to solve the transitivity problem. So the conclusion I can reach is *merely* that given a standard understanding of composition, if there is at least hope for SSA to be true, then my version of SSA is a promising one.

## 2. SSA and transitivity problem

Let me first make some background clarifications. The general lesson of SSA is that there are different criteria of composition corresponding to different types of material objects. Formally, SSAs are of this form:

> **SSA-Form**: For any xs (where xs are material objects), there is a further material object, y, composed of those xs iff the xs are of the kind F1 and stand in relation R1, or the xs are of the kind F2 and stand in relation R2, or …, the xs are of the kind Fn and stand in relation Rn.

For instance, there are some particles, and some cells. When particles stand in nuclear relation, or when cells stand in biochemical relation, material composition happens. More precisely, SSA-Form entails two theses, respectively about kinds and relations:

> **Level-kind Thesis**: There are kinds of material objects, where some of some kind (F1) can be composed of material objects of a distinct kind (F2, ..., Fn).

> **Exclusion Thesis**: There are some kinds and relations such that a relation sufficient for some composition to happen among some kind of material objects (F1), is not sufficient for composition to happen among some other distinct kind of objects (F2, ..., Fn) even if the F2s can stand in the same relation as the F1s when the F1s compose something[1].

Both theses satisfy our commonsense knowledge and intuitions about the world, where things of a kind can compose things of other kinds: cells are composed of atoms; organs are composed of cells; tables are composed of pieces of wood, etc. (Level-kind), and for relations, the biochemical relation for cells to compose organs is insufficient for the composition among atoms, pieces of woods, etc., to happen (Exclusion).

In order to construct it more precisely, I form SSA in mereological composition. In classical mereology, parthood is viewed as a partially ordered relation. Take x, y, z...as singular variables and P as a primitive predicate for parthood “is a part of”, then we have:

| | |
|---|---|
| **Reflexivity** | $\forall x(xPx)$ |
| **Transitivity** | $\forall x \forall y \forall z((xPy \wedge yPz) \rightarrow xPz)$ |
| **Anti-symmetry** | $\forall x \forall y((xPy \wedge yPx) \rightarrow x=y)$ |

Then, mereology investigates the features of the relation of parthood (P) and of some other connected relations, such as proper parthood (PP) and overlap (O). According to the standard definition:

[1] Silva (2013), p.72

| | |
|---|---|
| **Proper Parthood** | $x\ PP\ y =_{df} xPy \wedge x \neq y$ |
| **Overlap** | $x\ O\ y =_{df} \exists z(zPx \wedge zPy)$ |

As composition deals with relations between single variables and plural variables, we need to introduce a non-standard first-order logic that allows us to quantify over plural variables, where xx, yy...are plural variables and ≺is a predicate for the relation of "being one of", we can introduce an essential axiom of plural logic:

**Plural Comprehension** $\exists x\phi(x) \rightarrow \exists xx \forall u(u \prec xx \leftrightarrow \phi(u))$

Plural Comprehension says that, if there is at least one thing that is a ϕ, then there are some things such that something is one of them iff it is a ϕ. For example, if ϕ(x) stands for "x is a human being", then we can get a plurality Zs of "all human beings", that is, anything which is a human being is one of Zs, and anyone of Zs is a human being.

Then the definition of composition "∑" in plural language:

**Composition**[2] $xx\sum y \equiv_{df} \forall z(z \prec xx \rightarrow zPy) \wedge \forall z(zPy \rightarrow \exists w(wOz \wedge w \prec xx))$

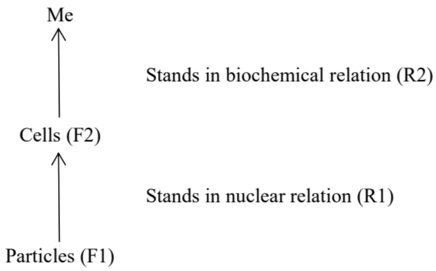
("y is something as a composition of some xs, iff anyone of xs is a part of y, and any part of y overlaps something that is one of xs") Now SSA-Form can be formulated in mereology with plural language:

| | | |
|---|---|---|
| **SSA-Form**$\forall xx \exists y, xx\sum y \leftrightarrow$ | $\forall z(z \prec xx \rightarrow F1(z)) \wedge R1(xx)$ | (SSA-R1 disjunct) |
| | $\vee \forall z(z \prec xx \rightarrow F2(z)) \wedge R2(xx)$ | (SSA-R2 disjunct) |
| | ...... | |
| | $\vee \forall z(z \prec xx \rightarrow Fn(z)) \wedge Rn(xx)$ | (SSA-Rn disjunct) |

Take an oversimplified example to show how SSA works. Assume that, within the spacial region of my body, there are only two kinds of things, particles, and cells (this three-layer case is simple but enough to show how SSA works).

When xx are particles, and R1 is "standing in nuclear relations" (the covalent bond happens and the positive nuclei from different atoms are held together by the attraction for the shared pair of electrons held between them), there are the composite cells, according to our oversimplified example. Further, when xx are cells, and R1 is "standing in biochemical relations" (adhere together through cell-surface proteins called CAM cell adhesion molecules, a category that includes the transmembrane adhesion proteins), they compose my body.

[2] Here I present the most common definition of composition. There are some alternative versions, see Varzi (2019), section 4.




(i) **SSA-Form** in this case: For any xs, there is some y such that the xs compose y, iff the xs are particles (F1) and stand in nuclear relation (R1), or the xs are cells and stand in biochemical relation (R2), or there is only one of the xs.

And then, we have the level-kind thesis and exclusion:

(ii) **Level-kind Thesis**: I am composed of cells, and cells are composed of particles.

(iii) **Exclusion Thesis**: Standing in biochemical relation R2, or any other relations rather than R1, is not sufficient for particles F1 to compose any composite things.

(iv) **Proper Parts**: x is a proper part of y, iff there are zs other than x that x and the zx compose y.

(v) **Transitivity**: Parthood is transitive.

The Transitivity problem reveals the inconsistency involved in SSA-Form through Plural Comprehension, (i)-(iii), and Transitivity. For instance, if we accept (i) and (v), mereologically, an x could be a part of a y which was itself part of a z (particles are parts of me):

| | |
|---|---|
| **(i)-R1 disjunct** | $\forall xx \exists y\ (xx \sum y \leftrightarrow\ \forall w(w \prec xx \rightarrow F1(w))\ \wedge R1(xx))$ |
| **(i)-R2 disjunct** | $\forall yy \exists z\ (yy \sum z \leftrightarrow\ \forall m(m \prec yy \rightarrow F2(m))\ \wedge R2(yy))$ |
| | $xPy \wedge yPz$ |
| **(v) Transitivity** | $\forall x \forall y \forall z((xPy \wedge yPz) \rightarrow xPz)$ |
| **Conclusion:** | $xPz$ |

On the other hand, as per the SSA, zs can only be composed of ys related by R2, but not the xs, therefore, xs are not parts of z. (i), (ii), and (iii) fail to include a condition for when particles can compose me. If there is a condition for particles (F1) plus other particles (F1) plus cells (F2) to compose me, what is it? SSA fails to tell this, since F1 and F2 are not of the same kind, and there is no condition given by SSA for their composition. If one wants to give a condition for F1s and F2s to compose something, his answer will be no longer in the Series style, and violates (iii) Exclusion Thesis.

(i)-R1 disjunct and (iii) says that particles only compose something when standing in R1 relation and (ii) says that when in R1 relation, they compose cells. Taken together, particles cannot compose anything when standing in other relations rather than R1, or with anything that are not F1s. Therefore, particles are not among anything that composes me. Then, from the right-to-left direction of (iv), we can conclude that no particles are proper parts of me (¬ xPz). SSA stops the transitivity between its different disjuncts.

| | |
|---|---|
| **(ii) Level-kind Thesis** | $\exists xx \exists yy \exists z\ (xx \sum yy \wedge yy \sum z)$ |
| **(iii) Exclusion Thesis** | $\forall xx \exists y, \forall w (w \prec xx \rightarrow F1(w)) \wedge R2(xx) \rightarrow \neg xx \sum y$ |
| **(iv) Proper Parts** | $\forall x \forall z\ (\neg (xPz \wedge x \neq z) \leftrightarrow \neg xPPz)$ |
| | $x \neq z$ (obviously, a particle is not me) |
| **Conclusion:** | $\neg xPz$ |

The consequence of SSA is unacceptable, as it conflicts with Transitivity. Transitivity says that x is a part of z "xPz", while SSA says that, x would not be part of z "¬ xPz".

Note that Van Inwagen's Transitivity problem *only* aims at the cross-disjunct cases of SSA, where different kinds and relations are involved, for example, xs are F1s and stand in R1 to compose ys (according to the R1 disjunct), while ys are F2s and stand in R2 to compose z (according to the R2 disjunct). It is not a problem about the transitivity when things are of the same kind and in the same relations. Within a single disjunct, the Transitivity clearly follows and poses no problems.

The root of this problem is that parthood, characterized by different disjuncts of SSA, seems to be non-transitive. The whole is only achieved by the last step, namely, the last disjunct closest to the composite objects. If SSA is true, then the smaller parts are not parts of the composite whole. However, not only Transitivity but also our commonsense intuition suggests that they actually *are* parts of the whole. Given the fact that particles are obviously my parts, SSA defenders seem to be trapped in a self-contradiction.

## 3. Carmichael's two-category SSA

Although intuitive, SSA is currently the most less-popular answer to SCQ, as very few philosophers gave proposals for it. The most representative SSA proposal is from Carmichael. In this inspiring article "Toward A Commonsense Answer to SCQ", Carmichael suggests a two-category SSA, which makes a distinction between event-based composition and lump-like composition, and claims to avoid the transitivity problem.

His SSA idea seems quite natural. Lump-like composition is the most common kind in daily talking. When lump-like objects (e.g. woods, metals), are bonded, they compose other things (tables, the Eiffel Tower). For lump-likeness, I will detail analyze it later in section 4. On the other hand, some events may impose a kind of composition on the objects involved in them. For instance, the atmospheric event imposes unity on those unbonded parts, winds, water droplets, etc., involved in a tornado.

The general lesson is, in Carmichael's words, "*there is an intuitive distinction between objects that are event-based and those that are lump-like*" (Carmichael, 2015: 479). Therefore. If we can figure out compositions into two kinds, and at the same time, find transitivity in each, it is possible for SSA to be true. Carmichael gives a two-category SSA, and his story begins by introducing two different kinds of composition:

> For any material objects xs, there is a further object y, composed of those xs, iff either
> (i) lump-like composition: the xs are lump-like objects (F1s) and the xs are bonded (R1)
> (ii) event-based composition: the xs are event-based objects (F2), and the activities of the xs constitute an event that imposes sufficient unity on the xs (R2)

There are different ways to define "bond". Van Inwagen tries different kinds of bonding, such as "contact"[3], "fastening"[4], "cohesion"[5], and "fusion"[6], each of which involves a greater strength of bond (roughly, the biochemical relation in our oversimplified example can be viewed as "fastening"). Here, Carmichael takes fastening, cohesion, and fusion as "bonding", as we will see this in his argument for Lump-like Transitivity

For the definition of "Event-based", Carmichael has two steps. First, some objects are event-based, when there are underlying events that pose sufficient unity to hold parts together. Moreover, for "sufficient unity", Carmichael technically defines it as "the activities of the xs constitute a self-sustaining[7] and homeodynamic[8] event".
Some examples will illustrate how this two-category SSA works. A model airplane is composed of several pieces of wood that are glued together. This is because the pieces of wood (and clumps of glue) are lump-like and bonded together, and so compose an object by condition (i).

In a hurricane, there are water droplets, quantities of air, debris, and so on, whose activities constitute the relevant atmospheric event. The event in question imposes sufficient unity on these objects. Thus, by condition (ii), these objects compose the hurricane.

---

[3] Define CONTACT: The xs are "in contact" if they do not overlap spatially and are "clumped together". That is, the xs are in contact if (1) no two of them overlap spatially, and (2) if Y and z are among the xs, then y is in contact with z, or y is in contact with w, which is one of the xs, and w is in contact with z- and so on. See Van Inwagen (1990). p.33.
[4] Define FASTENING: Objects are in contact and suppose that they are so arranged that, among all the many sequences in which forces of arbitrary directions and magnitudes might be applied to either both of them, at most only a few would be capable of separating them without breaking or permanently deforming or otherwise damaging each one of them. Then these objects are fastened to each other. See Van Inwagen (1990). p.56.
[5] Define COHESION: Fastened and cannot be separated. See Van Inwagen (1990). p.58.
[6] Define FUSION: objects are melted into each other in a way that leaves no discernible boundary. See Van Inwagen (1990). p.59.
[7] e is self-sustaining: earlier stages of e cause later stages of e. See Carmichael (2015). p.482.
[8] e is homeodynamic: possibly, there are xs such that the activities of the xs constitute e at one time, but none of the xs participates in e at another time (at which e occurs). See Carmichael (2015). p.482.

Why must rely on a new distinction between different kinds of composition? Indeed, according to the most natural understanding of composition, things compose something, just as (i) shows, when they are bonded together in some appropriate way. However, there are some different composition situations. If we glue two people together, although they are bonded, they compose nothing. Or, in a hurricane, parts are not bonded, but they still compose the whole. Solely having (i) cannot fully address these composition cases.

However, according to the new two-category SSA, there is a new "event-based" category. For event-based objects, bonding relation is not the condition for composition. In such a way, two people glued together do not compose anything. People are not lump-like—they are event-based objects and so do not meet condition (i). And merely gluing together the hands of two people does not result in an event that imposes sufficient unity on them. So they do not meet condition (ii), either. Therefore, according to Carmichael, the following two statements reflect composition in different manners:

| | | |
|---|---|---|
| **A:** | xLPy (lump-like) | *Cells are composed of particles.* |
| **B:** | xEPy (event-based) | *I am composed of my cells.* |

Having illustrated Carmichael's SSA, we can now start to discuss the transitivity within it. Namely, under what circumstances are some parthood relations transitive? And, under what circumstances should this transitivity stop? This indeed depends on how we find the transitivity within two categories.

At first, let me state Carmichael's solution:

**Lump-like Transitivity** (LT) $\forall x \forall y \forall z((xPy \wedge yLPz) \rightarrow xPz)$

("If x is a part of y, and y is a lump-like part of z, then x is a part of z.") Here is Carmichael's argument for LT (Carmichael, 2015: 481): according to LT, suppose that y1, ..., yn are lump-like and bonded, and that they compose object z. And suppose that x1 is among some objects x1, ..., xn that are lump-like and bonded, and that compose y1. Then we have (1) - (3):

(1) y1 is bonded to y2, ..., yn, and x1 is bonded to x2, ..., xn.
(2) If y1 is bonded to the rest of z, then it is impossible that all parts of y1 is unbonded to the rest of z, i.e., there must be some part, let it be x1, of y1 bonded to one of y2, ..., yn.
(3) Bonding is transitive[9].
From (1)+(2)+(3), we have
(4) Lump-like x1, ..., xn (all parts of y1) are bonded to lump-like y2, ..., yn.

From (4)+Definition of Lump-like composition,

[9] It is necessary to clarify what "Bonding" in LT precisely means. Recall Van Inwagen's different kinds of bonding. By saying "bonding is transitive", it is very obvious that Carmichael takes Fastening, Cohesion, and Fusion as "Bonding" here, but not Contact (Consider this counterexample against Contact transitivity. Three cubes are placed in this way: A contacts B, and B contacts C, A does not contact C, where Contact transitivity fails.

(5) x1, ..., xn and y2, ..., yn compose something
(6) It is extremely implausible to say that x1, ..., xn and y2, ..., yn compose something other than z. (for y2, ..., yn are parts of z; and x1, ..., xn compose y1, the remaining part of z.)
(7) x1, ..., xn and y2, ..., yn compose z, as its parts.

Carmichael arrives at:
(8) Conclusion: Parthood relation involved in a lump-like composition is transitive.

In Carmichael's LT argument, the lump-like transitivity follows from the transitivity of bonding. Parts of a lump-like object must be parts of the further lump-like object composed by those objects, because the transitivity of bonding bans the possibility for them to be scattered. For example, If I bond the right half of a model airplane to the left half, then every part of them must be bonded as well. This is how Carmichael's LT works.

**Event-based Transitivity** (ET) $\forall x \forall y \forall z((xPy \wedge yEPz) \rightarrow xEPz)$

Similarly, Carmichael finds the transitivity in event-based cases. ("If x is a part of y, and y is an event-based part of z, then x is an event-based part of z.") Here is how his argument goes:

(1) Event constitution (C) The activities of xs constitute event e iffdf the fact that e occurs is grounded in facts about the activities of xs.
(2) The facts about a composite object are grounded in facts about its parts[10].
(3) Grounding is transitive[11].
(4) Any parthood relation involved in an event-based composition is transitive.

For example, Some event e is constituted of composite objects x1 , ..., xn. Then, by (C), the occurrence of e is grounded in facts about x1 , ..., xn. According to (2), Facts about x1 , ..., xn that ground the occurrence of e are themselves grounded in their parts. As grounding is transitive, the occurrence of e is grounded in facts about the parts of x1 , ..., xn, in other words, parts of x1 , ..., xn are 'caught up' in the event as well. The event-based transitivity follows from the transitivity of grounding.

Taken together, Carmichael intuitively claims that transitivity follows and Van Inwagen's transitivity problem fails. But at least two objections might be raised. The first one is quite short and general: Carmichael's SSA is unclear in its ontology. First, its distinction between the two categories is unclear. Lump-like objects are defined as "*the non-event-based objects*" (2015: 479) and event-based objects are defined as "*objects united by an event*" (2015: 478). But, as shown in many cases, some

[10] The premise is an analogue for grounding of mereological supervenience. Proponents of the latter thesis include Horgan (1982); Kim (1984); Zimmerman (1997); Markosian (2005); Koslicki (2008). Strictly speaking, a weaker premise suffices for ET: if some facts about the activities of a composite object x partially ground the occurrence of event e, then these facts about x are grounded in facts about the parts of x.
[11] Proponents of this premise include Schaffer (2009); Fine (2010); Correia (2010); Whitcomb (2012); Cameron (2008). For some alleged counterexamples, see Schaffer (2012). For replies, see Litland (2013) and Raven (2013).

objects are also possible to be both lump-like and event-based. For instance, for Carmichael's SSA, when the simples compose particles, they are lump-like objects. But when the same simples constituting me (suppose that he really solves the transitivity problem), being united by the occurrence of my life, they are event-based objects. Therefore, this ambiguity shakes the root of his distinction.

Second, Carmichael's SSA is ambiguous in answering this question: which one is the "lump-like" or "event-based" object, the candidate objects for composition, or the composites after composition? He did not give a clear answer to this, saying that is the former, the latter, or both. When saying "*the existence of an (event-based) composite object is grounded in the occurrence of an event*" (2015: 478), and "*for some composite objects, there is no underlying event at all. For example, in the case of an object like a rock, there does not seem to be an underlying event that unifies its parts...rocks are lump-like*" (2015: 479), Carmichael seems to view the composite object after composition as the lump-like or event-based objects. But, as per his SSA, xs, as the candidate parts before the composition, should be those lump-like or event-based objects. If a clear material ontology is what we desire, this is not a small problem that can be ignored.

The second objection is much more substantial: Carmichael's LT fails. Formally, take F1 as "lump-like objects", R1 as "bonding", F2 as "event-based objects", and R2 as "united by events" we have:

| | | |
|---|---|---|
| **Carmichael's SSA** $\forall xx\exists y, xx\sum y\leftrightarrow$ | $\forall z(z\prec xx\rightarrow F1(z)) \wedge R1(xx)$ | (Lump-like disjunct) |
| | $\vee\forall z(z\prec xx\rightarrow F2(z)) \wedge R2(xx)$ | (Event-based disjunct) |

Then, within this two-category SSA, consider all the four basic transitivity cases that we have at hand now:

| | | |
|---|---|---|
| Lump-like first, then Lump-like | $xLPy \wedge yLPz$ | (LT) |
| Event-based first, then Lump-like | $xEPy \wedge yLPz$ | (No proof) |
| Event-based first, then Event-based | $xEPy \wedge yEPz$ | (ET) |
| Lump-like first, then Event-based | $xLPy \wedge yEPz$ | (ET) |

Examples are the best way to show these four cases. For "*Lump-like first, then Lump-like*" cases, a handle can be a part of a door, and the door is a part of the house; for "*Event-based first, then Event-based*" cases, I am a part of the football team, while my team is a part of a game; for "*Lump-like first, then Event-based*", a simple is a part of a particle, and the particle is a part of me; for "*Event-based first, then Lump-like*", it is hard to give an example, and I will provide mereotopological reasons for this later in section 4.4.

Here I attack LT that having (7) is not sufficient for getting (8). Given (1) - (3) and the definition of lump-like composition, we cannot arrive at (8), because one important situation is lacking for getting it. According to two categories, if x is a part of y, then x can be a lump-like or event-based part of y. This suggests that LT should involve two cases: "$xLPy \wedge yLPz$" and "$xEPy \wedge yLPz$". Only transitivity is found in both cases, can we say that LP is successful. However, Carmichael's argument for LP only deals with the former, while leaving the latter to be unsolved. If LT does not involve both two cases, it is not justified to claim that transitivity is saved, as the conclusion (8) does.

Take a closer look. According to SSA, "xEPy ⋀ yLPz" is:

Event-based objects (F2)→→→Lump-like objects (F1)→→→objects
Sufficient united by an event (R2) bonded (R1)

The remaining question for him to answer is that, "why event-based objects, through event-based composition, cannot compose lump-like objects?", in other words, "why the composite objects, from event-based compositions, cannot be bonded to compose further objects?" Carmichael's SSA may respond to this that, this case is actually impossible, because it is very unlikely for composite objects, from event-based composition, to be lump-like, and thus be appropriate candidates[12] for an upcoming lump-like composition. However, he just ignores this, without giving a justification for doing so.

In this situation, does the two-category SSA really offers a way to fully solve the transitivity problem? I doubt it. Since, in the first place, it lacks clarity in the fundamental distinction between two categories of composition, and has an unclear material ontology; in the second place, even if we adopt this SSA, lump-like transitivity is still uncompleted and unsolved. Taken together, my provisional diagnosis is that, Carmichael's SSA is a good step towards a commonsense answer to SCQ, but it is incomplete and imprecise, which means that more work needs to be done in his direction.

## 4. Mereotopological SSA

Through my argument in section 2, it seems that the two-category SSA does not fully solve the Transitivity problem. However, although I am not persuaded by Carmichael's argument, I am still happy to accept his general proposal, namely, the distinction between "lump-like" compositions and "event-based" ones. Different from the most common understanding of the "bonding" condition for composition, Carmichael's SSA finds a new category for material objects to be united by some events, as shown in the SSA Event-based disjunct, for which bonding is no more the criteria for composition at all.

The general lesson is, therefore, that as long as Carmichael's two-category SSA proposal is correct, transitivity needs to be found in its two categories. Thus I am motivated to update his proposal, to reach the same conclusion as Carmichael's, to save transitivity. And my strategy is quite straight: follow his direction and add some more precise work to fill his proposal.

[12] I use the word "candidates" rather than "participants", because I cannot confirm that the composition really happens. If I can confirm so, then I use the word "participants".

My plan has two steps: first, make a further distinction between *many-to-many* composition and *many-to-one* composition, within the two-category SSA, to save transitivity (aiming at my second substantial objection to Carmichael); second, add a clear definition to "lump-like" objects through "exterior boundary" characterization in mereotopology (aiming at my first short objection). In this way, to update Carmichael's proposal into a better SSA and give a clear material ontology.

To formalize this idea, the first step:

**(Many-to-one composition)** $xx\sum y \equiv_{df} \forall z(z \prec xx \rightarrow zPy) \wedge \forall z(zPy \rightarrow \exists w(wOz \wedge w \prec xx))$

Same as the standard mereological definition of composition, introduced in section 1.

**(Many-to-many composition[13])** $xx\sum{}^{*}yy \equiv_{df} \forall z\exists y(z\prec xx \wedge y\prec yy \rightarrow zPy)$
$\wedge\ \forall m(\forall y(y\prec yy \wedge mPy) \rightarrow \exists w(wOm \wedge w\prec xx))$
$\wedge\ \forall x1\forall x2(x1, x2\prec xx \rightarrow \neg(x1Ox2))$

("When many things compose many other things, the xs compose* the ys ≡df each one of the xs is a part of exactly one of the ys, every part of each one of the ys overlaps at least one of the xs, and no two of the xs overlap[14].") Similarly, we can have *many-to-many* lump-like composition "L∑*", *many-to-one* event-based composition "E∑", *many-to-many* event-based composition"E∑*", *many-to-one* lump-like composition"L∑".

The standard *many-to-one* composition deals with relations between plural variables and singular variables, while *many-to-many* composition deals with relations between plural variables. It is evident to see that, in all four basic cases of the Transitivity problem, they are the combination of *many-to-many* composition and *many-to-many* composition. If we can add a recursive clause to SSA "if xx *many-to-many* compose yy, and yy *many-to-one* compose z, then xx compose z", then the parthood transitivity from x to z, in our four cases, follows. This will be analyzed in detail in section 4.3.
The second step: I define "lump-like" objects as those objects have only 1 mereotopological exterior boundary in 3-dimensional space, at a time point t, where "EB" is "being an exterior boundary of" (I will explain EB in mereotopology later), and "∃!" is the uniqueness quantification "only one":

**Lump-like** objects** $\exists! z,\ zEBx$

("if z is an exterior boundary of x in space, and there is only one z, then x is a lump-like object")

Combining these two steps, my proposed SSA can be stated as follows:

**Mereotopological SSA**
$\forall xx\exists y,\ xx\sum y \leftrightarrow$ at t, $(\forall n(n\prec xx \rightarrow \exists! z,\ zEBn) \wedge \exists! z',\ z'EBxx)$ (Lump-like**)

[13] Here I borrow the many-to-many composition characterization of Silva (2013). p.78.
[14] See Silva (2013). pp.78-80.

$\vee$ in $\triangle t$, $(\forall n(n \prec xx \rightarrow \text{Event-based}(n)) \wedge \text{Sufficiently United}(xx)$ $\wedge \neg (\exists! z, zEBxx))$ (Scattered Event-based*)

$\vee \exists x \exists xx\ (xx=x)$

$\vee \exists zz \exists w\ (xx \sum^{*} zz \wedge zz \sum w)$ (Recursive[15])

Lump-like**ness, and Scattered event-based*ness will be analyzed in detail in sections 4.1 and 4.2.

For any xs, there is some y such that the xs compose y iff: at a time point t, 3-dimensionally, the xs are lump-like objects and they compose further lump-like objects (e.g. several pieces of wood compose a model airplane); or, during a time period, 4-dimensionally, the xs are event-based objects and are sufficiently united (e.g. winds, droplets, ..., compose a hurricane); or, there is only one of the xs; or, there are some zs and some w such that the xs *many-to-many* compose the zs and the zs *many-to-one* compose w (Recursive).

If we want to make SSA philosophically convincing, we need more precise work: not only should we offer a clear definition of what are the fundamental building bricks of our ontology, but also should we offer justified composition conditions for SSA, to use those bricks to build the material world. Only in this way can we have a fine material ontology. This is exactly the work I will do in the next section.

## 5. An argument through mereotopology

### 5.1 Why "Lump-like"?

My story begins with ordinary objects. Compared with the strange composite objects of Universalism, or the simples or gunk of Nihilism, we do have more commonsense reasons, empirical or not, to believe in the existence of ordinary objects, e.g., tables, chairs, walls, stones, and so on.

Among them, there are some lump-like objects, e.g. a grain of rice, a cube of metal, a wooden block, etc., which are generally used by us to compose other ordinary objects, just like we use wooden blocks to make a table, or bricks to build a wall, etc. Then, I pick these lump-likes as the building bricks to build a commonsense material ontology.

Why do so? Three reasons can justify my choice. First, this choice can make our material ontology as clear as possible. If we want to have a plausible ontology, we then want *everything* within it, speaking from a mereological point of view, the composite material beings or the building bricks, to be as safe as possible. Then, lump-like ordinary objects can maximally satisfy our requirements, compared with two camps of alternatives:

[15] See Silva (2013) for a precise articulation of the recursive clause. By adding this clause, he did not actually give a specific SSA proposal, but only a possible schematic *form* of SSA that is free from the attack of the transitivity problem.

Compared with postulated gunks and simples, or strange objects in Nihilism's or Universalism's ontology, lump-like objects are more concrete, because they are empirically accessible and trustworthy. For simples[16], at least, no one knows whether there are them. Moreover, even if there are them, it is uncertain that which entities, like quarks, Bosons, etc., are really simples, as there is the possibility that currently most successful theories about the building bricks may be defeated by later theories, and the postulated entities may later be found no more the simples.

Similarly, for Gunks, no one will be able to, via any instruments or not, get empirical access to every one of those gunks. And no one can exhaust empirical access to every one of the strange objects. But we can get access to tables, atoms, ice cubes, and other lump-likes. The ontological trustworthiness of the lump-likes often comes from an empirical basis. At least, whether a table exists is not as controversial as whether simples, gunks, and strange objects exist.

Compared with Carmichael's SSA, which admits the lump-likes and the event-baseds, plus their distinction, my choice is more ontological parsimonious, since it posits fewer types of building bricks of the universe. This leads to a more elegant and clear foundation layer of material ontology: lump-like only. In my picture, there is no longer any burden to admit the distinction between lump-like objects and event-based objects, as criticized in section 2.

Second, this choice does not contradict common sense. A lump-like metal cube in front of my eyes exists. Somebody might deny this existent. For instance, supporters of Nihilism definitely deny it, because they radically deny the existence of any composite objects (this is indeed contradictory to our commonsense intuition!). But people who admit this extent are not necessarily supporters of Universalism, as they can still say that this metal cube exists while composite objects from unrestricted composition, like "the Eiffel Tower and Donald Trump's hair bonded to the moon" (also contradictory to commonsense) do not.
Third, this choice has maximal explanatory power for ordinary objects. Obviously, it is hard for Universalists to explain what is the difference between "lump-like" compositions and "event-based" compositions, as for them, all compositions are unrestricted. Putting Universalism aside, compared with Nihilism which admits only gunks or simples, My choice does have stronger explanatory power for ordinary objects. What is there in the spatio-temporal region of the table? An answer like "four wooden legs plus one wooden desk bonded together" indeed seems more convincing than "a cluster of simples, or gunks".

Taken together, lump-like objects are now the starting point for my SSA to build a commonsense material ontology.

### 5.2 "Lump-like" in mereotolopogy

After justifying my choice, before I start my argument, however, it is necessary to make our understanding of "lump-like" more clear: when I say that something is lump-like, I

[16] Here I mean the *real* simples. Mereologicaly, Sx=df ¬∃y, yPPx ("entities with no proper parts"), but not the postulated building bricks, molecules, atoms, Bosons, etc., of physical theories.

am not saying that they are "non-event-based" (as what Carmichael's SSA does), rather, I wish to give a solid definition of "lump-like" objects.

What lump-like objects, like metal cubes, wooden blocks, etc., share in common? In formal ontology, mereotopologically, when saying some object is a "lump", technically we are saying that it occupies a region in space, while every part of a "lump" is connected (actually, a connected point set).

To articulate this view, in mereotopology, where C is "connected"[17]:

| | |
|---|---|
| **Reflexivity** | (C1) xCx |
| **Symmetry** | (C2) xCy→yCx |

Then "lump-like" can be characterized as:

**Lump-like* objects** $\forall x \forall m \forall n\ (mPx \wedge nPx \wedge mCn)$

("x is lump-like iff, for any two parts of it, they are mereotopologically connected[18]")
This seems to be the most intuitive *prima facie* characterization for lump-like material objects, through mereotopology. For instance, no matter how you find two parts in a metal cube, they are always mereotopologically connected. The path connectivity between m and n ensures that there must be a path inside the object (without leaving the object) connecting them. Lump-like* seems to be a proper definition for lump-likeness. But, is this true?

Unfortunately not, as there are some problematic cases against Lump-like*. Consider a hollow sphere with small balls inside, which is tangential to its inner surface, thus not connected. Is there one composite thing, or many things? When each of them is super small in a molecule-level size, actually, this is the case similar to a football, which itself is a hollow sphere, and contains many air molecules inside it. Lump-like* would answer "many things", and say that there is no composition, as parts are not mereotopologically connected. However, our commonsense intuition is likely to admit that there is only one thing, a football. At least, no one is likely to say that a football is not one thing, but many things, of the hollow sphere plus those air molecules inside it.

All lump-like objects satisfy Lump-like*, while Lump-like* fails to accommodate these problematic non-connected cases, thus it is a *necessary but not sufficient condition* for something to be lump-like, not strong enough. Therefore, we need to seek a more appropriate definition.

---

[17] I do not adopt any Transitivity axiom here. And (MT1)-(MT3) do not involve anything from Mereological transitivity. As I will use mereotopological treatment to prove mereological Transitivity in section 4.4. This is to avoid the circularity to use mereotopological transitivity to prove mereological Transitivity.

[18] Someone may deny lump-like* by giving examples, such as "the leg and the back are both parts of a chair, but they are not connected". But this is wrong. Lump-like* shows "*mereotopologically connectivity*" rather than "*connectivity in everyday talking*". A subset of a topological space is connected if it is connected in the subspace topology. A space is connected if any two points of it lie in some connected subset, that is, can be joined by some connected set. "mCn" is of *path connectivity*, a property of a space whereby any two of its points can be joined by a path, i.e., a continuous image of a segment.

From (C1) and (C2), we may define other mereotopological relations as follows (the definitions here follow the axiom system of Rachavelpula, 2017):

| | |
|---|---|
| **External Connection** | (MT1) $xECy := xCy \wedge \neg xOy$ |
| **Tangential Part** | (MT2) $xTPy := xPy \wedge \exists z(zECx \wedge zECy)$ |
| **Internal Part** | (MT3) $xIPy := xPy \wedge \neg xTPy$ |
| **Crossness** | $xXy := \neg xPy \wedge xOy$ |
| **Straddleness** | $xSTy := \forall z(xIPz \rightarrow zXy)$ |

Crossness says that "x crosses y whenever x is not a part of y and overlaps y", while Straddleness says "x straddles y whenever x is such that everything of which it is an interior part that crosses y". From here we can define Boundaries and Tangents:

| | |
|---|---|
| **Boundary** | $xBy := \forall z(zPx \rightarrow zSTy)$ |
| **Tangent** | $xTy: = \exists z(zPx \wedge zBy)$ |

Tangent says that "A tangent of y is an entity which has as part a boundary of y". From Tangent, we are able to prove that all tangents must be straddlers, and that every boundary of y must be a tangent of y and therefore cannot be an interior part of y. We can prove further (stronger) about boundaries, that:

| | |
|---|---|
| **Boundary Definition** | $xBy \equiv \forall z(zPx \rightarrow zTy)$ |

Intuitively, among boundaries, there are two different kinds: inside and outside[19]. Then, from which one, can we get the proper mereotopological definition of lump-likeness? Imagine a lump of cheese with air bubbles inside. Not only does it have an exterior boundary (EB), but it also involves many internal boundaries of those holes or internal cavities within it. But, as long as it is a lump, no matter how many interior boundaries of air bubbles inside it, after all, it can only have one exterior boundary. It is the exterior boundary that determines its "lump-likeness".

Exterior Boundary (EB) is what I want to define lump-likeness. However, it is mathematically very difficult, if not impossible, to formally distinguish the boundaries of the holes inside an object and its exterior boundary[20]. Several attempts are made by

---

[19] See Casati and Varzi (1994).

[20] I tried to mathematically distinguish EBs and the boundaries of inner holes in three possible ways, but each faced problems. Technically,

(a) Complementary approach: *the complementary set of the lump of cheese has two disconnected parts: internal holes, and external space. The external space is an infinite open set, while the holes and cavities are finite.*
Problem: although intuitive, it is hard to mathematically prove that the external space is infinite.
(b) Homeomorphism approach: *the holes are homeomorphic to a ball, but the external space is not.*
Problem: when the structures of the holes are more complex (e.g. in doughnut-like or Klein-bottle-like structures), they are not homeomorphic to a ball. (b) fails.
(c) 3-dimensional approach: *if the origin is within the object, 3-dimensionally, then any coordinates of any points (x,y,z) within the geometrical space of the cheese is within the EB, but not within the boundaries of interior holes.*
Problem: Which point should we choose as the origin? By saying "the origin has to be within the EB", this method to define EB seems to be trapped in circularity.

mereotopologists and topologists[21] to formalize EB, but this remains to be a controversial and currently unsolved[22] hard problem in mereotopology.

This means that I have to add some intuitions about EB here, more than the mereotopological characterization of Boundaries, to complete my lump-likeness definition. EBs of x are boundaries that separate x from the (outside) remainder of the universe. For instance, you can touch the EB of a lump of cheese without breaking it, but you cannot do this to the boundaries of holes inside it; or, the EB of the lump of cheese can be illuminated by the sunshine, but the boundaries of holes cannot. I believe that readers will understand this intuitive metaphysical distinction between them. Using EB to characterize lump-like objects, then we can achieve a new definition, through the uniqueness quantification:

**Lump-like** objects** $\exists! z, zEBx$

Unlike Lump-like*, the new definition Lump-like** does accommodate non-connectivity composition cases. It only defines a lump-like object in terms of *itself*, plus *its own exterior boundary*. An object is lump-like, when it has only one exterior boundary that separates it from the remainder of the universe. Therefore, Lump-like** is the mereotopological definition that I want for the building bricks of a fine material ontology. Here I get it. Then, Lump-like** should be put into SSA, to see how compositions, lump-like ones or event-based ones, work. This is exactly the work I will do in section 4.3.

## 5.3 Composition from the lump-like building bricks

Indeed, the most common motivation for SSA theorists is to divide compositions into different types, to which different criteria may apply. Speaking for myself, I feel the same motivation to apply different conditions for different composition cases. Having Lump-like** objects at hand, now it is time to get to the bottom of how other material objects are composed of them. Here I chose to follow Carmichael's proposal of a two-category SSA.

However, although I follow Carmichael's direction, my SSA ontology is quite different from his. As criticized in section 2, first, his route to building up the material ontology starts from ambiguous distinctions between lump-like objects and event-based objects; second, he uses the composition concept "being united by an event" to define "event-based" objects, and furthermore using "non-event-based" to define "lump-like". Both lead to an unclear material ontology.

In contrast, my SSA material ontology has solid building bricks—"lump-like" objects only. In other words, what exists there are only lump-like bricks, both lump-like composition and event-based composition can be imposed onto them. More precisely, my task is to use Carmichael's two categories to compose further objects from the Lump-like**s.

[21] For different kinds of boundaries, see Brentano (1988), Part one; Smith (1992); Casati and Varzi (1994).
[22] See Rachavelpula (2017). p.15

**(a) Lump-like** composition**

Now I can start my argument. Recall Carmichael's Lump-like disjunct "$\forall z(z \prec xx \rightarrow F1(z)) \wedge R1(xx)$", when xx are lump-like objects and bonded, then through lump-like composition, they compose a further object. Because of the transitivity of bonding relation, the new composite object is also lump-like.

Having Lump-like** and lump-like building bricks at hand, now, we can utilize them to grasp lump-like composition: the lump-like** objects, as building bricks, have only 1 exterior boundary in ordinary 3-dimensional space "$\exists !z, zEBx$", while the new composite object composed of them has only 1 exterior boundary "$\exists !z', z'EBxx$", then the Lump-like** composition happens.

**Lump-like** Composition** $\forall xx \exists y, xxL\sum y \leftrightarrow$ at t, $\forall n(n \prec xx \rightarrow \exists !z, zEBn) \wedge \exists !z', z'EBxx$

For instance, when we fuse two metal cubes to create a new one, two lump-like metal cubes each have only 1 exterior boundary. When they are fused, the new fused thing has only 1 exterior boundary too. Then, obviously, we can say that the lump-like** composition happens.

Actually, Van Inwagen's discussion on fusion, as the strongest kind of bonding relation, has already suggested or implied, more or less, this way of thinking compositions in an "exterior boundary" manner, between his words and lines. But so far, not much attention has been paid to this:

"It is possible to cause objects to be joined more intimately than this, so that they melt into each other in a way that leaves *No Discoverable Boundary*. If two very smooth pieces of chemically pure metal are brought together, for example, they become attached to each other in just this intimate way...Let us say that if two things are caused to "merge" in this way, they become fused or that they fuse." (Van Inwagen, 1995: 59)

By saying "*no discoverable boundary*" and "*attached in this intimate way*", actually, Van Inwagen means the boundaries between parts disappear. If I translate it into a mereotopological reading, as I have argued that exterior boundaries, but not interior boundaries, are the criteria for composition, then, taking the composite object as a whole, through my lump-likeness**, there is no new exterior boundary added in the composition. My work here is in line with Van Inwagen's view on bonding, and just to develop this idea further, through mereotopology.

After introducing the origin of my idea from Van Inwagen's work, let us consider a further objection. Someone may raise a quick "two EBs" rebuttal to my lump-like** composition: when I stick a small sticker to a cup, do they lump-like**ly compose something? Or, more radically, when a lump-like speck of dust is attached to a super big lump-like metal cube, do they compose something? If so, there are 2 EBs of the composite, including one of the bigger object and one of the smaller; if not, they are actually bonded, and this contradicts our commonsense intuition (e.g. they *do* compose a cup with a sticker on it, or a metal cube with one speck of dust). Lump-like** composition seems to be trapped in a dilemma.

The root of this rebuttal is that candidates for lump-like** composition can differ greatly in size. The cube and the dust are both lump-like**, with only 1 EB. When the candidates for composition are close in size, we are likely to ignore this "two EBs" problem. However, when candidates differ greatly in size, guided by the commonsense intuition, if I try to admit the composition, like a cup plus a sticker, then I have to admit that lump-like** composite objects can have 2 EBs. Similarly, if I bond more and more small lump-likes to a bigger one, I will get more and more EBs. And there would be "three EBs" problems, "four EBs" problems, ..., and so on. Then my lump-like** composition collapses.

To save my lump-like** composition, this problem must be solved. Luckily, the binary relation ">>" (much greater than) and "V" (3-dimensional volume) can offer a solution, where "xx-xn" is the remainder of xx out of xn:

**Approximation** ∀xn∃xa, 1≤a, 2≤n,
((∃!zn, znEBxn)
∧(xnEC(xx-xn))
∧((Vxa>>Vx1, ..., Vxa-1, Vxa+1, ..., Vxn) ∨(Vxx>>Vx1, ..., Vxn)))
→(∃!z, zEBxx))

("when one lump-like** candidate xn's, or the whole xx's volume is much greater than other ones of xx, and all the lump-like** candidates, in 3-dimensional space, then xx approximately has only 1 boundary".) The key point of Approximation is that, when an object has actually more than one EB, but there is one candidate much greater than all other ones, in 3-dimensional space, boundaries of the tiny ones can be ignored. The composite object still has only 1 EB. Lump-like** composition holds.

Take a closer look at composition situations where the lump-like** candidates differ greatly in size. What happens in this kind of composition, e.g. metal cube plus one atom, or a cup plus a sticker? First, there are more than one lump-like** ("∃!zn, znEBxn") candidates ("∀xn∃xa, 1≤a, 2≤n"); then, the candidates are externally connected ("xnEC(xx-xn)"), for, if the atom is away from the cube, or the sticker is not attached to the cup, our commonsense (not Universalists nor Nihilists) would not say that they compose something; moreover, among the candidates, one's 3-dimensional volume is much greater than the other ones ("Vxa>>Vx1, ..., Vxa-1, Vxa+1, ..., Vxn"), just like the volume of the cube is much greater than the speck of dust, or the volume of the cup is much greater than the sticker. Then, Approximation says that they compose further a lump-like** object.

My treatment of Lump-like** compositions here has two good implications. It will be helpful in solving the Transitivity problem, which we will see in section 4.4. What is more, the Approximation solution to Lump-like** can shed light on the Sorites Paradox, which we will see in section 5 for a more detailed analysis of Approximation.

**(b) Event-based* composition**

Here I agree with Carmichael's proposal on "event-based" composition and "sufficient unity". Limited by space and my argumentation purpose, I shall not repeat event-based compositions here.

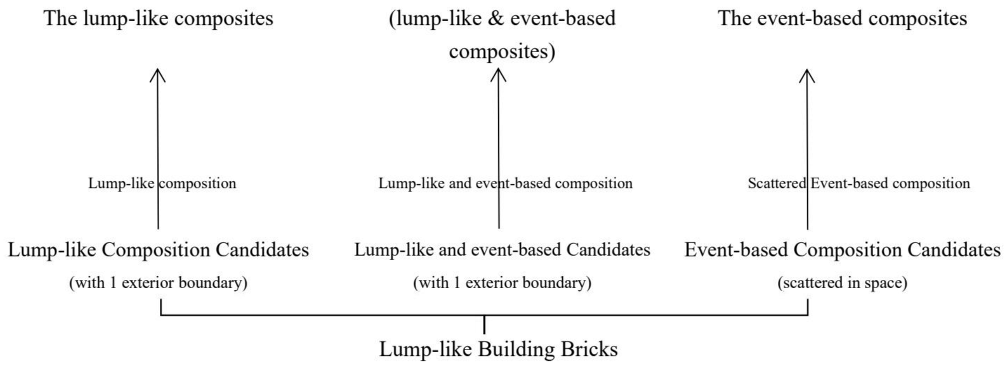
But two things need to be noted: first, unlike Carmichael's, in my SSA, my lump-like building bricks can undergo event-based compositions, as event-based participants. When being involved in event-based compositions, they are as participants for them. To put it bluntly, there are only lump-like building bricks as the keystones of the material ontology. But they can be put together in two different manners, lump-like** and event-based composition. When undergoing event-based ones, the building bricks serve as event-based object candidates.

Just like we can play different two games with the same deck of cards. Now, having all "lump-like" cards at hand, there are different kinds of composition games that we can play: the lump-like** one, the event-based ones, or the lump-like** and event-based ones.

What is more, obviously, as shown in the cases of players in a football team, or winds and droplets in a hurricane, event-based composite objects can have more than 1 exterior boundary. If they have 1 EB, according to my Lump-like** treatment, their compositions are both Lump-like** and Event-based. In other words, some composite objects are only results of event-based compositions, scattered in 3-dimensional space, but completely not results of lump-like** compositions. Let me define two kinds of Event-based* Compositions:

> **Scattered Event-based* Composition**: candidates are scattered in space, and only united by underlying events.
>
> **Lump-like and Event-based* Composition**: candidates taken together are with only 1 exterior boundary, united by underlying events, held together by 1 exterior boundary.

To avoid the confusion between "lump-like building bricks", "candidates for composition", "composition", and "composite objects", here I give a picture to present my material ontology:




At first, we have the lump-like building bricks as the keystones of our material ontology, then, some of them can participate in lump-like** compositions, as lump-like composition candidates, while some of them can undergo event-based compositions, as event-based composition candidates. And it is also possible for the same kind of objects, (e.g. particles, etc.), to have a double identity, to be both lump-like** composition candidates and event-based ones, undergoing lump-like & event-based compositions.

In Lump-like** compositions, composition candidates are with only 1 EB; in event-based composition, composition candidates can be scattered in space or with 1 EB. As an object can only have 1 exterior boundary, or not, namely, scattered in space, this proposal is completed enough to exhaust all the possible situations for composite objects to be. I arrived at a simple and elegant picture of material ontology.

Someone may ask about Carmichael's most representative example, which says that a person's parts are united by the occurrence of his or her life. Therefore, I should be something as a result of an event-based composition. Now, in my material ontology, I shall respond that, as long as I (my body) satisfy the condition of having only 1 EB, I am also from a lump-like** composition. I view it not merely as a case for event-based composition, but as a lump-like** and event-based one. Apart from my life, the lump-like**ness also plays its role in holding my parts together[23].

As suggested by Carmichael, the composite object in an event-based composition only exists in a time period $\triangle t$, not at a time point. Then, add two things, the time period "$\triangle t$" and the scatted in space condition "$\neg$ ($\exists$!z, zEBxx)", to the characterization of scattered event-based compositions:

| | | |
|---|---|---|
| **Scattered Event-Based* Composition** | $\forall xx \exists y, xxE\sum y \leftrightarrow$ | in $\triangle t$, $\forall n(n \prec xx \rightarrow$ Event-based (n)) $\wedge$Sufficiently United (xx) $\wedge\neg$ ($\exists$!z, zEBxx) |

Taking Lump-like** composition and (Scattered) Event-based* composition together, plus Recursive, now I arrived at my general picture of a more precise version of SSA:

**Mereotopological SSA**

| | | |
|---|---|---|
| $\forall xx \exists y, xx\sum y \leftrightarrow$ | at t, ($\forall n(n \prec xx \rightarrow \exists$!z, zEBn) $\wedge \exists$!z', z'EBxx) | (Lump-like**[24]) |
| | $\vee$ in $\triangle t$, ($\forall n(n \prec xx \rightarrow$ Event-based (n)) $\wedge$Sufficiently United (xx) $\wedge\neg$ ($\exists$!z, zEBxx)) | (Scattered Event-based*) |
| | $\vee \exists x \exists xx$ (xx=x) | |
| | $\vee \exists zz \exists w$ (xx$\sum$*zz$\wedge$zz$\sum$w) | (Recursive) |

## 5.4 Solve the transitivity problem of Carmichael's SSA

Indeed, through sections 4.1-3, I have detailed stated my mereotopological SSA, as a more precise version of the two-category SSA. Then, how do I save transitivity within my SSA to solve the Transitivity problem? We need to take a closer look.

Unlike Carmichael's SSA seeking transitivity from the transitivity of bonding and grounding, plus a series of premises about events[25], fortunately, My SSA is more elegant and palatable, entailing mereological Transitivity $\forall x \forall y \forall z((xPy \wedge yPz) \rightarrow xPz)$, as we will now see. Here I seek transitivity in the two cross-disjunct cases, as they are the only concern for Van Inwagen's Transitivity problem.

[23] Aristotle's famous case about part-whole relations, that a hand cut off from the body is not a part of the man, can also support this lump-like** intuition.

[24] Note that "lump-like and event-based" compositions can be treat as a special case included in lump-like**.

[25] See Carmichael (2015), pp.476-485

**(a) “Lump-like first, then Event-based”**

First, the “Lump-like first, then Event-based” cases (xLPy ⋀ yEPz). In this case, x is a lump-like part of y, while y is an event-based part of z. For instance, a particle is a part of a droplet, while the droplet is a part of a hurricane. Rewriting this case into the *many-to-one* composition and *many-to-many* composition, we have that, the particle is among those participants for the *many-to-many* lump-like** composition to compose the droplet, while the droplet is among those participants for the *many-to-one* event-based* composition to compose the hurricane.

Then, from the Recursive clause: $\exists xx \exists yy$ (xx L∑* yy∧yy E∑ z), we can get ∀xx, xx∑y[26]. The cross-disjunct composition happens, in which the particle is among the participants of the composition to get the hurricane. Now, it is time to say that, the particle is a part of the hurricane.

To formalize my argument:

| | | |
|---|---|---|
| **P1**: | xLPy↔ xx L∑* yy | (Lump-like** Many-to-many Composition) |
| **P2**: | yEPz↔ yy E∑ z | (Event-based* Many-to-one Composition) |
| **P3**: | ∃xx∃yy∃z ((xx∑*yy∧yy∑z)↔ xx∑z) | (Recursive) |
| **P4**: | ∀x∀z (xx∑z↔ xPz) | (Composition) |
| **C**: | ∀x∀y∀z((xLPy ⋀ yEPz)→xPz) | (Transitivity) |

My SSA offers an adequate solution to the Transitivity problem in this case. P1 and P2 just say that something composes some further things, then it is a part of it, and vice versa. P3 is the Recursive clause that passes the transitivity from *many-to-many* cases to the *many-to-one* cases. P4 is the widely recognized definition of composition in classical mereology.

It seems that given the basic concepts in classical mereology, this argument can be wrong only if P3 is wrong. Withing P3 the recursive clause, there are two elements, the distinction of *many-to-many* and *many-to-one* composition, and the recursive relation. The former is perfectly intuitive, and obviously, very few people may feel reluctant to admit that there are *many-to-many* and *many-to-one* compositions; while the latter is not. Why must admit the recursive relation between *many-to-many* and *many-to-one* composition? This recursion seems to suffer from the risk of being *ad hoc*.

To answer this, we need to get to the bottom of the definitions of *many-to-many* and *many-to-one* composition. If, for any *many-to-many* then *many-to-one* composition cases, the participants of *many-to-many* composition, and the outcome composite object of the *many-to-one* composition also satisfy the definition of *many-to-one*, then, this recursive clause holds, as it is just entailed by the basic definitions.

| | |
|---|---|
| **(Many-to-one composition)** | xx∑y≡df ∀z(z≺ xx→zPy) ∧∀z(zPy→∃w(wOz∧w≺ xx)) |
| **(Many-to-many composition)** | xx ∑ *yy ≡ df ∀z∃y(z≺xx∧y≺yy→zPy) |

[26] Here it is no more important to distinguish that this composition is event-based or lump-like, because for my purpose to address the Transitivity problem, xx∑y is enough.

$\wedge \forall m(\forall y(y \prec yy \wedge mPy) \rightarrow \exists w(wOm \wedge w \prec xx))$
$\wedge \forall x1 \forall x2(x1, x2 \prec xx \rightarrow \neg(x1Ox2))$

Now, suppose that we have xx "*many-to-many"ly* composing yy, and yy "*many-to-one"ly* composing z at hand. What we need to do is to examine whether xx and z satisfy the *many-to-one* definition.

According to the definitions above, we can make sure that now as "yy∑z" holds, anyone of ys is a part of z, and any part of z overlaps something that is one of the ys. Then, as "xx ∑ *yy" holds, each one of the xs is a part of exactly one of the ys, every part of each one of the ys overlaps at least one of the xs, and no two of the xs overlap.

Then, the composition definitions entail transitivity. For any one of ys is a part of z, plus the fact that, any one of the xs is a part of exactly one of the ys, within each part of z, there must be at least one of the xs. Why? Mereotopologically, I can give this ad absurdum argument:

(1) Supposition: xx are not lump-like** parts of z.
(2) If any one of xx (let it be x1) is not a lump-like part of z, then from the lump-like**ness, it locates outside the exterior boundaries of z.

(3) yy are parts of z, therefore, any one of yy locates inside the exterior boundaries of z.
(4) x1 is a part of exactly one of the yy (let it be y1).
(5) If x1 is a part of y1, x1 locates inside the exterior boundary of y1, furthermore, x1 locates inside the exterior boundaries of z.

From (3) - (5), we have
(6) x1 locates within the exterior boundaries of z.

Then (2) contradicts (6),
Contradiction: x1 locates outside and within the exterior boundaries of z.

It is logically impossible for x1 to locate outside and inside the exterior boundaries of z at the same time. As premises (2) - (5) come from the definitions of Lump-like**ness, *many-to-many* and *many-to-one* composition, then if these three definitions are correct, according to my SSA, then the supposition (1) must be wrong. It cannot be the case that xs are not lump-like** parts of z. In other words, xx *many-to-many* compose yy, then yy *many-to-one* z, then xs must be lump-like** parts of z. The parthood transitivity in the recursive clause follows.

The finding here is that, as for quantification, as long as there are more things, composing fewer things, the more things are always within the portions of fewer things, while every one of the more things must overlap with the portions of fewer things. Therefore, the recursive relation between *many-to-many* then *many-to-one* compositions always holds.

For anyone willing to accept *many-to-many* and *many-to-one* compositions, he has to admit the recursive relation between them. The recursive relation is not an additional supposition, but actually, a result that comes from the definition of two kinds of compositions. The Recursive clause is crucial for my SSA to save the Transitivity.

**(b) "Event-based first, then lump-like"**

Second, the "event-based first, then lump-like" cases (xEPy ∧ yLPz). They are just ignored by Carmichael's solution to find LP. I do understand that it is difficult to find examples for these cases in commonsense ordinary objects. But, now, from my more precise SSA, I am able to provide characterizations and explanations for what happens in this case.

With the help of my SSA, I can completely deny the possibility of "(Scattered) event-based first, then lump-like" compositions. Rewrite "xEPy ∧ yLPz" into composition sentences: x is among the things that event-based* compose y, and y is among the things that lump-like** compose z, then:

**Event-based first, then lump-like Composition** xx E∑* yy ∧ yy L∑ z (if possible)

Note that E∑* here is the scattered event-based* composition. If it is a lump-like**&Event-based* one, then we can easily prove the transitivity from "lump-like first, then lump-like" cases (xx L∑* yy ∧ yy L∑z).

According to the first two disjuncts of my SSA:

∀xx∃y, xx∑y⟷ at t, (∀n(n≺xx→∃!z, zEBn) ∧ ∃!z', z'EBxx) (Lump-like**)
∨ in △t, (∀n(n≺xx→Event-based (n)) (Scattered Event-based*)
∧Sufficiently United (xx)
∧¬ (∃!z, zEBxx))

At a glance, two difficulties of this case show their true faces. Speaking for the exterior boundaries, if this case is possible, then the composite objects from the first step, the strong event-based composition, will have only 1 exterior boundary in space, according to the definition of Lump-like**, while it will have more than 1 exterior boundaries, according to the definition of Strong Event-based*. Cases like "∃!z, zEByy∧¬ (∃!z, zEByy)" are logically impossible. In other words, event-based composite objects yy are scattered in 3-dimensional space, so they cannot be proper candidates for Lump-like** compositions, as they do not meet the 1-exterior-boundary condition to be involved.

Speaking for dimensional aspects, it is perfectly justified for 3-dimensional composite objects, through Lump-like** compositions, to compose event-based objects in 4-dimensionality. However, it would be impossible for 4-dimensional Event-based* objects to lose their temporal dimension to be proper candidates for 3-dimensional Lump-like** compositions. Therefore, from my point of view, I reject this case, as it is indeed impossible, both logically and 4-dimensionally.

Taken together, here is my general diagram[27] for all the four basic transitivity cases. Linking them together, we can get infinitely long sequences of Transitivity. Therefore, My mereotopological SSA resolves the transitivity difficulty. We got what we want.

| | | |
|---|---|---|
| Lump-like first, then Lump-like | $\forall x \forall y \forall z\ (xLPy \wedge yLPz)$ | $\rightarrow xPz$ |
| Event-based first, then Event-based | $\forall x \forall y \forall z\ (xEPy \wedge yEPz)$ | $\rightarrow xPz$ |
| Lump-like first, then (Scattered) Event-based | $\forall x \forall y \forall z\ (xLPy \wedge yEPz)$ | $\rightarrow xPz$ |
| (Scattered) Event-based first, then Lump-like | $\{x,y,z \mid (xEPy \wedge yLPz)\} = \emptyset$ | Impossible |

## 6. Conclusion

In this paper, I start from a contradiction following the standard version of SSA, say, the Transitivity problem. And I have argued that Carmichael's SSA cannot fully solve this problem while having an unclear material ontology. In order to raise a preciser SSA to fully solve it, I choose to use mereotopological Exterior Boundaries to define lump-like objects as the building bricks for a fine material ontology. Then I spend some time defending my mereotopological SSA and offering the positive reasons why it is a promising one, which solves the Transitivity problem. I do admit that there still can be many objections to SSAs, but to convince all the detractors of SSA is not my main work in this paper. In this sense, I have finished.

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# AI for Science：两种科学、两种 AI 和两种数学观

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**摘要：**

自牛顿时期以来，科学研究主要存在两种范式：开普勒范式和牛顿范式。开普勒范式即数据驱动的范式，得名于开普勒分析第谷的行星观测数据而归纳出开普勒三定律，在此范式下科学家通过数据分析来获取唯象的科学发现。牛顿范式即基本原理驱动的范式，得名于牛顿发现力学三定律以解决一切经典力学问题，在此范式下科学家试图发现支配研究对象的基本原理。自牛顿之后、直到上世纪近代物理学革命完成，基础科学研究主要遵循基本原理驱动的范式进行。

然而，自上世纪 50 年代至今，基本原理驱动的范式存在两重困境。困境之一是自然科学的基本原理越来越难以获得：一方面实验或观测经验的获取难度增大，如粒子物理学领域的实验数据获取依赖耗资巨大的粒子加速器，而对高能宇宙射线的观测难以获得稳定有效的结果；另一方面随着研究规模的增大、大科学装置的普遍应用，原始的经验数据集也以惊人的速度膨胀，从中归纳基本原理的难度增加。困境之二是基于基本原理解决具体科学问题的建模和计算难度越来越大，在具体科学问题中，随着系统复杂度的上升，对应科学模型求解的难度呈指数增长。如一个具有 79 个电子的金原子对应的薛定谔方程有 79×3 维、而即使数百数千原子组成的结构仍然只是极小的尺度，对这些体系的高近似度求解在数学上存在指数级的巨大困难。

AI for Science，简称 AI4S，意为人工智能驱动的科学研究。AI4S 于上世纪 50 年代诞生，早期发展几乎同步于母学科人工智能，但其早期的工作仅仅是基于人工智能的符号主义思想进行解题自动化的工作，对真实科学研究未做出显著贡献。直到 2015 年人工智能学家辛顿和萨拉胡迪诺夫基于人工智能的联结主义思想（也参考了部分行为主义思想）提出深度神经网络模型后，AI 领域的革命性进展辐射到了 AI4S 领域，为科学研究注入了强大的动力。

首先，AI4S 缓解了当代科学研究的两重困境。一方面，复杂科学模型的求解一般表现为高维度偏微分方程组的近似求解，依赖于使用初等方法逼近多变量函数的能力。不同于传统多项式逼近方法遭遇“指数爆炸”的困难，神经网络模型逼近方法的收敛速度和函数的维数是线性关系，从而大大降低了方程组近似求解的难度，类似的优势也存在于数值积分计算领域。另一方面，AI 也存在极强的从数据中发现基本原理的潜力，如卷积神经网络可以从真实的太阳系内天体运动数据中归纳发现万有引力定律和各个天体的质量比，这至少已经处于科学原理再发现的层次。

更重要的是，不限于曾占主流的基本原理驱动的范式，AI4S 同时导致了数据驱动的范式的复兴。这一方面的工作可以以吉姆·格雷提出的“第四范式”概念概括之。自科学诞生以来，我们已经依次实现了三种科学研究的模式：实验范式为第一范式、理论范式为第二范式和计算范式（计算机模拟范式）为第三范式。在此基础上，第四范式意为数据驱动的范式，通过特定的数据学习算法，可以实现在海量数据中高效地发现变量之间的相关性，从而得到唯象的科学定律。由于没有对本体论假设和模型简单性的要求，这一模式比上述的“从数据中发现基本原理”容易实现得多，其中最突出的成就莫过于 2022 年至今 DeepMind 公司开发的 AlphaFold 模型在蛋白质结构预测领域的巨大成功。在此背景下，以收集数据为目的的实验替代了假说-检验方法下具有明确目的的实验，用于算法运作的算力中心成为了新的大科

学装置的形式，数据共享成为了推动科学发现的科学共同体共识。可以认为，AI4S 深刻地重塑了当代科学研究的格局。

然而在 AI4S 取得辉煌成就的背后，也需要注意到，基于神经网络算法的 AI 方法运用于科学研究时，至少存在三大根本性的问题：

1. AI4S 方法的成功依赖于高质量的数据集，难以用于数据集获取困难、或是数据集好坏难以评定的情境。
2. AI4S 实现的科学发现的本质是在大数据中寻找相关性，即使发现了科学原理，也仅仅是发现了原理的数学形式，而不能深入到因果性、本体论层次。
3. AI4S 进行科学发现的过程难以被人类研究者直接检验或复现，换言之，其过程对人类来说是认知黑箱，从而引起了信任问题。

总而言之，当代 AI4S 带给我们的是一种"知其然而未知其所以然"的科学。这一点在科学哲学上表现为解释功能和预测功能的分歧，AI 带来的结果可以有效预测未来，但不能对现状背后的原理给出任何具有因果效力的解释。对于秉持理性原则的基础科学研究而言，这样的科学无疑是不够完备的。

对于 AI4S 引起的科学进步及其局限，有待建立一种新的科学哲学概念框架与之适配。在这一方面的基础性工作应归功于美国哲学家保罗·汉弗莱斯。通过对近代物理学和计算科学的考察，汉弗莱斯提出了一组互逆的命题："可处理数学的发展推动着科学进步，科学模型为可使用的数学方法所约束。" 论题中"可处理"的含义位于实践层面，即建立和求解数学模型的实践可行性。直观标准是，对于给定的经验科学问题，能够建立适切的模型，且具备有效的算法和足够的算力进行模型求解。从本质上说，从人工科学到计算科学到 AI4S，科学方法的进步本质就是可处理数学的进步。，可以以"可处理数学"作为线索重新梳理科学史，将科学史划分为理论-实验科学、计算科学和 AI4S 三个时期，从而提供一种新的视角来看待新背景下的科学哲学。

汉弗莱斯的工作集中于 AI4S 的前一时期，即计算科学时期的科学发现概念框架。汉弗莱斯注意到，传统科学哲学概念框架下的分析单元——理论、定律、范式、研究纲领等——对于计算科学的背景都太大了。在计算科学中，真正值得关注的分析单元是"计算模板"，即科学定律中可直接用于计算的数学形式。选择这一单元背后的思想可以归结为费曼的名言"相同方程组同解"——少数可处理性良好的计算模板（如三类基本数学物理方程）在科学实践中占据主导地位，计算科学的成功在于大大增加了解析不可解模板的可处理性，而适切性良好但不易处理的模板则很少使用。

同这一科学发现的概念框架相适配的，还有其独创的科学认识论"非人类中心主义认识论"。汉弗莱斯认为，现代科学的认知主体已经不再是单一的人类个体，而是人类与人工系统的联合认知系统。这一新认识论在提出之初是为了适配于计算科学时期人工系统计算和模拟的黑箱问题，而到了如今的 AI4S 时期，黑箱问题蔓延到了算法的不透明性、或者说是计算模板的不透明性，无疑具备了更高的哲学价值。

当前人工智能和科学哲学界对 AI4S 所取得成就及面临困难的相关哲学问题已进行了诸多探讨，探讨的内容包括"AI 是否真的发现了科学原理"、"如何对现有 AI4S 的结果进行尽可能可靠的解释"、"能否建构可解释的人工智能模型"等问题。学界对这些问题的分析和回答，往往基于的是人类和 AI 认知与行为模式的对比，从而不可避免地涉及了大量心智概念（如"理解"）的概念分析。然而，一旦问题涉及到心智领域，就必然要直面认知科学理论的局限和心灵哲学领域大量悬而未决的问题。更进一步来说，按照查尔默斯的观点，即使我们对人类神经系统已有充分的理解，仍然无法回答"意识如何产生"的问题，也就原则上无法从根源分析"心智"。对于理解心灵的困难导致了诸多哲学观点的分歧。

但是，当我们回归汉弗莱斯论题中的核心概念"可处理数学"，重新审视非人类中心主义

认识论，就会发现这些分歧未必不能得到解决、至少未必不能得到回避。学界对于人类心智和人工智能的种种分析和对比并非解决 AI4S 带来的哲学问题的必须，而可以转而分析和对比“人类智能”和“人工智能”。较之于心智，“智能”无疑是更容易厘清的一个概念。按照 GPT 的数学基础的提出者所罗门诺夫的观点，智能的本质是对信息的压缩和对未来的预测，而“对信息的压缩”又可以还原为信息熵的熵减。人类认知系统的本质是贝叶斯机，从先验知识和感官经验推知后验信念；基于神经网络算法的 AI 则是玻尔兹曼机，通过优化目标函数来学习潜在分布。由此，所有的相关概念都还原为数学概念，而数学概念显然比心灵哲学概念清晰得多。

汉弗莱斯所做的、围绕“计算模板”概念的相关概念框架工作适切于计算科学时期，在适当延拓后同样可以适切于 AI4S 时期。不同于依据固定计算模板进行的计算机模拟，AI 进行数据挖掘、模型求解等工作时采用的是生成式算法，计算模板是依据某些基本的原理、在运行过程中不断迭代生成的。因此，AI4S 时期的计算模板和具体计算过程对人类同样是不透明的，从而不能再继续使用这一分析单元用于哲学上的概念框架建构。作为对这种不透明性的让步，可以选择“计算原理”作为 AI4S 时期的新分析单元。无论模板如何变化，指导模板进行迭代调整的原理是固定的，而基础的计算原理往往决定了某一类 AI 适用于解决哪些问题。例如，基于最优化方法的监督学习适用于数据分类，基于密度建模的无监督学习适用于自然语言处理，基于贝尔曼方程的强化学习适用于自动驾驶、机器人控制等与环境交互的领域。

AI4S 当今的成就和困境，都可以在底层的计算原理中找到原因。回到上述基于神经网络算法的 AI 方法运用于科学研究时存在的三大根本性问题。从根本上说，这三项问题都是源于其最基础的计算原理，即神经网络算法，或者说联结主义思想。神经网络算法的数学本质是线性代数、概率统计和最优化，这三种十分基础的数学工具在极高复杂度的组织下呈现为玻尔兹曼机的形态。基于神经网络算法的 AI4S 能够实现的全部工作都只是通过大量的线性迭代来拟合一个对给定数据集的最好的近似表征，无论这种近似表征做得多么完善，都不可能以之实现严谨的逻辑推导。这一点在当前的 AI for Mathematics 领域中有明显的表现：主流的大语言模型，包括 OpenAI o3 和 DeepSeek r1，进行数学证明的本质都是“按照语词频率猜一个最可能的后继步骤”而非“按照逻辑规则推导”。因此可以认为，基于线性代数的 AI 不可能克服固有的三大问题。

与之形成对比的是 AI4S 初期基于符号主义的 AI。符号主义 AI 的底层数学原理是符号运算，原则上可以兼容绝大多数的数学形式，也就意味着可以以此实现“step by step”的逻辑推理过程。相比于联结主义，符号主义 AI 的思维方式更接近于人类，不依赖高复杂度迭代来逼近对象，从而摆脱了对大数据集的依赖、可以具备因果性和对人类的可检验性。神经符号 AI（Neuro-Symbolic AI）尝试综合二者的优点：通过神经网络从数据中提取特征并生成符号表示，再利用逻辑规则或程序语言进行推理与决策，实现了数据驱动学习与结构化推理的统一。神经符号 AI 在逻辑和数学推理、知识图谱、自动编程等领域取得了优于传统神经网络的成果，对于科学研究亦有 AI-Feynman 等成功案例。

对于符号主义和联结主义两类 AI 的解读，有必要进行哲学层次的深入探讨。数学哲学的一大基本问题是数学的定义问题，或者说“数学真理是什么”的问题。对于这一问题，历史上的诸多学者（恩格斯、康德、弗雷格、卡尔纳普、哥德尔等）都已做了丰富的探讨。总而言之，各种观点不外乎数学真理是分析真理、经验真理或二者的折中。于是可以给出两种对数学的定义，我们称之为数学I和数学 II，一切其他的数学定义都介于二者之间。

数学I：数学是有效逻辑推理形式的集合；

数学II：数学是有效经验表征形式的集合。

可以看出，符号主义思想对应数学 I，联结主义思想对应数学 II。数学 I 的外延要显著

广于数学 II，这限制了基于联结主义思想的 AI 的功能上限，但换来了针对具体现实问题表征的适切性。

上述的两种科学（基本原理驱动的科学和数据驱动的科学）、两种 AI（符号主义 AI 和联结主义 AI）、两种数学观（数学 I 和数学 II）事实上共同指向了哲学中的一组基本概念：理性主义和经验主义，也就因此有了两条脉络。理性主义的脉络至少可以追溯到笛卡尔、进一步可以模糊地追溯到柏拉图，以此串联起了基本原理驱动的科学、符号主义 AI 和数学 I；经验主义的脉络至少可以追溯到培根、进一步可以模糊地追溯到亚里士多德，以此串联起了数据驱动的科学、联结主义 AI 和数学 II。理性主义和经验主义中的任何一者都不足以支撑人类的数学和自然科学事业，但二者结合产生了惊人的效用，成为了一切可靠认知的根源。当前 AI4S 的困境及其解决方案，同样可还原为两条脉络的上升螺旋。

**关键词：**AI for Science；理性主义；经验主义；符号主义；联结主义

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# A Preliminary Exploration of Conceptual Frameworks of Scientific Discovery In the context of AI for Science

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**Abstract:**

Since Newton, scientific research has followed two main paradigms: the Keplerian (data-driven) and Newtonian (principle-driven) paradigms. However, since the 1950s, principle-driven science has faced two dilemmas: the increasing difficulty of acquiring empirical data and extracting governing principles, and the exponential complexity of modeling and computation for real-world systems. In response, AI for Science (AI4S) emerged, leveraging deep learning to address these challenges by approximating complex functions and enabling principle discovery from data. This shift revives the data-driven paradigm, exemplified by the success of AlphaFold, and reflects Jim Gray's "fourth paradigm" of data-intensive science.

Yet AI4S also raises concerns. Neural network models rely heavily on large, high-quality datasets, obscure causal explanations, and operate as cognitive black boxes. This divergence between prediction and explanation highlights the philosophical incompleteness of AI-led science. To understand this shift, Paul Humphreys proposed that scientific progress is driven by "tractable mathematics" and that AI4S marks a new phase after theoretical and computational science. His "non-anthropocentric epistemology" views scientific cognition as a human-machine partnership, suitable for handling the opacity of modern AI systems.

As neural network-based AI (connectionism) struggles with logic, causality, and verifiability, symbolic AI—grounded in logical reasoning—offers a complementary path. These two approaches correspond to distinct philosophies of mathematics: Mathematics I (analytical truths, aligned with rationalism and symbolic AI) and Mathematics II (empirical patterns, aligned with empiricism and connectionist AI). The tension and synthesis between these two traditions—rationalism and empiricism—mirror the dual challenges and future potential of AI4S.

**Keywords:** Rationalism; Empiricism; Symbolism; Connectionism

# 几何实在论的两次升华*

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[长　摘　要]

## 一、几何实在论的第一次升华：从单欧系统到多欧系统

欧几里得几何不仅统治人类文明的几何观念逾两千年，而且至今在各国教育作为推理典范而被传授。受普遍观念影响的人们深信，欧几里得几何即为现代教科书中成熟样式所展现的那唯一理论系统，它是欧几里得《几何原本》经改良精简而来的现代版本。至于解析几何，它看起来不过是欧几里得几何的坐标附庸而已。

对欧几里得几何的这种刻板单一理解，所带来的只是对唯一性和由此而来的权威性之渴望的虚幻满足。这种虚幻满足根源于如是成见：理性传统中向来如此的，便是不可更易的，因而也是神圣权威的。尚且不论非欧几何之于欧几里得几何独断性地位的颠覆，单就欧几里得几何内部而言，其可能系统亦绝非单一。若欧几里得几何诸系统及其关系尚处遮蔽之中而未厘清，遑论各式非欧几何及其相应诸系统。

我们称刻板单一理解中的欧几里得几何系统为**单欧系统**（**mono-Euclidean system**）。伴随单欧系统的几何实在论倾向于将点、线、面、体这些几何对象视为几何实体，而将属于、介于、全等于这些几何关系视为几何实体的属性或几何实体之间的关系。譬如，点属于线，共线三点中的一点介于另外两点之间，两个三角形全等。根据单欧系统的几何实在论，整个欧几里得几何学是关于点、线、面、体这些几何实体的性质描述。

事实上，欧几里得几何的可能系统绝非单一，而是呈现多样化的局面。从文本呈现来看，欧几里得《几何原本》系统和希尔伯特《几何基础》系统时隔两千多年，其公理化程度不一，但不可否认，二者所着力刻画的，是同一个欧几里得几何。从公理构成来看，欧几里得平行公理不必参与几何公理集的构成，它有许多等价替代，譬如三角形内角和定理。结合其余几何公理，欧几里得平行

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公理与三角形内角和定理可相互推导。**基本几何关系**不必限于全等关系，还可以是共球关系，由此衍生全等几何系统与共球几何系统，二者可以相互转换。不独基本几何关系可变，便是基本几何对象亦可变。**基本几何对象**不必为点、线、面、体之全体，择其一即可，由此而形成不同的欧几里得几何系统，譬如点几何、线几何、面几何、体几何。欧几里得几何甚至不必限于图形域，而可进至数域，这是因为解析几何不必作为综合几何的数域解释附庸，数域亦可作为满足欧几里得几何的对象集。凡此种种，欧几里得几何系统呈现多样化局面。我们称这些欧几里得几何系统的集合为**多欧系统**（**multi-Euclidean systems**）。

多欧系统中的一个几何系统与另一个几何系统之间具有**相互可转换性**，这种相互可转换性可被界定为**理论等价性**。多欧系统之间的相互可转换性可源于欧几里得平行公理及其等价替代的置换，可源于基本几何对象的相互转换，可源于基本几何关系的相互转换，可源于图形域和数域的相互转换，亦可源于前述各类相互可转换性的复合。

伴随多欧系统的几何实在论不同于伴随单欧系统的几何实在论。当几何观念从单欧系统升级至多欧系统，几何实在论经历**第一次升华**。这种几何实在论的升华如此隐秘而微妙，以至于升华之后的几何实在论在某些场合被界定为几何反实在论。普特南正是因为看到点几何与体几何之间的相互可转换性，亦即某种较为宽松的理论等价性，便倾向于取消点、线或体的实体性地位，从而发展出**反实在论的几何学论证**。普特南的几何反实在论同**哈沃森论题**交相呼应，后者是说，实在论者持相对保守的理论等价性标准，而反实在论者则持相对自由的理论等价性标准。哈沃森认为点几何与体几何虽不满足定义性等价关系，但满足稍弱的莫里塔等价关系，后者容许定义域基数的差异。然而，诚如此前分析，多欧系统之间的相互可转换性不独源于基本几何对象之间的相互可转换性，亦有其他诸多来源，从而呈现出相互可转换性的多重性和理论等价性的谱系性。假如普特南注意到多欧系统所呈现出来的理论等价性谱系，他很可能会在择取何种理论等价性来发展其反实在论的几何学论证这一问题上产生犹豫。择取其中任意一种理论等价性而忽略其余，都有为几何反实在论立场而特设某种理论等价性标准之嫌。假如哈沃森注意到多欧系统所呈现出来的理论等价性谱系，他很可能不会满足于实在论与反实在论对立的传统二元叙事，而是伴随理论等价性谱系发展出几何实在论谱系。于是，哈沃森论题可作如下修正：**所持理论等价性概念愈保守而严苛，则其实在论倾向愈强**；**所持理论等价性概念愈自由而宽松，则其实在论倾向愈弱**。就多欧系统而言，各式系统之间的差异性是对系统层级同一性的背离，从而造就系统层级的丰富样态；各式系统之间的等价性则确保这样的背离被约束在几何层级统一性的框架之内，不至于损害欧几里得几何的同一性。伴随并确保多欧系统之间理论等价性的是欧几里得几何这一理论层级的实体。欧几里得几何学犹如一个多面体，可立于众多不同基底。每当立于不同基底，欧几里得几何学则表现为多欧系统中的一个系统。欧几里得几何学作为理论实体，较之于点或体作为几何对象实体，自然更加抽象。从单欧系统到多欧系统，几何实在论经历了第一次升华。

## 二、几何实在论的第二次升华：从内欧系统到跨欧系统

实现相互可转换性、建立理论等价性的几何系统不独限于欧几里得几何内部的多欧系统，亦有欧几里得几何系统和非欧几何系统构成的更大几何系统。我们将单欧系统与多欧系统统称为**内欧系统**（**intra-Euclidean systems**），而将欧几里得几何系统与非欧几何系统构成的集合称为**跨欧系统**（**trans-Euclidean systems**）。非欧几何作为数学史上的思想革命，对欧几里得几何的否定性和颠覆性意味极强。然而，一方面为论证非欧几何相对于欧几里得几何的相对相容性，另一方面为论证几何公理作为约定真理的地位，庞加莱以可译性概念沟通了欧几里得几何与非欧几何，认为欧几里得几何词项与非欧几何词项之间可以实现犹如德法词典那般的翻译。既然内欧系统与跨欧系统都具有相互可转换性，由此产生非欧几何哲学疑难。非欧几何哲学疑难的外延型表现为划界难题，即在理论等价性视域下，内欧系统与跨欧系统如何划界；非欧几何哲学疑难的内涵型表现为共存难题，即非欧性与可译性如何共存。

**变换不相容性**概念是解决非欧几何哲学疑难的关键。内欧系统之间的不相容性实际上是**词项不相容**。譬如，基于两个点对全等的欧几里得几何系统和基于五点共球的欧几里得几何系统之间的不相容性为基本词项的不相容性。跨欧系统之间的不相容性则深入至**公理不相容**。譬如，欧几里得平行公理、黎曼平行公理、罗巴切夫斯基平行公理是互不相容的。公理不相容系统的相对相容性必定蕴涵变换不相容性。对于跨欧系统之间的相互可转换性，其所涉及的词项变换必定包含不相容性。譬如，以欧几里得几何中的直线对应黎曼几何中的测地线。**这种不相容变换之不可规避，一方面以其变换保系统的相对相容性，另一方面以其变换之不相容性保系统的互不相容性**。对于内欧系统之间的相互可转换性，其所涉及的词项变换是相容的，可视为几何系统中的定理。譬如，两个点对全等和五点共球之间的相互转换性实则为内欧系统中的一条定理。

从内欧系统至跨欧系统，几何实在论经历了**第二次升华**。从内欧系统之间的相互可转换性至跨欧系统之间的相互可转换性，由于后者容许变换不相容性而前者不容许变换不相容性，故跨欧系统之间的理论等价性标准更加自由而宽松。根据修正版的哈沃森论题，其所伴随的实在论倾向必然愈弱。具体而言，从内欧系统至跨欧系统，平行公理内容的变化集中反映非欧几何对欧几里得几何的颠覆性。根据欧几里得平行公理，过直线外一点可作已知直线的平行线的数目为一；根据黎曼平行公理，此数目为零；根据罗巴切夫斯基平行公理，此数目大于二。于是，若在跨欧系统之中仍像在内欧系统之中那样执著于欧几里得直线的实体性地位，便会面临矛盾的境地，直线的平行线数目不可能同时为零、一或多。从内欧系统进入跨欧系统，几何实在论的升华典型地反映在**直线的实体性地位**。欧几里得直线不再具有原先那般的实体性地位。更加抽象的测地线概念取而代之。测地线的距离计算方式取决于空间曲率，而欧几里得空间曲率为零，黎曼空间曲率恒正，罗巴切夫斯基空间曲率恒负。欧几里得直线只不过是测地线概念在欧几里得空间这种特殊空间中的特殊表现。

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# The Dual Sublimation of Geometric Realism

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## Abstract

We term the Euclidean geometric system within a singular interpretation as the mono-Euclidean system. According to geometric realism within mono-Euclidean systems, the entirety of Euclidean geometry describes properties of geometric entities such as points, lines, planes, and solids. In reality, potential Euclidean geometric systems manifest not as singular but pluralistic formations. We designate the collection of diverse Euclidean systems as multi-Euclidean systems. These systems exhibit mutual transformability, which can be rigorously defined as theoretical equivalence.

The first sublimation of geometric realism occurs through the conceptual evolution from mono- to multi-Euclidean systems. The strength of realist commitments correlates with the flexibility of theoretical equivalence criteria: more conservative and stringent equivalence standards correspond to stronger realist tendencies, whereas more liberal and flexible standards align with weaker realist inclinations. Within multi-Euclidean systems, inter-systemic differences manifest as deviations from systemic identity while generating diverse modal configurations. Simultaneously, systemic equivalences constrain such deviations within the framework of geometric unity, thereby preserving the essential identity of Euclidean geometry. Notably, Euclidean geometry as a theoretical entity demonstrates higher abstraction compared to geometric object-entities like points or solids.

We categorize both mono- and multi-Euclidean systems collectively as intra-Euclidean systems, while defining trans-Euclidean systems as the superset containing both Euclidean and non-Euclidean geometric systems. The transition from intra- to trans-Euclidean systems marks the second sublimation of geometric realism. This progression involves shifting from intra-systemic transformability (which prohibits transformational incompatibility) to trans-systemic transformability (which permits such incompatibility). Consequently, the theoretical equivalence criteria for trans-Euclidean systems become more liberal, corresponding to significantly weaker realist tendencies. This manifests paradigmatically through the conceptual realization that Euclidean straight lines represent merely specific instantiations of geodesic concepts within the particularized framework of Euclidean space.

# 克利福德的空间观：与黎曼、亥姆霍兹的对话

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19 世纪 60 年代，高斯（Carl Friedrich Gauss，1777—1855)）、罗巴切夫斯基（Nikolai Lobachevsky，1792—1856)）、黎曼（Bernhard Riemann，1826—1866)）等数学家在非欧几何上的工作被重新发现，引来了欧洲大陆数学家和哲学家对几何基础以及空间观的重新讨论——空间是处处平直的吗？空间的几何究竟是什么？然而，当时的英国对非欧几何以及相对空间的态度相对保守，绝对空间和欧氏几何不论是在研究上还是在学校教育中，都是需要坚守的铁律。物理学家即便在具体研究中考虑了高维几何，仍相信绝对空间的基本假设[①]；学校的几何教学严格遵循欧几里得《原本》冗长的证明方式，直到 19 世纪末才被简化[②]。

克利福德（William Kingdon Clifford，1845–1879）是英国最早接受非欧几何的数学家之一，他积极地推进非欧几何知识在英国的传播，除了将黎曼的就职演讲翻译为英文，还讨论了非欧几何对空间观的变革以及椭圆空间中的运动等问题。“克利福德-克莱因问题”和“空间曲率由物质运动决定”的相对论观点让克利福德成为非欧几何接受史上的重要人物。在 19 世纪英国重视欧几里得几何传统的环境下，是什么原因使得克利福德勇敢接受非欧几何的新思想？现有研究分析时代背景和人物性格等因素，如以进化论诞生为代表的自然主义兴起[③]，克利福德反对宗教教条的个性和行为等，也论及其受到黎曼和亥姆霍兹（Hermann von Helmholtz，1821—1894）的影响[④]。那么，克利福德从这两位德国数学家得到了哪些具体的启示，他因此做出了什么创新？本文将分析和比较黎曼、亥姆霍兹、克利福德在几何基础方面的工作，结合时代背景，从哲学和数学讨论克利福德的空间观。

## 一、黎曼关于几何基础的假设

1854 年，黎曼在哥廷根大学宣读了题为《论奠定几何基础之假设》的就职演讲[⑤]，该演讲于 1868 年出版，得到了数学家的广泛关注。黎曼受到高斯和赫尔巴特的影响，将曲面的内蕴性质推广到高维，提出更一般的几何量及其度量。在演讲伊始，黎曼指出几何是构造空间的基本理念，空间的本质由几何公理给出，因此，要明确空间的概念，首先要理清几何公理之间的关系，而现有几何研究都预设了空间观和空间构造的基本概念，这也是为什么几何基础仍“处于黑暗之中”。黎曼的目的就是要使得这种模糊得到澄清，他的研究计划如下：先构建流形（多重延伸量）的概念，然后讨论从几何度量关系得到空间的基本事实，最后，将这些观点应用在对空间的理解中。

由于演讲面向的是非数学家，黎曼省略了公式的所有推导过程，这些结论在接下来的一个世纪中被数学家们证明并发展[⑥]；黎曼还提出均匀空间假设的理想性和物质分布决定空间度量的现实性，这些工作使得黎曼成为几何学史上的重要数学家之一。黎曼演讲的出版引来

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[①] Max Jammer. Concept of Space: the History of Theories of Space in Physics[M]. New York: Dover Publications, INC, 1993: 140.

[②] John L. Richards. Mathematical Visions: the Pursuit of Geometry in Victorian England[M]. London: Academic Press, INC, 1988: 198.

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学界对几何学基础的重新审视，亥姆霍兹在通信和论文中多次提到黎曼，从物理学的角度探讨了空间的几何基础。

## 二、亥姆霍兹的可自由移动刚体

1860 年代，几何学被认为是描述空间的科学。德国数学家、物理学家、生理学家亥姆霍兹从描述物体运动出发阐释几何公理，从不同于黎曼的视角发展了非欧几何的理论①。对视觉感知的实证研究促使亥姆霍兹重新考虑几何基础，1868 年 5 月和 6 月，亥姆霍兹先后发表了题为“论几何的事实基础”和“论奠定几何基础的事实”②的两篇文章，后者是他看到黎曼就职演讲的复印本后，与当时负责编纂高斯全集的谢林（Christian Julius Schering，1824—1897）通信确认文章内容与黎曼的演讲有所区别，并要求将其发表在黎曼文章所在的那一期上③][363]。贝尔特拉米（Eugenio Beltrami，1835—1900)）看到该文后，指出其中的错误“三维无限的空间一定是欧氏空间”，原因是这样的空间有可能是罗巴切夫斯基非欧几何对应的空间。亥姆霍兹得知后主动与贝尔特拉米通信，于 1869 年发布了一条注释对文章内容做了修正。④同年，英国《Acadamy》杂志主编写信邀请亥姆霍兹写一篇关于几何基础的文章，1870 年 2 月 12 日亥姆霍兹的英文文章“几何公理”⑤发表在该期刊上。其中，他提到了贝尔特拉米、黎曼等人的工作，并认为非欧几何的发展不仅仅关乎数学，而且与人类认知本质的最重要的问题紧密关联。

亥姆霍兹的目的是寻求几何公理来源的“反先验”的解释。前人在证明几何公理尤其是平行公设上的失败，源自于使用欧几里得综合方法，强调只有能狗构造出图形的证明才是合法，而人们在图形构造的过程中会不自觉地承认一个事实，即刚体在空间中能够不改变形状和大小地自由移动。对这一事实的承认预设了与空间的几何是欧氏几何，亥姆霍兹从三个方面阐述几何公理的实证基础。

亥姆霍兹强调力学原理与几何公理的结合，以此摆脱公理的先验性；采用分析的方法，从特殊到一般地探究常曲率空间的性质，即刚性几何体可以不变形地移动，因此可能存在三种空间，只有通过经验考察力学原理，才能确定我们所处空间的属性。亥姆霍兹于 1876 年和 1878 年在英国杂志《Mind》上再次发表了关于几何基础的思考⑥，进一步扩大非欧几何在英国的传播。

## 三、克利福德的空间假设

在 19 世纪的英国，欧氏几何被认为是无可指责的必然真理，因此非欧几何在英国的接受并不顺利，克利福德在阅读了黎曼和亥姆霍兹关于非欧几何的论著后，积极参与非欧几何的传播和研究。他不仅在哲学上讨论了空间科学的基本假设，还从数学上研究了椭圆空间中的运动问题。1870 年，克利福德在题为“关于物质的空间理论”的讲座中提到黎曼关于通过经验来确定我们所处空间的观点，并提出以下空间假设：

（1）小部分空间的本质事实上与一个平坦曲面上的小山类比，即就是说，几何的一般定律在其上不满足；(2) 这种弯曲或扭曲的性质是连续地以波动的形式从空间的一部分传递到另一部分；(3) 空间曲率的这种变化是真实发生在我们称作为物质运动（*motion of matter*）

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① Jesper Lützen. Interactions between Mechanics and Differential Geometry in the 19th Century[J]. Archive for History of Exact Sciences, 1995, 49(01): 1-72

② Hermann von Helmholtz．On the Facts underlying Geometry [C]//Helmholtz, Hermann von, Epistemological writings[M]. Translated by Malcolm F. Lowe, edited by Robert S. Cohen and Yehuda Elkana. Dordrecht: D. Reidel Publishing Company, 1977: 39-71.

③ Cahan, David. Helmholtz: a life in science[M]. Chicago: The University of Chicago Press, 2018: 360-369.

④ 李东升.反先验论与亥姆霍兹非欧几何公理化思想的关系[J]. 科学技术哲学研究, 2010, 27(05): 21-25.

⑤ Hermann von Helmholtz. The Axiom of Geometry[J]. The Academy, 1870: 128-131.

⑥ H. Helmholtz. The Origin and Meaning of Geometrical Axioms[J]. Mind, 1876, 01(03): 301-321.

的现象中，不论是有重量物质还是没有重量的物质；（4）取决于连续定律，在物理世界中只有这种变化会发生。[①]

其中第三条假设被认为是对相对论的预测，但克利福德的讲座全文并没有被保存下来。1873 年 3 月，克利福德在皇家学会做了题为《纯粹科学的哲学》[②]的讲座，从对几何知识本质的理解和几何公设阐述了他的空间观。在克利福德看来，几何学知识建立在经验之上，是描述空间的物理科学，这不同于康德的先验论。克利福德显然知道罗巴切夫斯基的工作，他将黎曼的就职演讲译为英文，也了解亥姆霍兹对几何学基础的论述。在这些工作的基础上，根据对知识的定义，克利福德重新讨论了几何学的基础。

## 四、与两位数学家的对话

黎曼的演讲一经出版就引来了亥姆霍兹和克利福德的注意，同时亥姆霍兹在学术上的影响力促使他关于几何公理的观点进一步传播，因此，克利福德自然受到这两位数学家的影响。黎曼从内蕴几何的角度提出了决定空间性质的关键数学本质，即由线元表达式确定的空间曲率，现实空间是上述几何假设的特例，其线元表达式为二次微分平方根，空间曲率为常数。亥姆霍兹从力学和运动的角度刻画了三种常曲率空间中刚体移动的特点，并通过几何模型以及二维曲面生物的类比解释空间的本质。尽管黎曼和亥姆霍兹探究空间本质的进路不同，但他们都认为经验是检验空间本质的唯一途径。克利福德重新梳理了前人的研究，并用更富有逻辑且通俗的语言重新论述了空间的基本假设。按照克利福德提出的四条几何公设，可将三人的空间观点依次列举对比如下：

| 数学家 / 空间几何基础 | 黎曼 | 亥姆霍兹 | 克利福德 |
|---|---|---|---|
| 连续性 | 连续流形 | 点的运动可以由坐标的连续变化确定 | 空间中任何两部分的分界线是同一的 |
| 平坦性 | 线元表达式为二次微分平方根 | | 在三维空间中，任意点的方向角和立体角处处相同 |
| 叠加性 | 常数曲率空间 | 刚体的完全自由移动 | 空间具有常数曲率 |
| 相似性 | | 从日常观察中得到欧氏空间中几何图形的相似性 | 欧氏空间具有相似性；罗巴切夫斯基的几何不具备相似性 |
| 空间观 | 1.不同的空间对应不同的几何<br>2.无限小空间的曲率不一定是常数，取决于物质的分布 | 1.日常生活的空间是无限且平坦的<br>2.如果考虑球面空间或伪球面空间，那么所有的力学系统要完全改变 | 1.有限宇宙即正常曲率空间的可能性<br>2.空间的曲率由物质的运动决定 |

① William Kingdon Clifford. On the Space-Theory of Matter[C]// Robert Tucker(edited). Mathematical Papers by William Kingdon Clifford. London: Macmillan, 1882: 21-22.

② William Kingdon Clifford. The Philosophy of the Pure Sciences[C]//Leslie Stephen and Sir Frederick Pollock(edited). Lectures and Essays by the Late William Kingdon Clifford(Volume I). London: Macmillan, 2011: 254-340.

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[2]John L. Richards. Mathematical Visions: the Pursuit of Geometry in Victorian England[M]. London: Academic Press, INC, 1988: 198.

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[16]William Kingdon Clifford. The Philosophy of the Pure Sciences[C]//Leslie Stephen and Sir Frederick Pollock(edited). Lectures and Essays by the Late William Kingdon Clifford(Volume I). London: Macmillan, 2011: 254-340.

# On Clifford's Concept of Space: Dialogue with Riemann and Helmholtz

**Abstract:** In the 1870s William Kingdon Clifford became the first mathematician who accepted and propagated the idea of Non-Euclidean Geometry in Britain. He translated Riemann's famous lecture into English and proposed his own opinions on foundation of geometry and curved space. Klein was inspired by Clifford's 1873 lecture and his hypothesis that changing curvature of space manifests itself as the motion of matter, anticipating later geometrical interpretations of matter. Klein's advocacy made Clifford an important figure in the reception of Non-Euclidean Geometry. Historians have attributed Clifford's courage in accepting the new geometry in the context of the British conservative attitude towards geometry and space in the 19th century to his radical personality or the rise of naturalism, and also mentioned that Clifford was influenced by Riemann and Helmholtz. It is still necessary to dig more deeply to identify in detail the influences that Clifford got from the two mathematicians and what innovation he therefore made in the theory of space and geometry.

In this talk, I will compare the geometric ideas and work of Riemann, Helmholtz and Clifford, and then investigate the sources of Clifford's thoughts on the concept of space.

# “统一几何”的空间哲学
## ——以外尔的空间观为例

刘 杰　　王月儿

**摘　要：**空间哲学自康德以来一直是几何学哲学的基本问题之一，尤其在克莱因纲领所倡导的群统一几何的背景下，非欧几何与欧几里得几何的统一对空间概念提出了新的哲学挑战。外尔通过区分空间的本质与结构，综合了现代几何哲学和相对论的空间哲学。基于对外尔的构造主义与卡西尔整体主义观点的比较，可以发现统一几何在先验主义的立场上将连续性视为空间的本质。现代几何的空间哲学所面对的关键问题，在于如何解决空间的先天连续性和连续体的构造性之间的矛盾，这一矛盾最显著的表现就是连续性悖论。由此，阿基米德公理对几何认识论的影响已经超越平行公理，对非阿基米德几何的研究将成为现代几何学的关键。

**关键词：**空间哲学;外尔;连续性;纯粹直观;统一几何

## 1　引言

在欧几里得几何学与非欧几何学通过群论得到一定程度上的统一后，几何学的空间哲学受到了由相对论所引发的新空间观念的冲击，为它寻找一个恰当的哲学解释成为数学哲学最迫切的问题之一。作为深度参与相对论讨论的哲学家之一，外尔（H. Weyl）通过区分空间的本质与结构，试图在统一几何背景下协调数学结构与空间直观，进而提出一种同时具有先验性与构造性的空间概念。近十年间，外尔思想中的直觉主义倾向、其与构造主义间的紧密联系以及其对胡塞尔现象学的偏好已经得到了广泛的讨论。继此，伯纳德（J. Bernard）与洛博（C. Lobo）等人从数学和历史的视角开启了对外尔思想中空间问题的系统讨论，并着重强调了其对空间的本质和结构所作的双重描述。

然而，现有研究多集中于几何的公理化，对统一几何空间哲学基础缺乏系统的讨论。本文试图借助外尔的空间观念理解统一几何纲领的哲学本质，指出统一几何先天预设了空间的连续本性，并且为了回避连续性悖论，空间的本质和结构必须被分割。

本文首先以外尔的度量空间和本质空间为例，讨论现代几何学中的先验主义倾向。其次，通过外尔的无穷小邻域概念引出空间的连续性悖论，厘清连续性和无穷小概念的历史渊源。再次，比较空间的构造主义和整体主义，论证对空间先天连续性的信念如何导致空间的本质与结构的分离。最后，本文将指出，阿基米德公理的重要性已超越平行公理，对连续性的不同理解将会形成不同的空间观念，从而为现代几何学与空间哲学提供新的哲学

视角。

## 2 数学结构与纯粹直观在现代几何学中的并行

对康德而言，空间是纯粹直观的非经验形式，几何学则由先天综合判断组成。非欧几何的出现冲击了欧几里得几何在真理上的唯一性，但没有构成对康德数学哲学的彻底否定。在群统一几何的数学背景下，空间的经验度量结构与空间的先天直观本质并行不悖。外尔综合了亥姆霍兹（H. von Helmholtz）的经验论与对纯粹直观的理解，发展了克莱因（F. Klein）的空间观念在度量和纯粹直观上的两个起源[1]29，并把澄清空间的哲学问题视为自己工作的重心。伯纳德和洛博称外尔分别考虑了空间的结构层面和非结构层面[2]vii，即空间的数学结构和空间的直观本质，前者虽然包含射影几何，但外尔经常用度量结构或度量空间来统称它。

### 2.1 几何学在度量结构上的统一

在非欧几何的刺激下，黎曼断言“空间本身不过是一个没有任何形式的三维流形；只有通过充实它的物质内容的出现和确定它的度量关系，它才能获得一种确定的形式。”[3]78以此为核心的十九世纪度量认识论就是要用流形这一“带有良定义结构的概念”[4]30取代无结构的直观，并在有限的距离上使度量依赖于经验事实。

但是，概念与直观在空间哲学中并不像黎曼所陈述的那样泾渭分明。马堡学派称康德在一定程度上继承了莱布尼茨将空间视为共存事物的秩序的观点[5]32，因而当度量认识论要在空间观念上用外延构造取代直观时，它很难清楚地说明为其所主张的概念究竟属于何种哲学。黎曼流形与纯粹直观都受制于主体的感觉经验，区别在于是否它们将经验与直观解读在一起。唯一能够确定的是二者的先验主义立场，因为当莱布尼茨使用后继这一词语时，他认为它的含义是元逻辑的，这在康德的术语中意味着后继与先天的感性直观有关。

度量认识论的发展涉及了康德几何哲学的两个论点，即将空间视为非经验的纯粹直观和将几何学视为先天综合判断。虽然非欧几何的发现动摇了几何公理与实际知觉的关联，直观与逻辑不再能够必然推导出欧几里得几何，但是这一发现没有构成对康德哲学的否定。亥姆霍兹试图恢复康德直观在空间哲学中的作用，他称“康德关于空间学说的两个部分，即（i）空间是直观的纯粹的形式，（ii）……欧几里得几何学是先验成立的，并非很紧密地相互联系，（ii）并不是由（i）来的。”[6]167他承认欧几里得几何学不再是先天综合判断，但这一度量问题应该交由经验来研究，而不能触及空间的形而上学[7]357。同样，庞加莱认为既然不存在一种对经验空间必然为真的度量结构，那么应该反对康德使欧几里得几何必然为真的观点。但这一论述不是对“康德将空间视为几何学中全部先天综合命题的来源，它是经验真的，但却是先天理念”[8]129的反驳，毋宁说庞加莱一直在用现代术语重述这个主张。他想要区分感性直观空间和度量空间，后者由交换群产生，群作用于流形的方式决定了空间的性质[9]19。

几何学在经验与先验间的界限是现代几何学所处理的主要话题。帕斯（M. de Paz）称亥姆霍兹和庞加莱赋予了经验双重角色，一方面，经验使几何学适用于物理世界，另一方面，为了使几何学成为一门精确科学，经验元素需被抽象符号取代[10]5。比亚焦利（F.

Biagioli）认为庞加莱的约定论说明了几何公理既不是经验命题，也不是先天综合判断[11]191，并相信这一观点已经被新康德主义者所接受。伊万诺娃（M. Ivanova）[12]5 和福利纳（J. Folina）[13]27 指出，当庞加莱强调数学直观，并反对将几何命题视为先天分析判断时，他并不像自己所声称的那样反对康德思想中的先天综合部分。

外尔发展了康德的空间哲学，他用度量划定了先验和后验的边界，以三元的空间-度量-物质图示取代了传统的空间-物质的二元图示[2]xviii。外尔对先天综合判断的理解被伯纳德和洛博总结为“它们独立于任何经验知识，并对任何经验度量都是先天被假定的；并且，它们是经验可测的。”[2]xii 外尔将度量结构视为先天综合的，而仿射联络则是他空间观念的先天综合中最重要的部分[14]222。他并未详细阐明过如何区分先天综合判断和逻辑分析命题，而是在本体论的意义上接受了布劳威尔关于数学的真是先天综合的这一主张。

外尔也继承并发展了黎曼几何。在考虑他自己的仿射联络流形时，他陈述了度量空间的两条公理，即空间的性质对度量关系没有限制，并且仿射关系由度量关系唯一决定[3]112。这两条公理分离了空间的度量结构及其本质，即虽然度量受到经验的限制，但度量的本性则是先验的，这被福利纳称为将度量视为第三范畴的约定论立场[15]168。只有依赖于哲学的先验分析，才能阐明建立在欧几里得-毕达哥拉斯性上的度量结构的先天本质[2]xii。

**2.2 纯粹直观意义上的本质空间**

在外尔看来，空间问题在根本上是建立起表象的形式与表象的物质之间的统一关系。他将这一问题从黎曼到爱因斯坦的发展描述为空间表象的先天成分的逐步推广，并认为是这种先天成分使测量得以可能[7]353。由此，外尔对空间的先验与后验界限的划分就是对空间的先天直观形式和度量结构之间的区分。

外尔的先验主义倾向建立在 19 世纪几何学的一般立场之上。虽然黎曼声称要拒绝康德的空间观念，但是他承认空间的概念是“某种给定了的东西”[16]243。随后，克莱因在先天层面假定了空间概念[17]192，并将这个概念的起源归结于康德式的直观。“空间观念的起源有两个，一个是对空间观念的直觉，可以通过量度而直接意识到。另一个就完全不同了，它是主观的理想化的直觉。”[1]29 总体而言，19 世纪几何学的空间观念没有超出过德国观念论的范围。

对空间的本质（Wesen）一词的较早的使用出现在赫尔巴特（J. F. Herbart）的文本中。班克斯（E. C. Banks）[18]认为赫尔巴特从物理空间的性质中分离出空间表象，并融合了莱布尼茨和康德的空间观。赫尔巴特向康德式的空间表象理论中添加了记忆和想象，并削弱了莱布尼茨式的可理知的空间结构中的空间要素。空间的本质被赫尔巴特视为莱布尼茨式的单子，不可扩展并且是简单的。这一观点影响了黎曼，并传递到外尔那里。余下的问题是如何获得对空间本质的认识，外尔为此选择了胡塞尔的本质直观，而不是莱布尼茨的理性概念。

根据帕森斯（C. Parsons）[19]和丘德诺夫（E. Chudnoff）[20]对数学直观的历史所作的梳理，存在两种不同的数学直观：对数学对象的直观和对数学命题的直观。外尔对直观的使用经常在这两种语义间转换，他偶尔在庞加莱的语境上讨论“迭代直观”[14]174，但大多数时候站在现象学的语境里考虑“直观连续体”[6]366。其中，无穷迭代直观建立在布劳威尔式的二一性的时间直观之上，是关于命题的直观；而作为对数学对象的直观，直观连续体是对这个时间序列的整体直观。外尔的整个数学哲学在后期转向了希尔伯特的形式主义，

但他没有抛弃布劳威尔的直觉主义。他意图在这二者之间寻找一个中间立场，令直觉主义适用于有穷与可数无穷领域，形式符号构造适用于非可数无穷与理论物理领域[21]175。

外尔将空间概念划分为数学结构和直观本质，但却几乎没有讨论过该区分的必要性，这一区分与他在希尔伯特和布劳威尔之间的摇摆和他对连续体问题的持久关注有关。几何学无法回答空间的无穷小部分如何被组合为空间整体，为此需要借助对空间的本质的规定，因而空间的结构层与非结构层被分离为两个不同的论域。这一分离正是连续性悖论在近代几何学的空间哲学中的表现。

## 3 空间中的连续性悖论

不仅欧几里得几何建立在毕达哥拉斯定理之上，而且黎曼几何的关键就是获得以毕达哥拉斯定理为基础的局部的空间距离关系。外尔意识到在这里发生了一种距离上的转变，超距几何现在变成了近域几何，特别地，以他的术语而言就是无穷小几何。“（我）发现黎曼几何在实现纯无穷小几何理想方面只走了一半的路。我们仍然要消除几何学中的最后一个元素——距离。” [3]81 外尔所使用的方法是用仿射联络流形建立起无穷小几何，而在这个细致的数学方法之内起决定性作用的是他对连续、离散和无穷小之间关系的考察。赛罗卡（N. Sieroka）认为[22]100，外尔用无穷小邻域充当连续和离散之间的调解概念，这一概念受到了费希特哲学的影响。

综合考虑埃德加（S. Edgar）对无穷小和连续性的分析、赛罗卡的历史视角以及朔尔茨（E. Scholz）对外尔工作的研究，无穷小邻域不仅是先验的，同时也必须是连续的。这使空间哲学回避了自芝诺以来的连续性悖论，并说明由统一几何所预设的先验的空间的本质中必须包含连续性。

### 3.1 连续性之谜在近现代的发展

费弗曼（S. Feferman）[23]从概念结构主义的视角梳理了连续体的概念，他指出连续体在几何、算术与集合论中具有完全不同的含义，并暗示连续体的发展从几何学开始，经历了在算术中的概念化，而后在希尔伯特这里回到了几何学及其哲学。下文将谈论三种典型的连续体概念，并指出它们如何分别处理了连续体悖论。

（1）几何连续体：欧几里得传统与先天连续性

第一种连续体是传统的欧几里得式连续体，它被直线所例证，而不需要依赖于算术。这意味着对欧几里得式连续体而言，从直线中去掉一个点是无意义的说法，线并非由预先存在的点所组成。它将连续体和无穷小视为先天所予的实体，离散和连续之间不能互相推出。

（2）算术与集合论连续体：康托与戴德金的构造思路

第二种连续体是康托连续体和戴德金连续体，它们是混合了几何、算术与集合论思想的概念，对连续体的构造与离散点有关。康托和戴德金用有穷数逐步构造出无穷数，区别是康托连续体建立在有理数域上，他通过通过集合论方法引入实无穷，强调离散元素逐步逼近连续。而戴德金连续体建立在实数域上，他以连续性结构为前提，通过分割法定义无穷小量。

（3）希尔伯特连续体：阿基米德公理与完备性

第三种连续体是希尔伯特式连续体，它由度量公理或阿基米德公理（V1）和直线完备公理（V2）所共同组成，并试图调和几种不同的连续性概念。“连续的要求，在本质上，分成两个不同的部分：阿基米德公理和完备公理；前者的作用是替连续的要求作准备，后者为完成整个的公理系统作基础”[24] 24。费弗曼认为希尔伯特的完备公理（V2）等价于戴德金连续，但他没有对阿基米德公理（V1）作过多的评论，而是称希尔伯特的概念不是基础的，是几何学和集合论概念的混合。虽然希尔伯特赞扬过康托的集合论，但他自己却在阿基米德公理（V1）中否认了实无穷在几何连续体中的存在。通过完备公理（V2），希尔伯特约定了连续相对于无穷小的优先性。

克莱因和费弗曼对希尔伯特连续体的解读十分相似，虽然他曾认为希尔伯特连续体的要点是空间中的所有点都能够被还原到实数域上去[25]222，但当克莱因将希尔伯特与康托联系在一起时，他想强调的不是希尔伯特几何学中的算术性，而是要指出这种算术连续体的本质就是几何连续体。克莱因的主要观点是几何连续体先于算术连续体，他把戴德金分割看作由直线而来的原则，并认为康托公理“从直觉上看是不言而喻的”[1]28。

（4）连续体与无穷概念：柯亨的三种无穷分类

清楚地是，不同的连续体应与不同的无穷概念相对应，其中一并出现了连续性悖论和无穷的定义问题。对于前者，连续与无穷小的优先性是关键；对于后者，无穷和有穷的优先性是关键。从柯亨（H. Cohen）对历史的梳理出发，埃德加强调了三种不同的无穷概念[26]：任意小的有限小量、伽利略式无穷和莱布尼茨的不竭（inexhaustible）无穷。

任意小的有限小量促成了对传统微积分的极限概念的理解，曲线下的众多矩形的面积之和逼近曲线下的面积。伽利略式无穷则不同于极限，它意味着有穷量必须通过无穷量而得到定义，并且无穷在概念上应该独立于有穷。不竭无穷意味着有穷数永远不会被耗尽（exhaust），因而不存在最大和最小的有穷数。与伽利略式无穷相反，不竭无穷通过有穷定义无穷。

埃德加认为，柯亨通过充足理由律选择了不竭无穷，并通过连续和无穷优先原则默许了伽利略式无穷。他基本实现了对柯亨的无穷小方法的辩护，但其对“连续和无穷优先原则”的使用则比较含混。正如埃德加意识到的那样，莱布尼茨的连续律与无穷小概念在原则上蕴含了悖论，因而对无穷的定义问题很可能隶属于连续性悖论。康托连续体和戴德金分割作为构造的连续体，本质上是用有穷构造出无穷，它在莱布尼茨的意义与不竭无穷对应。欧几里得和希尔伯特的几何连续体不仅支持连续先于无穷小，而且能够与无穷先于有穷的观点相一致。

近现代对连续性的诸观点要么通过有穷构造无穷，要么通过无穷构造有穷，并都认为连续性蕴含无穷可分性。最重要的一点是，必须回避连续性和无穷小之间的悖论，即连续体决不能由点构成。为了解决这个悖论，外尔选择了无穷小邻域这个介于离散和连续之间的概念。

**3.2 外尔的无穷小邻域**

外尔所接受的空间连续体建立在戴德金公理（V2）之上，“这种逻辑上的完备性反映了空间中的点之间的连续性”[6]50。但他对阿基米德公理（V1）的态度是微妙的，他从来不曾支持过实无穷小的存在，但也没有明确反对过它的存在。

外尔评述了三种连续体，每一种都通过对无穷的讨论而得到说明。他所说的第一种连续体是作为任意小的有限小，外尔认为它仅仅是一种思辨。他所说的第二种连续体采用了伽利略式无穷，这是一种潜无穷，并且可以“按极限过程来解释”[6]56。当他引用欧多克索斯的话来描述这种无穷时，他也将其理解为莱布尼茨式的不竭无穷，即既没有无穷大，也没有无穷小。他所说的第三种连续体是集合论的连续体，“它是以实无穷为基础的”[6]58。外尔没有说明他具体站在何种立场上，在外尔后期的文章中，他也仍然认为“连续体这一构造性的概念，还未完全澄清并确定下来”[27]185。克罗西拉（L. Crosilla）和林内博（Ø. Linnebo）认为外尔支持了不竭无穷的概念，不过他们同时也指出，外尔的态度是没有证据证明实无穷的存在，而不是有证据反对它的存在[28]411。如果克罗西拉和林内博对外尔的解读是准确的，那么外尔显然将连续体理解为几何连续体。

不管在黎曼还是在外尔那里，无穷小邻域中不含任何经验概念[27]38，并且无穷小邻域不是点，因为外尔要令毕达哥拉斯度量在无穷小邻域中成立。“大自然的真正的规律性表现为近域作用定律（laws of nearby action），即这些定律都只把在时空点的紧邻处的物理量的值联系起来……我们只能期望在无穷小中遇到基本的而且一致的规律”[6]109。这意味着无穷小邻域具备毕达哥拉斯性和连续性。赛罗卡指出外尔与希尔伯特的关联，“所有的连续性概念都可以被还原为邻域的概念”[22]104。外尔在费希特的意义上类比了无穷小邻域和主体间性，由此使邻域概念成为离散和连续之间的桥梁。

总之，外尔仅仅明确支持了如下两个观点：首先，连续体必然无穷可分；其次，不管无穷小量是实在的还是潜在的，都可以通过极限概念得到理解。外尔采取了先验主义，将无穷小邻域当作“理想元素”引入几何学。他将连续体与无穷小邻域均视为先天所予，但是连续性悖论没有因此得到解决，横亘在连续和无穷之间的仍然是二者的关系问题，即如何从部分得到整体，或如何从整体得到部分。虽然外尔早期曾经认为不能否认“构成空间基础的实在必须形成一个离散的流形”[3]78，但他从未放弃过将连续性建立为空间的本质属性。

## 4　连续性视角下统一几何的空间哲学

对空间的先天连续性的信念和连续体在物理几何中并不存在的事实构成了现代几何的空间哲学的难题，由此延伸出的是体现在空间的无穷小部分和整体中的连续性悖论。

克莱因的埃尔朗根纲领是要将包括非欧几何学在内的几何学建立在射影几何的基础之上，而射影几何应通过射影变换群来理解。现代几何学在数学中的统一被视为围绕着平行公理展开的讨论，而它在空间哲学上的困难是应该如何理解连续性即阿基米德公理。这一点要从连续体和无穷小邻域之间的关系来考虑，它们之间的可能关系有两种，其一是令无穷小邻域彼此粘合为空间，它的核心是构造主义；其二是视无穷小邻域为空间的分割，它的核心是整体主义。

### 4.1 外尔的构造空间

外尔选择了构造主义的方法，虽然他在有穷范围内站在希尔伯特的立场上，但对于无穷对象的认识论，外尔则诉诸布劳威尔的直观。当外尔认为“连续统可以归入（布劳威尔

的）‘外延性整体’的概念下”[6]66 时，他不仅将连续体视为一个先验的概念，而且相信连续体的先天性是独立于逻辑并被直观给予的。

特别地是，外尔并不直接将空间整体视为连续的，而是将无穷小邻域视为满足连续性的先验的最小的单位。无穷小邻域向空间整体的构造如何可能，以及这个构造如何传递连续性成了外尔的空间哲学中最重要的问题之一，伯纳德和洛博称其为外尔的粘合问题。但是当他们在数学角度考虑无穷小几何的粘合时，他们没有注意到该问题的实质是连续性概念在空间的部分和整体中的分离。构造主义的基本思想能够被抽象为这样一条陈述，即对空间而言，无穷多个无穷小邻域之和是无穷大量，并且这个无穷大量应是完备的。该主张面临一定的困难。首先，虽然连续性蕴含无穷可分性，但无穷可分性并不必然蕴含连续性，例如狄利克雷函数处处无穷可分，但它在非零点处处不连续。因此，连续的无穷小邻域之和并不必然能够组成连续的空间整体。其次，由无穷小邻域的空间的性质是什么，以及为什么能够称其为空间，这一点在数学上不是清楚明白的。

这两个问题的本质仍然是对空间何以连续的追问，外尔的解决方式是将空间视为双重的连续体。一方面，它在直观本质的意义上是几何连续体；另一方面，在数学结构中，“为了对连续统进行数学处理”[6]113，外尔从无穷小邻域开始，构造出了空间的一个算术模式，它的连续性由构造所保证。当无穷多个无穷小邻域彼此粘合时，外尔得到了一张仿射联络之网，这张网上的每一个邻域以一种不可变易的方式保持其自身。空间整体仍然是连续的，但在这里，连续性是一个构造性质，即无论如何分割空间，必然切中一个无穷小邻域。他的方法遵循了自戴德金以来的算术传统，适合于克莱因的群论方法，并且物质可以自然地被视为落入一个相应的邻域之中。

然而，虽然外尔的所有讨论都是基于空间连续性的假定，在他的方法中连续性却最终成为了空间的冗余属性。外尔需要无穷小邻域来保证微分几何的成立，尤其是要在这个邻域中保持空间的黎曼-毕达哥拉斯性，以便于使度量几何被统摄在射影几何之下。但是预设邻域的连续性并不意味着预设空间的连续性，尤其是空间的整体结构与无穷小邻域并不具备相同的几何属性，非欧几何只在无穷小部分中满足毕达哥拉斯度量，在整体上，黎曼几何不与欧几里得几何同构。抛弃无穷小邻域的连续性只是一个哲学上的抉择，将其视作离散点也仍然能够满足外尔的数学需要。

在认识方法上，构造主义的空间观念的特点是建立在直观之上，它的连续性的思想尤其依赖于时间直观。对康德空间观念而言，外尔不仅将空间的直观本质视为纯粹直观形式，而且通过度量结构为几何学的先天综合提供了可能。他在空间的本质中融合了空间直观与时间直观，因而为黎曼-爱因斯坦意义上的四维时空流形提供了可能的哲学解释。一方面，他需要纯粹直观来确定一个被称为空间的形式，以便向其中添加几何与物理的内容；另一方面，他对时间直观的依赖无法被转换为概念，尤其是他需要直观来解释空间的连续性。与此相反，卡西尔（E. Cassirer）希望实现一种以概念取代直观的新康德主义，数学对象必须在整体中加以把握。

**4.2 卡西尔的整体主义**

卡西尔和外尔都在数学上受到了亥姆霍兹、戴德金和克莱因等人的影响，但他思想中的先验主义成分要比外尔更多。当外尔将连续性视为由先天直观所给出的概念时，卡西尔则更明确地将其理解为“它（几何空间）的连续性和无穷性，都建立在一个类似的基础

上……依赖于我们在这些（空间）感觉中假定的完整理想”[29]114。他抛弃了康德的感性直观和纯粹直观在几何哲学中所起的作用，转而将时空建立在莱布尼茨式的抽象概念之上。

有别于外尔的局部连续性，卡西尔将连续空间视为一个统一的整体。卡西尔很少讨论空间的数学细节，他对空间的理解不是从无穷小邻域出发的，他对空间中最小单位的描述从来仅仅是“点”。唯一被需要的是单子的概念，它保证了空间作为一个整体的连续性。不过，鉴于他曾称“时空连续域内……每个点都在一定程度上参照自身及与自身无限接近的点，而不是参照外面的固定不变的刚性参考系”[29]425，可以看出卡西尔和外尔都接受了莱布尼茨的近域作用定律。

卡西尔借助几何结构来规定空间，而不是描述在空间中的结构。空间在这里已经沦为背景，他所依赖的是函数这个一般的概念。他的主张和希尔伯特的公理化纲领非常相似，度量空间被视为由一组公理集或一群函数组成的聚集，绝对空间则是一个由纯粹直观确保的理想元素。比亚焦利[5]39将卡西尔的思想解读为对康德哲学的深入发展，并认为借由函数，卡西尔将众多数学概念统摄在数学结构之下。

整体主义[1]与结构主义的立场十分相似，它们都把目光聚焦在抽象的数学结构之上，并且都强调了结构对象的关系性质。卡西尔仅仅否认了直观知识的存在，而没有否认对象能够通过直观而被感知，抽象掉直观后遗留下来的是莱布尼茨式的概念。结构主义同样认为不可能具有对数学对象的直观知识。区别在于，在卡西尔的整体主义中，时空结构首先是先验的整体，其次才被描述为函数结构。而结构主义并不非常关心空间的本体论，数学结构是其关心的首要对象。

整体主义与构造主义代表了埃尔朗根纲领的两个面向，前者从连续整体得到局部无穷小，后者从局部连续性得到整体连续性。他们都将连续性视为空间的本质属性，外尔更重视如何在度量空间中构造出算术连续体，而卡西尔则更重视如何在哲学上解读几何连续体。但是，他们都没有成功地建立起连续性在空间结构和空间本质间的统一解释，整个现代几何学空间哲学的核心问题不是别的，正是对连续性悖论的解决。从黎曼开始形成、到外尔这里变得清晰的对空间的本质与结构的划分就是该悖论的直接影响。

## 5 总结

空间的先天连续性既是统一几何的前提，也是其理论困难的来源，对连续性的理解一并影响了对空间概念的理解。虽然连续性悖论的存在迫使空间被区分为结构的与本质的这两个层面，但是建立在无穷小邻域上的构造主义和依赖于函数概念的整体主义都尝试对统一几何做出合适的哲学解释。

这一结论意味着阿基米德公理在空间哲学中的重要性超过了平行公理，几何学的核心

[1] 整体主义被埃斯菲尔德定义为：“An S is holistic if and only if the following condition is satisfied by all the things which are its constituents: with respect to the instantiation of some of the properties that belong to such a family of properties, a thing is ontologically dependent in a generic way on there actually being other things together with which it is arranged in such a way that there is an S.”（M. Esfeld, 2001, 16）

仍然是解决连续性悖论，而不是回避或绕过它。就连续体概念而言，虽然克莱因和费弗曼都认为几何连续体比算术连续体更基本，但困难在于如何为它先于或独立于算术连续体提供一个逻辑上的证明，集合论连续体与它们的关联尚不明确。在现代几何学哲学中，结构主义、构造主义与新康德主义的立场是否存在根本性的差异也有待研究，它们不仅都需要约定空间的数学结构，而且在不同程度上要求了空间的先验性，区别仅仅是如何获得关于空间的知识。

总之，在成为物理学哲学问题之前，空间概念不仅是数学的，而且首先是哲学的研究对象。只有仔细审查空间概念，才能意识到对它的理解受到对连续性理解的影响，并且意识到现代几何的空间概念与代数思想的融合。

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# Philosophy of Space in “the Unification of Geometry”
## ——Taking Weyl’s view on space as an example

Jie Liu, Yueer Wang

**Abstract**

Philosophy of space is one of the fundamental issues in philosophy of geometry. With the unification of geometry as the background, Weyl distinguished the essence and structure of space, and his perspective can be regarded as a synthesis between modern geometric philosophy and philosophy of space in relativity. By comparing Weyl’s constructivism with Cassirer’s holism, it can be found that the unification of geometry, from a transcendentalist standpoint, regards continuity as the essence of space. The difficulty faced by the philosophy of space in modern geometry is to resolve the contradiction between the a priori continuity of space and the constructiveness of the continuum, and the most prominent manifestation of this contradiction is the paradox of continuity. Thus, the influence of the Archimedean axiom on geometric epistemology has surpassed that of the parallel axiom, and the study of non-Archimedean geometry will become the key to modern geometry.

# 理念追求与经验根基之纠葛
## ——重审古希腊数学之知识形态

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**摘要**

谈及希腊数学，传统数学史叙事为希腊数学标志着现代意义的、抽象的纯数学之诞生。这一观点在学界广为流传且具有其合理性：首先，希腊数学坚持**演绎方法**，开创了数学命题证明的先河，出现了诸如阿波罗尼《圆锥曲线》等演绎式的数学论著；其次，毕达哥拉斯学派与柏拉图学派对**数学实体抽象本质**之强调，为数学研究确立了超越经验世界的本体论基础。这种叙事逻辑也深刻影响了后世对希腊数学家工作的历史定位，形成了以现代数学符号体系重新阐释希腊数学文献的编史学纲领。典型例证为始于 19 世纪中叶的对希腊数学的**"几何代数"**（geometric algebra）式的解读模式，即认为希腊数学中的几何与算术都是变相的或隐蔽的现代代数，两者在本质上并无差异，只是使用的语言体系不同。如《几何原本》权威译者托马斯·希思爵士（Sir Thomas Heath）在评价第二卷时指出欧几里得在此给出了许多现代代数公式的几何证明。

将古希腊数学简化为现代抽象数学之雏形的线性进步史观，本质上折射出深层的阐释困境。此叙事范式通过预设古今数学的同一性，不仅在知识论层面实施着强制性的范式移植，更在史学维度上消解了希腊数学特有的认知结构与问题域。当我们跳出现代数学框架，将这种看似自明的宏大叙事置于思想史的解剖台上，一系列根本性追问便随之浮现：古希腊数之原初知识形态究竟呈现何种特质？mathematics 与其古希腊语词源 μάθημα 之间是否存在不可通约的语义裂隙？更进一步，若古希腊数学与现代抽象数学存在差异鸿沟，导致这种不同的关键成因何在？其背后究竟潜藏着怎样的本体论根基与认识论取向？

探究古希腊数学的原初形态，需从"数"(arithmos)与几何学的双重维度切入。在考察数概念时，应遵循雅各布·克莱因（Jacob Klein）的教导，回到源初语境理解对于希腊时期的数学家而言使用数时的意向（intentionality）如何。克莱因的语源学考察解揭示，ἀριθμός 始终关联着可计数对象的集合，毕达哥拉斯学派"万物皆数"的命题，本质意味着我们看到或听到的一切都可以被计数。这透露出两个层面的观念：首先，**被计数对象始终是感官经验中的"切近之物"**；其次，这里的**"数"指"计数"，**是数出具体的一、二、三的过程。亚里

士多德与欧几里得的定义共同确证，希腊人理解的数始终是某些事物的数——**单位**(monas)作为不可分割的存在基底，使数概念必然维系于具体对象。这形成独特的数之形态：其一，希腊数学家不承认“一”（因其作为单位而非数）、分数（以不同数量单元之关系替代）及无理数（破坏可公度性）的存在性；其二，度量性主导数学实践，如化圆为方问题实质是对可度量性的执着。柏拉图虽在理念论框架中区分理论数艺与实践算艺，但其“数本身”(auto to arithmos)仍受制于单元不可分割性——“所有分割终止于一”的设定，使柏拉图式“抽象数”依然保持着对具体事物的指涉。由此，希腊之“数”始终未突破自然态度的认知基底——数作为可感世界的秩序化显现，其单元始终锚定在日常经验中的“这一个”存在者。这样的“数”绝非抽象符号系统中的“数字”（number），而是根植于具体对象的度量实践。

与算术领域的经验性根基相呼应，希腊几何学的“抽象性”亦显现出直观构造的鲜明烙印。其首要特征在于有限直观框架的建构性。希腊几何中的“空间”并非独立实体，而是具体几何对象（点、线、面）的关系总和。欧几里得体系中的图形研究，本质是**对理想化“切近之物”的操作**：直尺圆规作图法限定了几何对象的生成方式，使“空间”成为可触知形式的集合。这种构造性特质与现代拓扑空间形成根本分野——当后者将空间抽象为无限维结构时，希腊人仍在通过有限工具捕捉经验对象的形式真理。第二特征体现于**指称系统的在场性**。希腊几何术语并未出现现代数学的普遍化表述：不使用“某个/任意”，而采用“这一个/每一个”。这种语言现象折射出深层的认知范式——几何对象必须作为在场者被直接指涉，其普遍性通过具体个例的完全枚举实现。

通过对希腊时期“数”概念与几何学的剖析可知，希腊数学之抽象性并非现代符号系统的形式化操作，而是对具体对象的理想化提纯，从而形成了独特的**具象抽象**之范式。此范式体现出受限的符号化进程：既通过理想化切割经验杂质，又因拒斥自由符号操作而止步于形式革命的门槛。同时，正是这种未完成的认知姿态，使其成为思想史的关键枢轴——在经验基底上建构理念世界，既终结了古代数学的具象传统，又以未彻底符号化的公理体系为现代数学埋下解辖域化的种子。

应进一步追问，形成此种知识形态的根由何在？首先，作为人类理性最早的体系化表达，它在本体论层面承诺数学对象的超验性——毕达哥拉斯将“数”神圣化为宇宙本原，柏拉图将几何形式升华为理念世界的完美原型，试图以三角形、圆等抽象实体建构超越感官的永恒真理。然而，这种理念化追求却始终伴随着经验根基之张力影响：数始终绑定不可分割的单元，几何依托尺规作图的直观构造。这种**本体论承诺与认识论方法的张力**，反映了古希腊哲学中理念与现象二元对立的数学具象化。亚里士多德以“抽象实体”概念试图调和这一张力：

数学对象虽脱离质料而纯粹，但其抽象过程必须根植于感性经验。这种辩证性使希腊数学兼具双重面相——希帕霍斯的天文测算既建构几何模型又依赖观测数据，阿波罗尼奥斯的圆锥曲线研究既回应倍立方体难题又探索纯粹形式。这一角度实则暴露出希腊理性之困境——其试图用有限工具把握无限真理，却拒绝为认知自由支付符号抽象的代价。

其次，海德格尔对希腊概念“数学因素”（τα μαθήματα）的解析也从存在论角度为理解这一现象提供了关键线索。海德格尔指出，希腊人通过 μαθήματα（可学之物）把握世界，其本质上是人与物相遇时预先携带的认知框架。数学在此并非现代意义上的形式系统，而是一种**“对物的基本态度”**。例如，当毕达哥拉斯学派宣称“数是万物的本原”，实则是将“数性”作为物之存在的敞开领域——非数本身构成世界，而是世界通过数的可计量性被理解。这种思维预设了“物”在希腊语境中的特殊意涵：物不是被置于抽象时空中的客体，而是作为**“切近之物”**直接呈现于人的周围世界。几何学中的尺规作图，正是这种“切近性”的操作化表达。海德格尔称此为**“生产性观看”**（herstellendes Sehen），即真理通过手的操作向视觉开显。同时，这也揭示出 μάθημα 与 mathematics 之间存在着的根本语义断裂——前者指向对象之可把握性，后者则演变为自主符号系统。

同时，可透过这一存在论视角反观并重申现代数学：现代数学将希腊的 μαθήματα 窄化为符号操作，实则遗忘了数学原初的“让存在显现”的功能。在希腊传统中，数学是存在之真理的揭示方式；而在现代性框架下，数学沦为了技术座架（Gestell）的计算工具。

对古希腊数学知识形态的当代重审，不仅为解构“抽象性进化史”的现代迷思提供批判性视角，更促使我们重返数学本质的哲学追问：数学不是或不仅是价值中立的符号游戏，更是文明认知范式的具象化呈现。希腊数学的“未完成性”由此显现出新的哲学意蕴：其理念与经验的纠葛，恰构成对现代数学形式主义霸权的潜在批判——它提醒我们，数学真理的生成永远需要直观与逻辑、经验与理念的辩证运动。

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# The Intertwining of Ideal Pursuit and Empirical Grounding: Re-examining the Epistemological Form of Ancient Greek Mathematics

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## Abstract

Traditional historiography of mathematics positions Greek mathematics as the origin of modern abstract mathematics, based on its deductive methodology and idealistic pursuits: The Pythagorean and Platonic schools emphasized mathematics' transcendent essence, while the Euclidean system pioneered axiomatic geometry. The 19th-century "geometric algebra" interpretive framework further reduced Greek mathematics to geometric variants of modern algebra. Such linear progressive historiography obscures the original characteristics of Greek mathematics, necessitating an intellectual-historical dissection to reveal its unique cognitive paradigm.

The Greek concept of number (arithmos) remained anchored in the empirical world. Jacob Klein's etymological analysis demonstrates that ἀριθμός refers to countable object collections, with its indivisible unit (monas) maintaining concrete referentiality. This led Greek mathematics to reject the existence of "one" (as unit rather than number), fractions (replaced by relations between heterogeneous units), and irrationals (contravening commensurability). Although Plato distinguished theoretical arithmetic from practical calculation, his "number itself" (auto to arithmos) remained constrained by indivisible units, forming a paradigm of concrete abstraction—numbers as ordered manifestations of the sensible world rather than an autonomous symbolic system.

Greek geometry similarly exhibited intuitive-constructive traits: Geometric space was not abstract entity but the totality of figural relations. Straightedge-and-compass constructions delimited object generation, rendering Greek geometry an operational science of idealized "proximate things". Its terminology employed deictic references rather than universal quantification, achieving generality through exhaustive instantiation. This finite intuitive framework fundamentally diverges from the infinite-dimensional structures of modern topology, revealing Greek abstraction as purified idealization of empirical objects.

Ontologically, Greek mathematics manifested tensions between transcendental ideals and empirical grounding: Pythagoras deified numbers as cosmic principles, while Plato elevated geometric forms to ideal prototypes. Yet numbers remained bound to empirical units, and geometry relied on constructive intuition. Aristotle's concept of "abstract entities" mediated this contradiction, asserting mathematical objects' independence from matter while rooting abstraction in sensory experience. This dialectic exposes the Greek rational paradox—aspiring to grasp infinite truth through finite means while rejecting symbolic abstraction.

Heidegger's ontological analysis of "the mathematical" (τὰ μαθήματα) provides profound insights: Greek mathematics was not formal system but fundamental disposition toward things. The Pythagorean dictum "all is number" essentially positioned measurability as the horizon for Being's disclosure, while geometric construction as "productive seeing" (herstellendes Sehen) revealed truth through manual operations. This cognitive paradigm fundamentally diverges from modern mathematics' symbolic systems—where mathēmata denoted graspable objecthood, "mathematics" has degenerated into Gestell (technological enframing).

Re-evaluating Greek mathematics holds dual contemporary significance: Firstly, it deconstructs the "abstract evolution" mythos, revealing its concrete-abstraction paradigm's critical potential against formalist hegemony; Secondly, it reasserts mathematics' philosophical essence as civilizational cognitive schema, emphasizing mathematical truth's dialectical genesis through intuition-logic and experience-idea interactions. The "incompleteness" of Greek mathematics constitutes a warning against modernity's crisis: When mathematics becomes fully symbolicized as computational instrument, have we forgotten its primordial mission as revelator of ontological truth?

# 跨学科共同体的制度化构建：从 ZAMM 到 GAMM 的战间期德国应用数学革命（1921-1933）

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摘要：本文以 1921 年《应用数学与力学杂志》（ZAMM）的创刊及 1922 年应用数学与力学学会（GAMM）的成立为核心线索，探讨第一次世界大战后德国科学共同体在跨学科制度化建设中的思想博弈与实践路径。通过分析数学家理查德·冯·米塞斯（Richard von Mises）的创刊策略、学科宣言及组织实践，本文揭示了应用数学在战后德国从边缘话语走向学科主流的动态过程，并着重论述了 ZAMM 与 GAMM“双轮驱动”模式下学术共同体构建的制度创新及其历史意义。研究结合科学社会史与思想史方法，依托原始档案、通信记录及刊物文本，阐明以下核心议题：第一，ZAMM 的创办如何回应战后德国科学重建的民族主义需求与学科整合愿景；第二，工程师与数学家的联合如何通过期刊命名、栏目设计及方法论宣言实现话语权再分配；第三，GAMM 的协同建设如何推动应用数学从知识生产向制度共同体的转型；第四，政治环境（如纳粹崛起）对这一跨学科实验的冲击与遗产重构。研究表明，ZAMM 与 GAMM 不仅是德国应用数学现代化的里程碑，更是 20 世纪科学制度化进程中“学科交叉”范式的早期典范，其经验为理解科学共同体的权力协商、知识生产机制与政治边界提供了重要案例。

一、战后科学重建与 ZAMM 的创刊：学科整合的政治隐喻

第一次世界大战的惨败使德国陷入物质与精神的双重危机，科学被赋予民族复兴的使命。冯·米塞斯敏锐捕捉到这一时代需求，试图通过创办 ZAMM 弥合纯数学与工程实践间的断裂。作为哥廷根数学传统与维也纳技术教育的“混血儿”，他批判德国大学体系对应用数学的轻视，主张将数学建模、图示计算与工程问题结合，构建“服务于国家竞争力”的新学科范式。这一理念与德国工程师协会（VDI）的资源支持形成合力，最终促成 ZAMM 于 1921 年诞生。值得注意的是，刊物名称从《应用数学杂志》到《应用数学与力学杂志》的妥协，揭示了学科命名的权力博弈：冯·米塞斯通过纳入“力学”一词，既安抚了工程师群体对理论数学的警惕，又为数学方法渗透工程领域预留接口。这种策略性命名成为跨学科制度化进程的缩影。

二、从方法论宣言到编辑实践：ZAMM 的学术革命

ZAMM 创刊号的核心文献《应用数学的任务与目标》不仅是一篇社论，更是一份跨学科方法论宣言。冯·米塞斯在此提出六大纲领，包括拒绝纯数学的霸权、倡导问题导向的研究范式、复兴工程几何教学、以及将力学塑造为“工程数学主轴”。尤为重要的是，他赋予图解计算与数值方法以合法地位，宣称“图解能完成数值计算的一切”，这一论断在无计算机时代具有革命性意义。编辑实践中，冯·米塞斯通过设立科学顾问团、分类出版综述与短评、强调多语言摘要等制度设计，将 ZAMM 打造为理论家与实践者的对话平台。其“每篇稿件按张付酬”的机制，更体现了对工程实用性的倾斜。然而，这种“数学家

主导”的编辑方针也引发争议，如普朗特尔批评其忽视实验科学的地位，暴露出学科整合的内在张力。

三、ZAMM 与 GAMM 的协同体构建：制度化的双轮驱动

若说 ZAMM 是知识生产的媒介，GAMM 则是学术共同体的制度载体。两者在人事交叉（如冯·卡门、普朗特尔同时参与编委会与学会）、议题呼应（年会报告与特刊联动）及青年培养（会议-发表链条）上的协同，构成战间期德国应用科学的独特生态。GAMM 的命名争议（“技术力学”vs.“应用数学与力学”）进一步凸显了学科主导权的争夺：冯·米塞斯以名称妥协换取制度生存空间，同时通过 ZAMM 的内容控制维持数学方法论优势。这种“表面平衡、实质主导”的策略，使 GAMM 迅速成长为涵盖数学家、力学家、冶金工程师的 500 人共同体，其“工程科学共和国”的理想超越了传统职业协会的局限。然而，这一共同体的跨国性（吸纳奥地利、捷克斯洛伐克成员）与冯·米塞斯的犹太背景，最终在 1933 年纳粹上台后遭遇政治清算，标志着科学自治理想在极端民族主义下的挫败。

四、遗产与反思：科学制度化的历史逻辑

尽管 ZAMM 与 GAMM 的“黄金时代”因政治变局戛然而止，其遗产仍深远影响了 20 世纪应用数学的发展。苏联的《应用数学与力学杂志》与美国的《应用数学季刊》均借鉴了 ZAMM 的栏目架构与学科定位；GAMM 的协同体模式则为二战后国际应用数学联盟（ICIAM）提供了组织蓝本。从科学技术史视角看，这一案例揭示了学科制度化进程中三重动力的交织：个体能动性（冯·米塞斯的企业家精神）、学科话语权争夺（数学 vs.力学）与政治经济环境的制约（战后重建与纳粹崛起）。它同时提醒我们，跨学科共同体的成功不仅依赖理念创新，更需在命名策略、资源整合与权力平衡中展现制度设计的艺术。冯·米塞斯的实验表明，科学进步既需要赫拉克利特式的“此处亦有神灵”的信念，也离不开对现实妥协的清醒认知。

**关键词：**应用数学制度化；ZAMM；GAMM；理查德·冯·米塞斯；跨学科共同体

# Institutional Formation of an Interdisciplinary Community: ZAMM, GAMM, and the Interwar Revolution in German Applied Mathematics (1921－1933)

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**Abstract:** This study examines how the Journal of Applied Mathematics and Mechanics (ZAMM) and the Society for Applied Mathematics and Mechanics (GAMM) facilitated the institutionalization of applied mathematics in post-World War I Germany through a synergistic "journal-society" model. Drawing on archival documents, correspondence records, and textual analyses of publications, the research reveals that mathematician Richard von Mises initiated an interdisciplinary integration experiment by founding ZAMM (1921) and establishing GAMM (1922). ZAMM bridged the divide between pure mathematics and engineering through methodological manifestos (e.g., advocating graphical computation and problem-oriented research) and editorial innovations (e.g., a scientific advisory board). Meanwhile, GAMM institutionalized a scholarly community via annual conferences, membership networks, and international collaborations. Their synergy manifested in content complementarity (linking conference reports with journal special issues), power-balancing strategies (negotiating nomenclature and dual disciplinary positioning), and mentorship for early-career scholars. However, tensions between mathematicians and engineers over disciplinary dominance persisted. The Nazi regime's political purges disrupted this process after 1933, yet its institutional legacies—such as international membership frameworks and practical analytical paradigms—shaped postwar global applied mathematics. This study demonstrates that ZAMM and GAMM not only served as institutional blueprints for interdisciplinary integration but also reflected the complex interplay of nationalism, disciplinary discourse, and individual agency within scientific communities, offering critical insights into 20th-century scientific institutionalization.

**Key words:** applied mathematics; institutionalization; ZAMM; GAMM; Richard von Mises; interdisciplinary community

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# Reanalysis of Penrose's New Argument and a Modified Version of it Formalised in DTK System

Yujiang Long

**Abstract.** Penrose's new argument tries to show that the idealised human mind outstrips the idealised finite machine. It is an anti-mechanistic argument and is based on Gödel's incompleteness theorems. The DTK system is a type-free theory of truth, which eliminates the semantic paradoxes that might arise from the predicate T's self-reference, through determinateness conditions. In this paper, I propose a modified version of the argument and try to formalise it within the DTK system. I compare two different reconstructions of Penrose's new argument, showing their difference and defects, and gaining insight into my modified version. The key innovation lies in showing how the core insight of Penrose can survive formalisation when properly bounded by language constraints and a proper modification of Penrose's formulation.

## 1 Introduction

In his 1951 Gibbs Lecture, Gödel proposed his famous disjunction:

> ...that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified (where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives). (Gödel, 1995, p. 310)

This disjunction is known as Gödel's Disjunction in which the first disjunct expresses an anti-mechanistic thesis. There are many rephrases of GD in the literature. Arnon Avron summarised these rephrases(Avron, 2020), and I list some here:

Rephrases of the first disjunct(anti-mechanistic thesis):

1. The human mind cannot be reduced to the working of the brain. (Gödel, 1951)
2. The human mind is not equivalent to a (finite) machine. (Lucas, 1961)
3. The operation of the mind in the field of arithmetics cannot be simulated by a machine. (Krajewski, 2020)
4. The human mind is not a Turing machine. (Penrose, 1989; 1994)

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5. There is no algorithm that can produce all the theorems that the human mind is capable of producing. (Horsten & Welch, 2016)
6. The mathematical outputs of the idealised human mind cannot coincide with the mathematical outputs of an idealised finite machine. (Koellner, 2016; 2018a; 2018b)
7. The collection of humanly knowable theorems cannot be recursively axiomatised in some formal theory. (Horsten & Welch, 2016)
8. No well-defined system of correct axioms can contain the system of all demonstrable mathematical propositions. (Gödel, 1951)

Rephrases of the second disjunct:

1. There are mathematical truths that cannot be proved by the idealised human mind. (Koellner, 2016; 2018a; 2018b)
2. There are objective (mathematical) truths that can never be humanly demonstrated. (Feferman)
3. There exists a particular true arithmetic statement that is impossible for human mathematical reasoning to master. (Charlesworth)

We can observe that many of these formulations differ to varying degrees, while some are essentially equivalent. First, I would like to declare that the version of Gödel's disjunction I adopt is as follows and I'll use **GD** to denote this specific version:

> The collection of humanly knowable sentences cannot be recursively axiomatized in any formal theory, or there exist absolutely undecidable sentences.

I maintain that this formulation faithfully captures Gödel's original intent. His first disjunct aims to establish that no finite machine (by which he means Turing machine) can extensionally equal or surpass the set of all true propositions that the human mind can in principle know. Since we know Turing machines and formal systems are equivalent, Gödel's terms 'finite machine' and 'recursively axiomatised formal theory' are indeed interchangeable. When Gödel refers to 'diophantine problems' he essentially means number-theoretic or arithmetic problems. Therefore, 'absolutely undecidable diophantine problems' specifically denote arithmetic propositions that humans can neither prove nor refute in principle.

Gödel argued that his disjunction was a result of his incompleteness theorems and provided an informal argument and saying his disjunction is a mathematically established fact(Gödel, 1951, p.310). From Gödel's inclinations and Hao Wang's records, we know that he held an anti-mechanistic stance, meaning he believed the first disjunct was true. However, at the time, he thought there was no way to provide a rigorous proof of the first disjunct. Still, he suggested that a satisfactory resolution to the intensional paradoxes might establish the first disjunct in the future.(Wang, 1996)

Nevertheless, many others have attempted to formulate anti-mechanistic arguments based solely on the incompleteness theorems. Notable among these are the Lucas-Penrose argument and Penrose's new argument (or Penrose's second argument). The core idea of the Lucas-Penrose argument is that for any sufficiently strong, consistent formal system, the incompleteness theorems imply the existence of true propositions that the system cannot prove, yet humans can recognise their truth through reflection. The literature primarily identifies a key issue with this argument: it requires the additional premise that we must also know the consistency of all such consistent systems, which is arguably impossible. Penrose's new argument is far more refined, and we are solely dealing with this argument in this paper. Like GD, this argument mainly involves three key concepts and we will treat them as three predicates.

The first is a formal system (F), a recursive set of axioms closed under a set of inference rules, where F must satisfy the conditions for applying the incompleteness theorems. Thus, F is a strictly precise concept, and since F is known to be equivalent to a Turing machine, the two terms are used interchangeably in this context. The second concept is the idealised human mind, or more specifically, a set of humanly knowable sentences, often referred to in the literature as absolute provability (K). So K represents the collection of propositions that idealised humans(e.g. an idealised mathematician) can, in principle, prove.[1] There is no unified principle for precisely characterising K in the literature, as it remains a somewhat vague notion. The third concept is truth(T), referring to all mathematically true propositions, or more broadly, all true propositions.

Penrose's new argument involves self-reference of K and T, which risks introducing semantic and also intentional paradoxes, such as the liar paradox. Typically, theories of truth are hierarchical (typed), prohibiting self-reference of T to avoid such paradoxes. However, typed truth theories cannot formally capture Penrose's new argument. To address this, Peter Koellner, building on Feferman's type-free truth theory DT (Determinate Truth), constructed the DTK system to evaluate Penrose's new argument and Gödel's disjunction. DTK permits self-reference of T and K but uses a predicate D to filter out semantic paradoxes. These paradoxes have an indeterminate truth value. But sentences satisfying D have a determinate truth value—either true or false—and are called determinate.

The following discussion will initially examine Penrose's new argument, then

[1] For this kind of idealisation, refer to Koellner, P., "On the question of whether the mind can be mechanized, I: From Gödel to Penrose", *The Journal of Philosophy* CXV, 7: 337–360.

its two reconstructions by Koellner, and Corradini and Galvan(henceforth CG). Subsequently, I will modify Penrose's new argument to propose a novel anti-mechanistic argument and attempt to formalise it within the DTK framework.

## 2 Penrose's New Argument

Penrose's new argument is an anti-machanistic argument, which was first introduced in his book *Shadows of the Mind*, but the exposition there is not centralised and is somewhat circuitous. Later, in *Beyond the Doubting of a Shadow* (a response to criticisms of his earlier work), he summarised his argument as follows, where F denotes a formal system:

> Though I don't know that I necessarily am F, I conclude that if I were, then the system F would have to be sound and, more to the point, F′ would have to be sound, where F′ is F supplemented by the further assertion "I am F". I perceive that it follows from the assumption that I am F that the Gödel statement G(F′) would have to be true and, furthermore, that it would not be a consequence of F′. But I have just perceived that "if I happened to be F, then G(F′) would have to be true", and perceptions of this nature would be precisely what F′ is supposed to achieve. Since I am therefore capable of perceiving something beyond the powers of F′, I deduce that, I cannot be F after all. Moreover, this applies to any other (Gödelizable) system, in place of F. (Penrose, 1996)

First, it should be noted that in Koellner's papers (2018b, 2016), his direct quotation of this passage from Penrose contains an error. In Koellner's quotation, the last two sentences read:

> But I have just perceived that 'if I happened to be F, then G(F+) would have to be true,' and perceptions of this nature would be precisely what F is supposed to achieve. Since I am therefore capable of perceiving something beyond the powers of F, I deduce that I cannot be F after all.

However, in Penrose's original text, the two instances of 'F' should actually be 'F′'. This discrepancy may have influenced the differences between Koellner's and CG's reconstructions, as CG's paper quotes the passage correctly. It is also possible that Koellner intentionally made this change, believing it would not affect the essence of Penrose's argument—but if so, why did he not explain it in his paper?

Second, the phrase "I am F" is, in Penrose's words, merely shorthand for "F encapsulates all the humanly accessible methods of mathematical proof." Here, "encapsulate" could mean either that F contains or equals K (i.e. $K \subseteq F$) or that F equals K ($F = K$), and these two interpretations are significantly different. However, based on Penrose's actual argument above, it seems he intends $F = K$, because only if $F = K$ (rather than merely containing K) can one derive the first statement's claim that F is sound (assuming, of course, that K is sound[2]). Both Koellner and CG also interpret "I am F" as K = F. If Penrose indeed holds this view, it constitutes a critical flaw. His argument concludes that $K \neq F$, meaning there is no formal system coincide with the idealised human mind, but this does not rule out the possibility of a sound F such that K is a proper subset of F, meaning there is a machine not equal but stronger than humans—a scenario that would actually support a stronger mechanist thesis than the claim that "there exists an F such that F = K." Later in this paper, we will propose a modified argument that addresses this very weakness.

The following sections will first introduce Koellner's DTK system and then analyse his and CG's reconstructions of Penrose's new argument from the view of DTK.

## 3 Koellner's DTK System[3]

The DTK system extends Feferman's DT system, which in turn extends PA (Peano Arithmetic).

### 3.1 The Language of DTK

The language $L_{DTK}$ expands the language of PA ($L_{PA}$) by including the usual logical symbols (connectives, quantifiers, identity) along with the following non-logical symbols: An individual constant '0', A unary function symbol 'S'(successor), Binary function symbols '+' and ' $\cdot$ ' (addition and multiplication), A binary predicate symbol '<'. Terms are constructed from '0' and individual variables by iterated application of 'S', '+', and ' $\cdot$ '. Standard numerals are '0', 'S0', 'SS0', etc. For brevity, we use '0', '$\bar{1}$', '$\bar{2}$', etc., as shorthand.

$L_{DTK}$ further extends $L_{PA}$ by adding: A primitive predicate T (for truth) and a primitive predicate K (for absolute provability). Additionally, DTK introduces a defined predicate D(determinateness), where $D(x)$ abbreviates $T(x) \vee T(\dot{\neg}x)$. Well-formed formulas in $L_{DTK}$ are defined recursively in the standard way. $\ulcorner \varphi \urcorner$ denotes the Gödel number of formula $\varphi$. For convenience, we introduce a notation to represent arithmetic operations on Godel numbers, making them reflect the syntactic operations

[2]"K is sound" means that all the propositions belong to K is true.

[3]The content of this section is mainly from (Koellner, 2016)

on the corresponding formulas. For this purpose, we will adopt Feferman's 'Dot Notation', which represents arithmetic operations on Gödel numbers by adding dots below the relevant syntactic symbols. For example, '$\underset{\cdot}{\neg}$' represents: for any formula $\varphi$, we have $\underset{\cdot}{\neg}\ulcorner\varphi\urcorner = \ulcorner\neg\varphi\urcorner$

We need one more piece of notation. Notice that although it makes sense to write $Prov_{PA}(\ulcorner\varphi\urcorner)$, it does not strictly speaking make sense to write $(\forall x)Prov_{PA}(\ulcorner\varphi(x)\urcorner)$ since x ranges over natural numbers but the intension here is to say that every numeral substitution instance of $\varphi(x)$ is provable in PA. This is where we use the notation '$\dot{-}$' with the understanding that $(\forall x)Prov_{PA}(\ulcorner\varphi(\dot{x})\urcorner)$ means that for every natural number x, if you take the canonical numeral for x, substitute it for the dot in $\varphi(\cdot)$, then the Gödel number of the resulting expression is in the range of the arithmetical relation $Prov_{PA}$.

For a given language L, let $Var(x)$, $At\text{-}Sent(x)$, and $Sent(x)$ be the arithmetical formulas indicating that x is the Gödel number of a variable, that x is the Gödel number of an atomic (quantifier-free) sentence of L, and that x is the Gödel number of a sentence of L, respectively. Strictly speaking we should indicate 'L' in the notation. But we will omit this since the relevant language will always be clear from context.

Finally, if x is the Gödel number of a formula, z is the Gödel number of a variable, and y is a natural number, then $x(\dot{y}/z)$ is the Gödel number of the formula obtained by substituting the canonical numeral for y for (the variable numbered by) z in (the expression numbered by) x.

### 3.2 Axiomatic system of DTK

In addition to the axioms and inference rules of first-order logic, DTK includes four groups of axioms:

**I. Arithmetic Axioms**

These are the axioms of PA, but the induction schema is extended to all formulas in $L_{DTK}$ (so T and K may appear in induction).

**II. Axioms governing D**

$(D_1)$ $(\forall x)[At\text{-}Sent_{L_{PA}}(x) \to D(x)]$.

$(D_2)$ $(\forall x)[Sent(x) \to (D(\underset{\cdot}{\neg}x) \leftrightarrow D(x))]$.

$(D_3)$ $(\forall x)(\forall y)[Sent(x) \wedge Sent(y) \to (D(x \underset{\cdot}{\vee} y) \leftrightarrow D(x) \wedge D(y))]$.

$(D_4)$ $(\forall x)(\forall y)[Sent(x) \wedge Sent(y) \to (D(x \underset{\cdot}{\to} y) \leftrightarrow D(x) \wedge (T(x) \to D(y)))]$.

$(D_5)$ $(\forall x)(\forall z)[Var(z) \wedge Sent((\underset{\cdot}{\forall} z)x) \to (D((\underset{\cdot}{\forall} z)x) \leftrightarrow \forall y D(x(\dot{y}/z)))]$.

$(D_6)$ $(\forall x)[D(\underset{\cdot}{T}(\dot{x})) \leftrightarrow D(x)]$.

$(D_7)$ $(\forall x)[D(\underset{\cdot}{K}(\dot{x})) \leftrightarrow D(x)]$.

**III. Axioms governing T**

$(T_1)$ for every atomic formula $R(x_1,\ldots,x_n)$ of $L_{PA}$:

$(\forall x_1)...(\forall x_n)[T(\underset{\cdot}{R}(\dot{x}_1,...,\dot{x}_n)) \leftrightarrow R(x_1,...,x_n)]$.

$(T_2)$ $(\forall x)[Sent(x) \wedge D(x) \to (T(\underset{\cdot}{\neg}x) \leftrightarrow \neg T(x))]$.

$(T_3)$ $(\forall x)(\forall y)[Sent(x) \wedge Sent(y) \wedge D(x \underset{\cdot}{\vee} y) \to (T(x \underset{\cdot}{\vee} y) \leftrightarrow T(x) \vee T(y))]$.

$(T_4)$ $(\forall x)(\forall y)[Sent(x) \wedge Sent(y) \wedge D(x \underset{\cdot}{\to} y) \to (T(x \underset{\cdot}{\to} y) \leftrightarrow T(x) \to T(y))]$.

$(T_5)$ $(\forall x)(\forall z)[Var(z) \wedge Sent((\underset{\cdot}{\forall} z)x) \wedge D((\underset{\cdot}{\forall} z)x) \to (T((\underset{\cdot}{\forall} z)x) \leftrightarrow \forall y T(x(\dot{y}/z)))]$.

$(T_6)$ $(\forall x)[D(x) \to (T(\underset{\cdot}{T}(\dot{x})) \leftrightarrow T(x))]$.

$(T_7)$ $(\forall x)[D(x) \to (T(\underset{\cdot}{K}(\dot{x})) \leftrightarrow K(x))]$.

**IV. Axioms governing K**

$(K_1)$ $(\forall x)[Sent(x) \to (K(x) \to T(x))]$.

$(K_2)$ $(\forall x)(\forall y)[Sent(x) \wedge Sent(y) \to (K(x \underset{\cdot}{\to} y) \wedge K(x) \to K(y))]$.

$(K_3)$ $(\forall x)[Sent(x) \to (K(x) \to K(\underset{\cdot}{K}(\dot{x})))]$.

The inference rules are:

$$\frac{\varphi \wedge D(\ulcorner \varphi \urcorner)}{K(\ulcorner \varphi \urcorner)} \qquad \text{(DK-Intro)}$$

$$\frac{\varphi \wedge D(\ulcorner \varphi \urcorner)}{T(\ulcorner \varphi \urcorner)} \qquad \text{(DT-Intro)}$$

### 3.3 Some results of DTK[4]

**Theorem 1.** *DTK is consistent.*

**Theorem 2.** *For each $\varphi \in L_{PA}$, we have:*
$DTK \vdash D(\ulcorner \varphi \urcorner)$

**Theorem 3.** *For each $\varphi \in L_{DTK}$*
$DTK \vdash D(\ulcorner \varphi \urcorner) \to (T(\ulcorner \varphi \urcorner) \leftrightarrow \varphi)$

**Theorem 4.** *For each $\varphi \in L_{DTK}$ such that $DTK \vdash \varphi \leftrightarrow \neg T(\ulcorner \varphi \urcorner)$, we have:*
$DTK \vdash \neg D(\ulcorner \varphi \urcorner)$

*Proof* *Let $\varphi$ be a sentence of $L_{DTK}$ and $DTK \vdash \varphi \leftrightarrow \neg T(\ulcorner \varphi \urcorner)$. Suppose that $DTK \vdash D(\ulcorner \varphi \urcorner)$. Then we have $DTK \vdash T(\ulcorner \varphi \urcorner) \leftrightarrow \varphi$ by Theorem 3. And then the contradiction $DTK \vdash T(\ulcorner \varphi \urcorner) \leftrightarrow \neg T(\ulcorner \varphi \urcorner)$* ■

**Theorem 5.** *For each $\varphi(x) \in L_{DTK}$, we have:*
$DTK \vdash D(\ulcorner \exists x \varphi(x) \urcorner) \leftrightarrow \forall x D(\ulcorner \varphi(\dot{x}) \urcorner)$

[4] Refer to (Koellner, 2016), pages 170-174 to see the proofs of theorem 1-3

*Proof* Let $\exists x\varphi(x)$ be a determinate formula of $L_{DTK}$. Then $\neg\forall x\neg\varphi(x)$ is determinate. Then $\forall x\neg\varphi(x)$ is determinate by $D_2$. Then we have for all x, $\neg\varphi(x)$ is determinate by $D_5$. Then, by $D_2$ again, we have for all x, $\varphi(x)$ is determinate. The proof of the opposite direction is similar. ■

## 4 Two Reconstructions of Penrose's New Argument

We compare the reconstructions between Koellner's and CG's, showing their difference and gaining insight into our anti-mechanistic argument.

### 4.1 Koellner's Reconstruction

Koellner's rephrasing of Gödel's disjunction(GD′) is:

> The mathematical outputs of the idealized human mind cannot coincide with the mathematical outputs of an idealized finite machine or there are mathematical truths that cannot be proved by the idealized human mind.

We use ¬WMT(for Weak Machanistic thesis) to denote the first disjunct in his rephrasing. Following Koellner(2016), we show his reconstruction line by line, step by step. But first, we need some abbreviations:

**Definition 1** $K = F := \forall x(Sent(x) \to (K(x) \leftrightarrow F(x)))$

**Definition 2** $Sound(F) := \forall x(Sent(x) \to (F(x) \to T(x)))$.

His reconstruction goes like:

1. Though I don't know that I necessarily am F, I conclude that if I were, the system F would have to be sound.

$K = F \to Sound(F)$ by $K_1$

2. More to the point, F′ would be sound, where F′ is F supplemented by the further assertion "I am F".

$K = F \to Sound(F')$ where $F' = F \cup \{K = F\}$ by DT-Intro

3. It follows from the assumption that I am F that the Gödel statement G(F') would have to be true.

$K = F \to G(F')$ by 2 and the nature of Gödel sentence

4. Furthermore, that it would not be a consequence of F′.

(a)$K = F \to F' \nvdash G(F')$ by 2 and the incompleteness theorem

(b)$K = F \to F \nvdash (K = F \to G(F'))$ by 4a

5. But I have just perceived that "if I happened to be F, then G(F′) would have to be true"

$K(\ulcorner K = F \rightarrow G(F')\urcorner)$ by 3 and DK-Intro

6. and perceptions of this nature would be precisely what F is supposed to achieve. Since I am therefore capable of perceiving something beyond the powers of F

$K = F \rightarrow K \neq F$ by 4b and 5

7. I deduce that I cannot be F after all.

$K \neq F$ by logic

The above presentation aligns with Koellner's 2016 paper, except for a minor difference in the ordering of step 4b, which he places after step 5—an inconsequential variation. Additionally, as previously noted, the two instances of F in step 6 appear as F′ in Penrose's original text.

As mentioned in the introduction part, formalising Penrose's new argument requires a type-free theory of truth and a type-free notion of K. In step 5, K applies to '$K = F \rightarrow G(F')$', thereby applying to itself. At first glance, this formulation doesn't appear to involve self-reference of T. However, upon closer examination, we can observe that since K possesses an introduction rule[5] (as evidenced by step 5), we may derive sentences like $K(Sound(K))$ from $Sound(K)$. Since $Sound(K)$ involves T, thereby K applies to T. Furthermore, in step 2, T applies to K, which means T is ultimately applying to itself. This creates a situation where both K and T exhibit self-referential behaviour, which may cause paradoxes like the liar sentence. And this is why Koellner employs a type-free truth theory to analyse Penrose's new argument. The DTK system, as a type-free truth theory, not only eliminates the semantic paradoxes that might arise from T's self-reference but also resolves the intensional paradoxes potentially caused by K's self-reference.

Koellner identifies two critical flaws in Penrose's argument: First, in step 2, T applies to $K = F$ (employing the DT-Intro rule). However, in DTK, legitimate application of the DT-Intro rule requires that the operated sentence ($K = F$) be determinate. Koellner demonstrates that $K = F$ is precisely indeterminate—by considering the substitution of liar-like statements into the K predicate, one obtains an indeterminate instance.[6] Second, in step 5, K applies to '$K = F \rightarrow G(F')$' using the DK-Intro rule, which similarly requires the operated sentence to be determinate for a valid reasoning within DTK. Yet '$K = F \rightarrow G(F')$' is likewise indeterminate.[7] Consequently, Koellner concludes that Penrose's argument fails under DTK's framework.

[5]ie., from $\varphi$ deduce $K(\ulcorner\varphi\urcorner)$

[6]Proof: For any liar-like statements $\varphi$, i.e. the statements satisfy $\varphi \leftrightarrow \neg T(\ulcorner\varphi\urcorner)$, we have $\neg D(\ulcorner\varphi\urcorner)$ by Theorem 4. Then $K(\ulcorner\varphi\urcorner)$ is indeterminate by $D_7$. Then $K = F$ is indeterminate by $D_5$ and $D_4$.

[7]Proof: Since $K = F$ is indeterminate, $K = F \rightarrow G(F')$ is indeterminate by $D_4$.

Koellner then examines whether there is any proof of GD$'$, $\neg$WMT, and the second disjunct(AU) in DTK. Different versions of GD and $\neg$WMT can be distinguished. By restricting the predicates K, T, and F to the sub-language of $L_{DTK}$, we obtain the restricted versions of GD$'$, $\neg$WMT, and AU.

He formalised the disjunction and disjuncts as follows:

$$\neg\mathrm{WMT}_L := \neg\exists e\forall x(Sent_L(x) \rightarrow (K(x) \leftrightarrow F_e(x)))$$

$$\mathrm{AU}_L := (\exists x)(Sent_L(x) \wedge T(x) \wedge \neg K(x) \wedge \neg K(\dot{\neg} x))$$

$$\mathrm{GD}'_L := \neg\mathrm{WMT}_L \vee \mathrm{AU}_L$$

Where the subscript 'L' indicates language condition. When there is no subscript sign, it indicates that it is in the full language version. Koellner demonstrates that:

**Koellner 1** $\quad$ DTK$\vdash$ GD$'\wedge\neg D(\ulcorner\neg\mathrm{WMT}\urcorner) \wedge \neg D(\ulcorner \mathrm{AU}\urcorner)$

It means that the full-language versions of Gödel's disjunction and its disjuncts are indeterminate, hence no valid argument for GD$'$, $\neg$WMT and AU in DTK. But how about restricting predicats K, F, and T to the arithmetic language? These restricted versions are:

$$\neg\mathrm{WMT}_{PA} = \neg\exists e\forall x(Sent_{PA}(x) \rightarrow (K(x) \leftrightarrow F_e(x)))$$

$$\mathrm{AU}_{PA} = (\exists x)(Sent_{PA}(x) \wedge T(x) \wedge \neg K(x) \wedge \neg K(\dot{\neg} x))$$

$$\mathrm{GD}'_{PA} = \neg\mathrm{WMT}_{PA} \vee \mathrm{AU}_{PA}$$

And Koellner shows:

**Koellner 2**$^*$ $\quad$ DTK$\vdash$ $\mathrm{GD}'_{\mathrm{PA}} \wedge D(\ulcorner \mathrm{GD}'_{\mathrm{PA}}\urcorner) \wedge K(\ulcorner \mathrm{GD}'_{\mathrm{PA}}\urcorner)$.

But here Koellner made a mistake. $\mathrm{GD}'_{\mathrm{PA}}$ is not determinate, since $\mathrm{AU}_{\mathrm{PA}}$ is not determinate:

**Theorem 6.** *$AU_{PA}$ is indeterminate.*

*Proof* $\quad$ *To show $AU_{PA}$ is indeterminate, we substitute liar sentence $\lambda$ into $T(x)$, and then we have $T(\ulcorner\lambda\urcorner)$ indeterminate, by **theorem 4** and $D_6$. Then $AU_{PA}$ is indeterminate by **theorem 5** and $D_2$ and $D_3$. Although the component $Sent_{PA}(\ulcorner\lambda\urcorner)$ is false and hence determinate, the rest components are indeterminate and render the entire conjunction indeterminate.*

What Koellner intended to demonstrate with his **Koellner 2**$^*$ is that at least we have a legitimate proof of $\mathrm{GD}'_{\mathrm{PA}}$ in DTK. But this cannot be due to **theorem 6**. However, Koellner is right about:

**Koellner 3** $\quad$ $DTK \vdash D(\ulcorner\neg WMT_{PA}\urcorner)$

And Koellner proves that both the restricted $\neg\mathrm{WMT_{PA}}$ and $\mathrm{AU_{PA}}$ remain independent of DTK. That is:

**Koellner 4** Assume that DTK is correct for arithmetical statements. Then DTK can neither prove nor refute either $\neg\mathrm{WMT_{PA}}$ or $\mathrm{AU_{PA}}$.

He therefore concludes that no valid proof of the anti-mechanistic thesis exists within DTK's framework.

### 4.2 Corradini and Galvan's Reconstruction

Koellner achieved a determinate result of ¬WMT within the restricted language. Antonella Corradini and Sergio Galvan (CG) observed that the illegitimacy of Koellner's reconstruction of Penrose's argument in DTK stemmed from its inclusion of steps involving indeterminate sentences, while the application of inference rules (namely DK-Intro and DT-Intro) requires these sentences to be determinate. This led them to propose that by similarly restricting Penrose's argument to the language of PA, these indeterminate sentences could be rendered determinate, thereby potentially legitimising Penrose's new argument within DTK. This approach mirrors Koellner's strategy restricting ¬WMT to $\neg\mathrm{WMT_{PA}}$, applying similar constraints to maintain validity within DTK while preserving the argument's essential structure. The key innovation lies in showing how Penrose's core insight can survive formalisation when properly bounded by language constraints. Below, we present their reconstructed version line by line, and first, we need some abbreviations:

**Definition 3** $\mathrm{K} =_{\mathrm{PA}} \mathrm{F} := \forall x(Sent_{PA}(x) \rightarrow (K(x) \leftrightarrow F(x)))$

**Definition 4** $Sound_{PA}(K) := \forall x(Sent_{PA}(x) \rightarrow (K(x) \rightarrow T(x)))$.
$Sound_{PA}(F) := \forall x(Sent_{PA}(x) \rightarrow (F(x) \rightarrow T(x)))$.

1. Though I don't know that I necessarily am F, I conclude that if I were, the system F would have to be sound.

$K =_{PA} F \rightarrow Sound_{PA}(F)$ by $K_1$

2. More to the point, F′ would be sound, where F′ is F supplemented by the further assertion "I am F".

$K =_{PA} F \rightarrow Sound_{PA}(F')$ where $F' = F \cup \{K =_{PA} F\}$ by (1) and DT-Intro

3. It follows from the assumption that I am F that the Gödel statement G(F′) would have to be true.

$K =_{PA} F \rightarrow G(F')$ by (2) and the nature of Gödel sentence

4. Furthermore, that it would not be a consequence of F′.

$K =_{PA} F \rightarrow F' \nvdash G(F')$ by (2) and the incompleteness theorem

5. But I have just perceived that "if I happened to be F, then G(F′) would have to be true"

$K(\ulcorner K =_{PA} F \rightarrow G(F')\urcorner)$ by (3) and DK-Intro

6. and perceptions of this nature would be precisely what F′ is supposed to achieve. Since I am therefore capable of perceiving something beyond the powers of F′

(a) $K =_{PA} F \rightarrow F \nvdash (K =_{PA} F \rightarrow G(F'))$ by (4a) and the deduction theorem

(b) $K =_{PA} F \rightarrow \exists x(x = \ulcorner K =_{PA} F \rightarrow G(F')\urcorner \wedge K(x) \wedge \neg F(x))$ by (6a) and 5

7. I deduce that I cannot be F after all.

$K \neq F$ by logic

A more detailed deduction of step 7 goes like this: From step (6b) we have:

$K =_{PA} F \rightarrow K \neq F$

and by pure logic:

$K \neq_{PA} F \rightarrow K \neq F$

So the result of step 7 follows.

Note that CG's citation of Penrose's new argument aligns the same with the original text. However, their reconstruction appears to overlook that the formal system in step 6 should be F′ rather than F—though this ultimately doesn't affect their reconstruction's result. The primary distinction from Koellner's reconstruction lies in restricting all sentences from steps 1 through 6 to the language of PA. Consequently, steps 2 and 5, which were invalid in Koellner's version, become legitimate in DTK, as they have proven:

**CG 1** DTK$\vdash D(\ulcorner K =_{PA} F\urcorner)$

**CG 2** DTK$\vdash D(\ulcorner K =_{PA} F \rightarrow G(F')\urcorner)$

This demonstrates that $K =_{PA} F$ in step 2 is now determinate, thus permitting the use of the DT-Intro rule to derive $Sound_{PA}(F')$. Similarly, the sentence operated by DK-Intro in step 5 becomes determinate, rendering step 5 valid in DTK. The remaining steps don't involve reasoning that requires determinate sentences. Therefore, their reconstruction remains legitimate within DTK.

Crucially, their conclusion doesn't contradict Koellner's results, as by **Koellner 4** we have:

DTK$\nvdash \neg\exists e\forall x(Sent_{PA}(x) \rightarrow (K(x) \leftrightarrow F_e(x)))$

Whereas by the step 7 they established:

DTK$\vdash \neg\exists e\forall x(Sent(x) \rightarrow (K(x) \leftrightarrow F_e(x)))$

Note the crucial distinction between language conditions. Koellner's results did not rule out the possibility that there is a non-arithmetical sentence that K knows, but F do not. Therefore, they have established:

**CG 3** $K =_{PA} F \vdash_{DTK} \exists x(Sent_{\neg PA}(x) \wedge D(x) \wedge K(x) \wedge \neg F(x))$

where $Sent_{\neg PA}(x)$ menas that x is the Gödel number of a non-PA sentence. Unfortunately, we cannot fully accept this result, as the sentence in it is indeterminate—substituting liar-like statements $\lambda$ into predicate K yields an instance which is indeterminate, and this existential quantified statement becomes indeterminate. And what they can actually establish in view of DTK is:

**Theorem 7.** *DTK proves:*

$K =_{PA} F \to K(\ulcorner K =_{PA} F \to G(F')\urcorner) \wedge \neg F(\ulcorner K =_{PA} F \to G(F')\urcorner))$

*and it is provably determinate.*

It says: If there exists a formal system F that knows as many arithmetic sentences as same with the idealised human mind, then there exists at least one non-arithmetic statement that the idealised human mind knows but F doesn't. This precise statement is non-arithmetic because it contains predicates K and F that don't belong to the language of arithmetic. I believe that the reason that **CG 3** is indeterminate, while **Theorem 7** is determinate, is due to the insufficiency of the DT system. The indeterminateness of **CG 3** is merely a purely technical reason rather than substantive. Therefore, I think that due to the establishment of **Theorem 7** (which actually says the same as **CG 3** with respect to the anti-mechanistic thesis, though they are not equivalent), we can actually accept **CG 3**.

Note that there exists a fundamental difference between CG's reconstruction and Koellner's, particularly evident in their respective final steps. Koellner's reconstruction attempts to prove $K \neq F$ through reductio ad absurdum by assuming K=F and deriving a contradiction. In contrast, CG's reconstruction assumes $K =_{PA} F$ and concludes with the existence of a non-PA language statement that belongs to K but not to F—this argument does not follow the reductio structure. Therefore, they contend that this is the true insight of Penrose's new argument, though it only partially supports the anti-mechanistic thesis: "The argument does not reach a result that shows there is no formalism capable of deriving all the true arithmetic propositions known to man. Instead, it shows that, if such formalism exists, there is at least one true non-arithmetic proposition known to the human mind that we cannot derive from the formalism in question."

Since Penrose's original argument assumes $K = F$, even if we could validly conclude $K \neq F$, this would not exclude the possibility of there existing a sound formal system F for which K is a proper subset of F. To put it in other words, this would not rule out the scenario where a sound formal system (F) exceeds the capabilities of the idealised human mind—which would in fact represent a stronger mechanist thesis. Similarly, CG's conclusion does not exclude the existence of a sound F of which K is a proper subset regarding the arithmetic domain. That is, it does not eliminate the possibility of there existing an arithmetic proposition that F knows but K does not. In the latter case, $K =_{PA} F$ would not hold, and consequently, we could not guarantee the existence of a non-arithmetic statement where K knows but F does not. This would allow for the possibility that F comprehensively surpasses K, meaning that in the full language version, K would be a proper subset of F. Of course, the opposite possibility also remains—that K might be stronger than F in the arithmetic domain. To sum it up, they conclude:

**CG 4** $\vdash_{DTK} \exists x(Sent_{\neg PA}(x) \wedge D(x) \wedge K(x) \wedge \neg F(x)) \vee \exists x(Sent_{PA}(x) \wedge K(x) \wedge \neg F(x)) \vee \exists x(Sent_{PA}(x) \wedge \neg K(x) \wedge F(x))$

This is a direct corollary from **CG 3**.

Thus, we observe that Penrose-style arguments based on either the $K = F$ or $K =_{PA} F$ assumption are fundamentally flawed. Setting aside Koellner's refutation, neither Koellner's reconstructed conclusion of $K \neq F$ nor CG's derived **CG 3** actually succeeds in establishing the anti-mechanistic thesis. Both approaches leave open possibilities that are actually compatible with mechanist positions, including some that would represent even stronger forms of mechanism than those they attempt to refute. This crucial limitation suggests the need for alternative argumentative strategies if one aims to rigorously defend anti-mechanism.

## 5 A Modification of Penrose's Argument

We argue that Penrose's new argument has a critical flaw: His initial assumption is "There exists an $F$ such that $K = F$", leading via reductio to "For all $F$, $K \neq F$." However, "$K \neq F$" is not equivalent to the anti-mechanistic thesis, as it allows for cases where $K$ is a proper subset of F—a scenario that could support a stronger mechanistic claim. Koellner's formulation of GD$'$ and the anti-mechanistic thesis also uses $K \neq F$, which deviates from Gödel's original phrasing "K surpasses F". Like Koellner, CG also frame the anti-mechanistic thesis as $K \neq F$ in their reconstruction of Penrose's argument and we deem that this is the true reason why their reconstruction of Penrose's argument is valid but inconclusive. We propose that the most accurate formulation of the mechanistic thesis should be: **"There exists an F such that K is a subset of F and F is sound."** Conversely, a weaker but more defensible anti-mechanistic thesis would be: **"There exists a proposition belonging to K but not to any F."**

**Definition** $K \subseteq_{PA} F := \forall x(Sent_{PA}(x) \rightarrow (K(x) \rightarrow F(x)))$

$F \subseteq_{PA} T := \forall x(Sent_{PA}(x) \rightarrow (F(x) \rightarrow T(x)))$

I propose that by modifying the assumption in CG's reconstruction from $K =_{PA}$

$F$ to $K \subseteq_{PA} F$, we might obtain the conclusion that K is not a subset of F in the full language, i.e., "If $K \subseteq_{PA} F$, then there exists a non-arithmetic statement that K knows but F does not." This proposition eliminates the third disjunct in **CG 4**, establishing that in no case can F completely reduce K, thus constituting a genuine anti-mechanistic thesis. However, with this modification, the first step of CG's argument, $Sound(F)$, can no longer be established via rule $K_1$. To apply the incompleteness theorem to F′, we require additional assumptions. For this purpose, we assume that F is sound—a reasonable assumption, since a mechanist argument would naturally prefer a sound F that equals or exceeds K. Moreover, our assumption only needs $Sound_{PA}(F)$, meaning every arithmetic sentence F produce is true.

The natural-language formulation of this new argument is as follows:

> If there exists an F that matches or exceeds my arithmetic capacity, where F is sound, then F′ must also be sound, where F′ extends F by adding the assertions "F matches or exceeds me" and "F is sound." I recognise that, under these hypotheses, the Gödel sentence G(F′) must be true and further cannot be a theorem of F′. Consequently, F cannot derive the implication "if F matches or exceeds me and is sound, then G(F′)." But I just perceived that "if F matches or exceeds me and is sound, then G(F′)." Thus, under the assumed conditions, I am capable of knowing something beyond the power of F.

I will first briefly demonstrate how to prove this argument in DTK, then explain why the conclusion it yields is valid.

1. $(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to F' \subseteq_{PA} T$

where $F' = F \cup \{K \subseteq_{PA} F,\ F \subseteq_{PA} T\}$. This line follows from the DT-Intro rule, and is legitimate, as $K \subseteq_{PA} F$ and $F \subseteq_{PA} T$ both are determinate.

2. $(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to G(F')$

This line follows from 1 and the nature of Gödel sentences.

3. $(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to F' \nvdash G(F')$

This line follows from 2 and the incompleteness theorem.

4. $(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to F \nvdash ((K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to G(F'))$

This line follows from 3 by the deduction theorem.

5. $K(\ulcorner (K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to G(F') \urcorner)$

This line follows from 2 by DK-Intro, and is legitimate, since line 2 is determinate.

6. $(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \to K \nsubseteq F$

This line follows from 4 and 5 by logic.

7. $K \nsubseteq F \vee F \nsubseteq_{PA} T$

This line follows from (6) by logic.

8. $F \subseteq_{PA} T \rightarrow K \nsubseteq F$

This line follows from (7) by logic.

Our conclusion demonstrates that if a formal system (Turing machine) F, of which every arithmetic sentence it produces is true, then there exists at least one statement $\varphi$ in K that lies beyond F's capabilities. In other words, for any finite machine F that is arithmetically sound, the idealised human mind necessarily surpasses it(at least in some area)—we have thereby proved this anti-mechanistic thesis in DTK. But, is it determinate?

**Theorem 8.** *$F \subseteq_{PA} T \rightarrow K \nsubseteq F$ is indeterminate.*

*Proof* *By $D_4$, it sufices to show that $K \nsubseteq F$, i.e. $\exists x(Sent(x) \wedge K(x) \wedge \neg F(x))$ is indeterminate. It can be achieve through a substitution of a liar sentence for x in $K(x)$.* ■

But we have:

**Theorem 9.** *The sentence*

$(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \rightarrow (K(\ulcorner(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \rightarrow G(F')\urcorner) \wedge \neg F(\ulcorner(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \rightarrow G(F')\urcorner)$

*is provably determinate and provable in DTK.*

*Proof* *(1)$K \subseteq_{PA} F$ is determinate as a result of* ***CG 1****. (2)$F \subseteq_{PA} T$ is determinate because for any x, $F(x)$ is determinate(since x in F is recursively enumerable), and T(x) is determinate for any x which is a Gödel number of an arithmetic sentence, and iteratively using $D_4$ we have $F \subseteq_{PA} T$ is determinate. (3)$G(F')$ is determinate because it is arithmetical. Adding all together, we have shown the sentence is determinate. It is provable in DTK because of steps 4 and 5 in the above argument.* ■

Again, as the reason mentioned above, due to the insufficiency of the DT system. The indeterminateness result of **Theorem 8** is merely a technical reason rather than substantive. Therefore, I think that due to the establishment of **Theorem 9** (which says the same as step 8 in the above argument with respect to the anti-mechanistic purpose, though these two sentences are not equivalent), we can accept $F \subseteq_{PA} T \rightarrow K \nsubseteq F$ as a conclusive result.

Furthermore, our argument explicitly identifies a particular $\varphi$ that, under the given premise, is known by K but not by F. This $\varphi$ is:

$(K \subseteq_{PA} F \wedge F \subseteq_{PA} T) \rightarrow G(F')$

While $\varphi$ is a non-arithmetical statement, it nevertheless provides a robust anti-mechanistic conclusion according to our understanding of the thesis. That is:

The collection of humanly knowable sentences cannot be recursively axiomatised in any reliable formal theory

The assumption of the arithmetic soundness of F is entirely reasonable—if a finitary machine cannot even guarantee reliability in arithmetic, it would be incapable of replicating the idealised human mind. After all, how could we possibly accept that a machine prone to arithmetic errors could match the capabilities of an idealised human mind? It is worth emphasising that while our argument requires assuming F's reliability for arithmetic statements, it makes no parallel assumption about K's soundness. This distinction may represent an advantage in our approach.

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# The Coding Conception of In

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**Abstract**

A large amout of justifications for ZF have been produced by philosophers and mathematicians. Every such work focus on what is set, and then why our theory about sets captured the nature of our interpretation of sets before. Extending ideas in [4], we provide a new justification focus on the $\in$ relation we use in the language of set theory, and sets are just abstract entities that allows the $\in$ relation acts on and above it.

The $\in$ relation happens between first order objects and second order objects a long time ago. However, in arithmetic people come up with a function that can 'code' bounded second order objects into a single number. So here comes a 'first-order-ization' of the second order $\in$ over bounded second order objects: a number is '$\in$' another number iff the latter codes a bounded 'subset consists of numbers' with the former an element in it. Generalizing this idea, we can consider the following extension of any first order theory in first order language $\mathcal{L}$ with a highlighted binary relation $\lhd$: adding a new binary relation symbol $\in$, then the axiom schema of seperation asserting for every first order formula $\varphi(x, v_1, \ldots, v_n)$ in the extended language

$$\forall v_1 \ldots v_n \forall p(\forall x(\varphi \to x \lhd p) \to \exists! q \forall x(\varphi \leftrightarrow x \in q))$$

says there's enough first order objects to code every definable bounded second order objects. If we're considering 'purely coding objects', which we'll denote by 'sets' in short, the language is only $\mathcal{L}_\in$ with this $\lhd$ just $\in$, this axiom is equivalent to seperation and extensionality in ZF together. If we're considering natural numbers with $\lhd$ be $<$, this is clearly how the $\in$ relation reacts from Ackermann interpretation(see [2]).

Notice that if there are at least two different first order objects (that is, axiom $\exists x \exists y(x \neq y)$) making our theory non-trivial, we must

surely have infinitely coding objects, so intuitionally every finite second order object should be potentially bounded. We want to make it explicitly by using $\in$ for bounding them, which is the axiom schema of finite collections: for every natural number $n$,

$$\forall x_1 \ldots \forall x_n \exists y (x_1 \in y \wedge \cdots \wedge x_n \in y)$$

There are other maximality principles that says there are abundant coding objects for some concepts, first of which is collection principle: for every $\varphi(x, \bar{v})$ in the extended language,

$$\forall \bar{v} \forall p (\forall x \lhd p \exists y (\varphi) \rightarrow \exists q \forall x \lhd p \exists y \in q (\varphi))$$

which states that we have abundant coding objects to collect evidences for every single formula with parameters taking boundedly many values. For $\lhd$ we define $x \sqsubseteq y$ iff $\forall z (z \lhd x \rightarrow z \lhd y)$, and $x \ll y$ iff $\exists z (z \lhd y \wedge x \lhd z)$. The first relation, called $x$ is virtually not larger than $y$, means $x$ preceeds $y$ in any linear order extending $\lhd$. The second relation, called $x$ is far smaller than $y$, needs no explanation. Now each of them leads to an axiom:

$$\forall p \exists x \forall y (y \sqsubseteq p \rightarrow y \in x)$$

$$\forall p \exists x \forall y (y \ll p \rightarrow y \in x)$$

stating that coding objects are enough for them. All these axioms are reasonable no matter $\lhd$ is $<$ for natural numbers or $\in$ for sets, and can be part of formalizations for $\mathsf{PA}$ or $\mathsf{ZF}$. In fact, the collection principle in arithmetic and set theory share exactly the same model-theoretic characterization, see [3] and [1] for detailed analysis.

The intuition that well-ordering captures our intuition for a generating process is useful in the justification based on iterative conception of sets, where it takes the form of ordinals. Now we just adds the axiom schema of $\in$-induction: for every $\varphi(x, \bar{v})$,

$$\forall \bar{v} (\forall x (\forall y (y \in x \rightarrow \varphi(y, \bar{v})) \rightarrow \varphi(x, \bar{v})) \rightarrow \forall x (\varphi(x, \varphi)))$$

That's because we hope the whole coding can be implicitly done from a generating process, which means $\in$ should be a subrelation of a global well-ordering, so we have $\in$ is a well-founded relation and the corresponding induction principle on it.

Taking everything here together, in arithmetic after adding definitions for $0, \mathrm{S}, +, \times$ we just recover $\mathsf{PA}-(3)_{\mathsf{Q}}$, with $(3)_{\mathsf{Q}}$ the third axiom in Robinson's arithmetic that claims every positive natural number is successor of some natural number, it's a property of $0, \mathrm{S}$ instead of a definition. In set theory we obtain $\mathsf{ZF} - \mathsf{Inf}$. So to get the full $\mathsf{ZF}$, we must confirm a form of Inf. In fact, there is a reason why someone choose to live in theories weaker than PA while others live in theories stronger than ZF, named the indescribablility principle, more oftenly called the reflection principle in set theory: for every formula $\varphi(\bar{x}, \bar{v})$,

$$\forall \bar{v} \exists c(\bar{v} \in c \wedge \forall x \in c \forall y \in x(y \in c) \wedge \forall \bar{x} \in c(\varphi \leftrightarrow \varphi^{\in c}))$$

which states that every formula with fixed parameters can be checked coded inside of some $c$. The first two assertions makes sure the relativization that makes every quantifier $\in$-bounded by $c$ don't lost objects when we do such to quantifiers that have been already $\in$-bounded by other things. PA rejects it: $(3)_{\mathsf{Q}}$ is equivalent to negation of this principle on $\forall y \exists z(z = 2^{2^{2^y}})$ with parameter 0. ZF accepts it: it's equivalent to axiom of infinity from everything else.

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# A belated foundational role of set theory

The present work sets out to achieve something relatively straightforward: to sketch a particular function served by set theory, and then attempt to characterize it as foundational, perhaps on a par with those sketched in Maddy's insightful analysis of discourse regarding foundations (Maddy, 2017, 2019). The challenges to be surmounted, however, are non-trivial; so let me set the stage by saying something a little more precise about what I aim to do.

The narrative in this paper will revolve around descriptive set theory and the technical branch of set theory known today as Borel equivalence relations theory (or invariant descriptive set theory, when the focus on Borel-ness is suppressed. I shall use these two terms more or less interchangeably, *cf.* Gao (2008) and Hjorth (2010)). More specifically, I intend to convey the idea that, throughout its development, the theory has come to serve a peculiar foundational purpose - which I shall call **Brickwalling**. I will do this by isolating a latent thread that ran through the early disputes about the axiom of choice, as well as in the pre-history of Borel equivalence relations theory.

To really draw out the thread, I return to the early days of descriptive set theory, when the Borel sets were first introduced in Borel (1898). By studying the introduction of Borel sets in its historical context, I argue that they were introduced as a way to restrict attention to the tractable problems in analysis. This point remains salient (although not always explicit) in the later development of (descriptive) set theory amidst controversies surrounding the axiom of choice, for instance in Luzin (1927), where abstract equivalence relations were first considered. Particular attention will be paid, in this case, to the common context and motivations of Borel and Luzin. The takeaway is eventually summarized in the following maxim: *intractable problems are hopeless.*

This thread is echoed in the more modern developments of Borel equivalence relations theory, in particular in its applications to classification problems in mathematics. The aforementioned maxim is attested in various corners of the technical literature, e.g., Foreman et al. (2011) and Ros (2021). Now, if Borel-ness is taken to be the benchmark measure of tractability, as it seems to be in the early days of the subject and more so recently, then I claim the theory of Borel equivalence relations can be said to play a foundational role in mathematics. In fact, it plays a two-fold role: one of organizing and relating various structures from diverse fields of mathematics and their attendant classification problems (akin to category theory's Essential Guidance), and the other of delineating the boundaries of the tractable and intractable such problems.

If I am successful, I will have shown that the theory of Borel equivalence relations (and its earlier “spiritual" ancestors in descriptive set theory) has come to serve as a guide for when specific types of mathematical problems are tractable or intractable. This is a role that is not often associated with set theory in its capacity to provide interpretations for the whole of math, and as it will become clear, it is rather with set theory's initial, properly mathematical goals. Nonetheless, I will argue that this is a role that can be considered foundational.

More precisely, I will attempt to come to a conditional conclusion: this role is as foundational as some of the other candidates considered in Baldwin (2018, 2024) and Button et al. (2018). In other words, I will argue that insofar as category theory and model theory can be said to play a foundational role in mathematics, as evidenced in its providing the kinds of services outlined by Maddy and Baldwin, the same kind of role is being played by set theory.

A few disclaimers in closing: I do not intend to argue in set theory's favor as a foundational theory or anything of that sort. The ulterior motive of this little exercise here is really to reflect on what it means for a mathematical purpose to be foundational. The bulk of philosophy here will center around this, albeit scattered throughout the paper, sometimes in passing. Ultimately, the objective is to sharpen the notion of a foundational role/purpose and the ways it can be served by the mathematics. Of course, I should also stress that almost all of the technical results that appear in this paper are known to the specialists. Aside from a bit of intellectual history, not much is new in terms of the relevant mathematics. My contributions here are solely about the practice.

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# 古典数学的公理化及其确定性

刘力恺

**摘　要：**本文给出了四元数的二阶公理化理论HQ。在含完整概括公理的二阶演算中证明了HQ具有内在范畴性和内在不容忍性。又在一个模态三阶逻辑系统 TOM 中，以“必然存在现实世界”为前提，证明了HQ具有内在必然不容忍性。本文使用公理化学派的方法，将古典数学（与现代数学相对）奠基在了HQ之上，并利用内在主义克服了 Skolem-Gödel 二律背反，论证了古典数学的确定性，利用约定主义论证了古典数学的真理性。

**关键词：**四元数;内在主义;确定性

## 1　四元数理论何以为古典数学之基

明珠蒙尘！距 Hamilton 创造四元数已逾一百八十年，然而，自从在与矢量分析的争锋中落败后，四元数的重要性至今仍未获得最充分的重视。本文却以为，四元数理论足以作为古典数学的一个大统一理论而为之奠定坚实基础。

本文所谓之古典数学，乃是与现代数学相对的概念。十九世纪是古典数学向现代数学的过渡期，Galois 理论和 Riemann 几何等革命性工作的诞生，标志着数学的现代转向，数学迈向了以抽象结构为核心研究对象的新纪元。起初，数学家们从具体的数学问题中抽象出群、域、流形等等具有一般性的结构以解决具体问题；到了二十世纪中叶，正处鼎盛的 Bourbaki 学派则试图以结构主义的视角统一数学，他们将数学研究的对象归结为以三大母结构（代数结构、序结构、拓扑结构）为骨架不断衍生的各种结构以及结构间的态射。三大母结构虽然源于对数系和时空的抽象，但一旦成为确定的数学概念，就完全脱离了对数和时空的直观和具体内容，成为普遍的存在；而数学家可以根据研究需要，在母结构的基础上人为定义一些复合的或具体性更强的结构。正如 M. Kline 所言，十九世纪数学的发展状况“迫使人们承认数学是一种人为的并且多少带有任意性的创造物，而不仅仅是从自然界里引导出来的本质上是真实事物的一种理想化”[1, p. 106]。完全超越对自然界的依赖，一般性地研究抽象结构，构成了现代数学最鲜明的特征。也正是在这样的背景下，集合论成为了数学的基础，因为抽象结构本质上就是集合。

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古典数学面对的则是一些基于时空直观抽象出的具体结构：自然数系$\mathbf{N}$（初等数论）、实数系$\mathbf{R}$和复数系$\mathbf{C}$（算术、初等代数、一元微积分与无穷级数、单复变函数论）、三维欧氏空间$\mathbf{R}^3$（综合的与解析的欧氏几何、多元微积分、古典微分几何、数学物理方程）。而四元数系恰好能将上述结构一并囊括。

全体四元数的集合

$$\mathbf{H} \overset{\text{def}}{=} \{s + x\mathrm{i} + y\mathrm{j} + z(\mathrm{ij}) \mid s, x, y, z \in \mathbf{R}, \mathrm{i}^2 = \mathrm{j}^2 = -1, \mathrm{ij} = -\mathrm{ji}\}$$

显然$\mathbf{C} \subseteq \mathbf{H}$。同时每个四元数均可视作标量＋三维矢量的形式，即$\hat{q} = s + \vec{v} \in \mathbf{R} \oplus \mathbf{R}^3$；由于四元数的乘法满足等式：

$$\widehat{q_1}\widehat{q_2} = s_1 s_2 - \overrightarrow{v_1} \bullet \overrightarrow{v_2} + s_1\overrightarrow{v_2} + s_2\overrightarrow{v_1} + \overrightarrow{v_1} \times \overrightarrow{v_2}$$

故容易定义矢量分析中极为重要的点乘和叉乘运算：

$$\widehat{q_1} \bullet \widehat{q_2} \overset{\text{def}}{=} (\widehat{q_1}^*\widehat{q_2} + \widehat{q_2}^*\widehat{q_1})/2 = s_1 s_2 + \overrightarrow{v_1} \bullet \overrightarrow{v_2}, \quad \widehat{q_1} \times \widehat{q_2} \overset{\text{def}}{=} (\widehat{q_1}\widehat{q_2} - \widehat{q_2}\widehat{q_1})/2 = \overrightarrow{v_1} \times \overrightarrow{v_2},$$

其中$\hat{q} = s + \vec{v}$的共轭$\hat{q}^* \overset{\text{def}}{=} s - \vec{v}$。综上数和矢量都只是特殊的四元数，故可在四元数理论中统一地研究它们。此外$\mathbf{R} \oplus \mathbf{R}^3$正好可以理解为一维时间＋三维空间，故四元数在物理上有其独特的价值：如可将$\mathbf{H}$视为 Minkowski 时空，而双线性函数

$$g(\widehat{q_1}, \widehat{q_2}) \overset{\text{def}}{=} \widehat{q_1} \bullet \widehat{q_2} - 2(\widehat{q_1} \bullet 1)(\widehat{q_2} \bullet 1)$$

在基$\{1, \mathrm{i}, \mathrm{j}, \mathrm{ij}\}$下的表示矩阵即闵氏度规；Maxwell 方程组亦可简洁地表示为

$$\left(-\frac{\mathrm{i}}{c}\frac{\partial}{\partial t} + \overrightarrow{\nabla}\right)\left(\vec{E} + \mathrm{i}\vec{H}\right) = -\rho + \frac{\mathrm{i}}{c}\vec{J};$$ [2], p. 1008

等等。一旦完成四元数的公理化，那么范后宏[3], pp. 6-7 所说的四种基本的数学思维方式——数、形、逻辑、自然理性——就将在此形式理论中完美地统一起来。

众所周知，数和矢量在集合论中都可被定义为某个具体的集合。那为什么本文反对用集合论为古典数学奠基呢？一方面，如果认为数和矢量就是集合，一则不符合数学实践中的通常认知；二则将面临著名的 Benacerraf 问题（$\{\varnothing, \{\varnothing\}\}$和$\{\{\varnothing\}\}$谁才是 2？）。另一方面，如果认为那些作为定义项的集合仅仅是数和矢量的形式对应物，一则承认了数和矢量的本体论地位；二则本体论承诺溢出，对古典数学来说，并不需要纯集合，集合论只发挥语言作用，二阶四元数语言此时完全能承担这种语言功能，它能自如地言说数集、函数等对象，足以应对古典数学的问题域。

那为什么不在本体论更简洁的二阶算术中发展古典数学呢？十九世纪后期的分析算术化运动不就旨在把古典数学最终奠基在自然数理论上吗？二阶算术同集合论一样也面临归约的问题，所有数学对象都要归约为自然数或自然数子集，例如$(-1)_{\mathbf{Z}} \overset{\text{def}}{=} (0,1) \overset{\text{def}}{=} (0+1)^2 + 0 = 1$，$1_{\mathbf{Z}} \overset{\text{def}}{=} (1,0) \overset{\text{def}}{=} (1+0)^2 + 1 = 2$，$1_{\mathbf{Q}} \overset{\text{def}}{=} (1_{\mathbf{Z}}, 1_{\mathbf{Z}}) = (2,2) \overset{\text{def}}{=} (2+2)^2 + 2 = 18$，我们显然不可能把这种出于证明论目的的技术性编码视为本体论意义上的归约。

本文力图在保持数学对象原貌的前提下建立古典数学的统一理论。四元数理论确实保持了数的原貌，但它不也要把几何对象归约为代数对象吗？事实上直到复数的几何表示出现后，欧洲数学界才彻底接受了负数和虚数，正如 Gauss 所说"如果$1, -1, \sqrt{-1}$原来不称为正、负、虚单位而称为（坐标系的）直、反、侧单位，

那么人们对这些数就可能不会产生一些阴暗神秘的印象。”[4], pp. 7-8 因此与其说几何对象被归约为了代数对象，毋宁说四元数不过是时空点的一种精确表征。在此意义上，可以接受用二阶四元数理论统合古典数学。

## 2 四元数系的公理化

二十世纪初 Huntington[5]二阶公理化了复数系。对此稍加推广，不难得到四元数系的公理化，因为四元数最大的特征无非是$\mathrm{ij}=-\mathrm{ji}$导致乘法不满足交换律。美中不足的是，由于实数集$\mathbf{R}$在复数域$\langle\mathbf{C},+,\cdot,0,1\rangle$中不是一阶可定义的[6], p. 23，Huntington 不得不以实数谓词$R$为初始符号。由于复数$z$是实数的充要条件是$z=z^*$，以“共轭”这个一元函数为初始符号会更加自然方便。与复数域不同的是，$\mathbf{R}$在四元数体$\langle\mathbf{H},+,\cdot,0,1\rangle$中一阶可定义，即四元数$q$是实数的充要条件是$\forall p(qp=pq)$。但为了公理更简洁美观，本文仍引入共轭算子作初始符号。

用$\mathscr{L}_2^{\mathscr{H}}$表示非逻辑符号集$\mathscr{H}=\{+,\cdot,0,1,\mathrm{i},\mathrm{j}\ *,<\}$的二阶语言，该语言的项和公式递归定义如下：

$$t::=x\mid 0\mid 1\mid \mathrm{i}\mid \mathrm{j}\mid (t_1+t_2)\mid (t_1\cdot t_2)\mid t^*\mid ft_1\dots t_n$$
$$\phi::=Xt_1\dots t_n\mid t_1=t_2\mid t_1<t_2\mid \neg\phi\mid(\phi_1\to\phi_2)\mid\forall x\phi\mid\forall X\phi\mid\forall f\phi$$

基本二阶演算中除有联结词和等词的公理模式和变形规则外，还有：

$\vdash_2\forall v^\sigma\phi\to\phi(u^\sigma/v^\sigma)$；若$v^\sigma$不在$\phi$中自由，则由$\vdash_2\phi\to\psi$可得$\vdash_2\phi\to\forall v^\sigma\psi$。

其中$v^\sigma$表示类型为$\sigma$的变元，个体变元、$n$ 元谓词变元、$n$ 元函数变元的类型分别为$\iota$、$\iota\to\cdots\to\iota\to\tau$、$\iota\to\cdots\to\iota\to\iota$。有时会根据需要援引如下二阶有效式：

$$\exists X\forall\vec{x}(X\vec{x}\leftrightarrow\phi), X \text{ does not occur in } \phi \qquad \text{(CA)}$$
$$\forall X(\forall\vec{x}\exists!\,yX\vec{x}y\to\exists f\forall\vec{x}X\vec{x}f\vec{x}) \qquad \text{(FC)}$$

在此语言中建立公理化的四元数理论HQ，它有如下四组公理：

(I) 除环公理（去掉$0\neq 1$）： $\forall xy(x+y=y+x)$，

$$\forall xyz((x+y)+z=x+(y+z)\wedge(xy)z=x(yz))，$$
$$\forall x(x+0=x\wedge x\cdot 1=x)，$$
$$\forall x\exists y(x+y=0)\wedge\forall x(x\neq 0\to\exists y(xy=1))，$$
$$\forall xyz(x(y+z)=xy+xz\wedge(x+y)z=xz+yz)；$$

(II) 共轭和虚数单位的公理：

$$\forall xy((x+y)^*=x^*+y^*\wedge(xy)^*=y^*x^*\wedge x^{**}=x)，$$
$$\mathrm{i}^2+1=0\wedge\mathrm{j}^2+1=0，\mathrm{i}^*+\mathrm{i}=0\wedge\mathrm{j}^*+\mathrm{j}=0，$$
$$\mathrm{ij}+\mathrm{ji}=0，\forall x(x=x^*\leftrightarrow\forall y(xy=yx))；$$

(III) 序公理： $0<1$，$\forall xyz(x<y\wedge y<z\to x<z\wedge x\neq z)$，

$$\forall xy(x\neq y\to(x=x^*\wedge y=y^*\leftrightarrow x<y\vee y<x))，$$
$$\forall xyzw(x<y\wedge z=z^*\wedge 0<w\to x+z<y+z\wedge xw<yw)；$$

(IV) 实数的确界原理：

$$\forall X(\exists x Xx \wedge \forall x(Xx \to x = x^*) \wedge \exists y \forall z(Xz \to z < y)$$
$$\to \exists y(\forall z(Xz \to z < y \vee z = y) \wedge \forall x(x < y \to \exists z(Xz \wedge x < z)))).$$

如果能证明如下命题，就基本能说明此公理化是正确的。

**关键命题．** $\mathrm{HQ} \vdash_2 \forall q \exists!\, sxyz(\bigwedge_{h\in\{s,x,y,z\}} h = h^* \wedge q = s + x\mathrm{i} + y\mathrm{j} + z(\mathrm{ij}))$

**证明：**唯一性：只需证若$s + x\mathrm{i} + y\mathrm{j} + z(\mathrm{ij}) = 0$（$\bigwedge_{h\in\{s,x,y,z\}} h = h^*$），则$s = x = y = z = 0$。设$s + x\mathrm{i} + y\mathrm{j} + z(\mathrm{ij}) = (s + x\mathrm{i}) + (y + z\mathrm{i})\mathrm{j} = 0$ ①，①式两边先右乘i，再左乘−i得，$(s + x\mathrm{i}) - (y + z\mathrm{i})\mathrm{j} = 0$ ②。① + ②可得，$s + x\mathrm{i} = 0$ ③，③式两边取共轭得，$s - x\mathrm{i} = 0$ ④。③ + ④得，$s = 0$；③ − ④可得，$x = 0$。① − ②可得，$y + z\mathrm{i} = 0$，同理可推出，$y = z = 0$。

存在性：显然$q = \frac{1}{2}(q - \mathrm{i}q\mathrm{i}) + [\frac{1}{2}(q + \mathrm{i}q\mathrm{i})(-\mathrm{j})]\mathrm{j}$。令$a = \frac{1}{2}(q - \mathrm{i}q\mathrm{i})$，$b = \frac{1}{2}(q + \mathrm{i}q\mathrm{i})(-\mathrm{j})$，

$$-\mathrm{i}a\mathrm{i} = -\mathrm{i}\frac{1}{2}(q\mathrm{i} + \mathrm{i}q) = \frac{1}{2}(-\mathrm{i}q\mathrm{i} + q) = a,$$

$$-\mathrm{i}b\mathrm{i} = -\mathrm{i}\frac{1}{2}(q + \mathrm{i}q\mathrm{i})\mathrm{ij} = -\mathrm{i}\frac{1}{2}(q\mathrm{i} - \mathrm{i}q)\mathrm{j} = -\frac{1}{2}(\mathrm{i}q\mathrm{i} + q)\mathrm{j} = b。$$

对任意$c$，$\left[\frac{1}{2}(c + c^*)\right]^* = (c^* + c^{**})\left(\frac{1}{2}\right)^* = \frac{1}{2}(c + c^*)$；若还有$c = -\mathrm{i}c\mathrm{i}$，则$\mathrm{i}c = c\mathrm{i}$，则$\left[\frac{1}{2}(c - c^*)(-\mathrm{i})\right]^* = \frac{1}{2}(-\mathrm{i})^*(c^* - c^{**}) = \frac{1}{2}(\mathrm{i}c^* - \mathrm{i}c) = \frac{1}{2}(c^*\mathrm{i} - c\mathrm{i}) = \frac{1}{2}(c - c^*)(-\mathrm{i})$；显然又有$c = \frac{1}{2}(c + c^*) + \left[\frac{1}{2}(c - c^*)(-\mathrm{i})\right]\mathrm{i}$。因此取$s = \frac{1}{2}(a + a^*)$，$x = \frac{1}{2}(a - a^*)(-\mathrm{i})$，$y = \frac{1}{2}(b + b^*)$，$z = \frac{1}{2}(b - b^*)(-\mathrm{i})$，即可满足条件。∎

下一节将进一步说明上述公理化的正确性。在此之前先简单谈谈四元数的一阶公理化。根据 Niven-Jacobson-Baer 定理（右代数闭的中心有限非交换除环恰好是实闭域上的四元数体）[7], p.654，可以用如下语句公理化四元数体$\langle \mathbf{H}, +, \cdot, 0, 1\rangle$：

(I) 除环公理；(II) 乘法非交换：$\exists xy(xy \neq yx)$；

(III) 右代数闭：$\forall a_1 \ldots a_n \exists x(x^n + \sum_{i=1}^{n} a_i x^{n-i} = 0)$；

(IV) 维数$= 4$：$\exists x_1 \ldots x_4(\bigwedge_{1\le i<j\le 4} x_i \neq x_j \wedge \forall a_1 \ldots a_4\left(\bigwedge_{i=1}^{4} \mathrm{Ctr}\ a_i \wedge \sum_{i=1}^{4} a_i x_i = 0 \to \bigwedge_{i=1}^{4} a_i = 0\right) \wedge \forall q \exists c_1 \ldots c_4\left(\bigwedge_{i=1}^{4} \mathrm{Ctr}\ c_i \wedge q = \sum_{i=1}^{4} c_i x_i\right))$，其中$\mathrm{Ctr}\ x \stackrel{\mathrm{def}}{=} \forall y(xy = yx)$。

记这个一阶理论为$\mathrm{HQ}^-$。由于实代数数域是实闭域，实代数数域上的四元数体$\mathfrak{H}(\mathbf{A})$一定是$\mathrm{HQ}^-$的模型，这意味着$\mathrm{HQ}^-$并不能真正刻画四元数。若引入自然数谓词$N$，并添加 Peano 和 Archimedes 公理，将$\mathrm{HQ}^-$加强为$\mathrm{HQ}_1$，则$\langle \mathfrak{H}(\mathbf{A}), \mathbf{N}\rangle$是$\mathrm{HQ}_1$的模型。由此可知“级数$\sum_{n=0}^{\infty}\frac{1}{n!}$收敛于实数”独立于$\mathrm{HQ}_1$，这意味着无法在$\mathrm{HQ}_1$中从事分析学研究。因此要建立古典数学的统一理论，必须诉诸二阶语言。

## 3 HQ 的内在范畴性

显然四元数结构$\mathfrak{H} = \langle \mathbf{H}, +, \cdot, 0, 1, \mathrm{i}, \mathrm{j}, *, <\rangle$是HQ的模型，若想进一步说明HQ刻

画了$\mathfrak{H}$，还需证明HQ在同构意义上只有$\mathfrak{H}$这一个模型，即HQ具有范畴性。在此性质下，我们可以把$\mathfrak{H} \vDash \phi$等价地转化为$\vDash \mathrm{HQ} \to \phi$。然而标准语义下的二阶逻辑是不可完全的，我们无法进一步把“有效”（$\vDash$）概念等价转化为“可证”概念，从而古典数学的研究还不能归结为HQ中的形式证明。虽然可以为基本二阶演算（$\vdash_2$）构建可靠完全的 Henkin 语义，但是在此语义下 Löwenheim-Skolem 定理成立，使得HQ不可能有范畴性。

事实上所有语义概念只有转化为句法上证明概念才能被理解。既然二阶逻辑不可能有可靠完全的形式演算，我们只能在形式系统ZFC 中理解“$\vDash \mathrm{HQ} \to \phi$”。“HQ具有范畴性”的证明本质上也是在ZFC 中实现的：

$$\mathrm{ZFC} \vdash_1 \forall xy(\mathscr{L}_2^{\mathscr{H}}\text{-Str.}(x) \wedge \mathscr{L}_2^{\mathscr{H}}\text{-Str.}(y) \wedge x \vDash \mathrm{HQ}^{\mathrm{Set}} \wedge y \vDash \mathrm{HQ}^{\mathrm{Set}} \to x \cong y)$$

因此我们尚未实现脱离ZFC而只通过HQ为古典数学奠基的目标。

通过 Väänänen[8, p. 98] 提出的内在范畴性概念，上述目标其实可以在带概括公理（CA）的二阶演算中实现。这种内在主义思想源于 Putnam[9, p. 482]“模型是我们理论自身的造物”的主张，内在主义者不再将“HQ有模型”言说为集合论语句$\exists x(\mathscr{L}_2^{\mathscr{H}}\text{-Str.}(x) \wedge x \vDash \mathrm{HQ}^{\mathrm{Set}})$，而是用纯二阶语句$\exists H \exists \mathscr{V}\, \mathrm{HQ}^H(\mathscr{V}/\mathscr{H})$言说之，其中$\mathrm{HQ}^H(\mathscr{V}/\mathscr{H})$是将HQ中所有非逻辑常项统一替换为同类型的变元，再相对化到一元谓词$H$所得公式的合取。在这样的理解下会出现一种比范畴性更强的现象。

接下来的证明中需要处理部分函数，但二阶语言只允许量化全函数，为了方便，我们考虑将非逻辑符号集$\mathscr{H}$替换为$\mathscr{H}' = \{\mathrm{H}, \mathrm{S}, \mathrm{P}, 0, 1, \mathrm{i}, \mathrm{j}, \mathrm{K}, <\}$，其中$\mathrm{S}, \mathrm{P}, \mathrm{K}$是函数符$+, \cdot, *$相对应的关系符，H是一元谓词。然后将HQ改造为$\mathscr{L}_2^{\mathscr{H}'}$的语句集$\mathrm{HQ}'$：(1) 添加公理：$\mathrm{S}, \mathrm{P}, \mathrm{K}$具有函数性；(2) 将涉及函数符$+, \cdot, *$的公理等价地转换为关于关系符$\mathrm{S}, \mathrm{P}, \mathrm{K}$的公理（方法见[10], pp. 111-113）；(3) 将上述公理都相对化到H上，并添加公理：$\mathrm{H}0, \mathrm{H}1, \mathrm{Hi}, \mathrm{Hj}$。用$\mathrm{HQ}(\mathscr{V}_n)$表示把$\mathrm{HQ}'$中的所有$\mathscr{H}'$-符号统一用同类型的变元$\mathscr{V}_n = \{H_n, S_n, P_n, o_n, e_n, i_n, j_n, K_n, L_n\}$替换后所得公式的合取。

我们将证明如果$\mathrm{HQ}(\mathscr{V}_1) \wedge \mathrm{HQ}(\mathscr{V}_2)$成立，那么$\mathscr{V}_1$与$\mathscr{V}_2$“同构”，即存在关系$I$使得$\forall xy(Ixy \to H_1x \wedge H_2y) \wedge \forall x(H_1x \to \exists!\, yIxy) \wedge \forall y(H_2y \to \exists!\, xIxy) \wedge Io_1o_2 \wedge Ie_1e_2 \wedge Ii_1i_2 \wedge Ij_1j_2 \wedge \forall x_1x_2y_1y_2z_1z_2(Ix_1x_2 \wedge Iy_1y_2 \wedge Iz_1z_2 \to (S_1x_1y_1z_1 \leftrightarrow S_2x_2y_2z_2) \wedge (P_1x_1y_1z_1 \leftrightarrow P_2x_2y_2z_2) \wedge (K_1x_1y_1 \leftrightarrow K_2x_2y_2) \wedge (L_1x_1y_1 \leftrightarrow L_2x_2y_2))$，将这个长公式记为$\mathrm{ISOM}(I, \mathscr{V}_1, \mathscr{V}_2)$。

**定理.**（$\mathrm{HQ}'$的内在范畴性）

$$\mathrm{CA} \vdash_2 \forall \mathscr{V}_1 \forall \mathscr{V}_2(\mathrm{HQ}(\mathscr{V}_1) \wedge \mathrm{HQ}(\mathscr{V}_2) \to \exists I.\, \mathrm{ISOM}(I, \mathscr{V}_1, \mathscr{V}_2))$$

**证明：**首先定义$\mathscr{V}_n$中模拟自然数、整数、有理数、实数、复数的谓词如下：

$$N_nx \overset{\text{def}}{=} \forall X(Xo_n \wedge \forall yz{:}\, H_n(Xy \wedge S_nye_nz \to Xz) \to Xx),$$

$$Z_nx \overset{\text{def}}{=} N_nx \vee \exists y(N_ny \wedge S_nyxo_n),$$

$$Q_nx \overset{\text{def}}{=} \exists yz(Z_nz \wedge N_ny \wedge y \neq o_n \wedge P_nxyz),$$

$$R_nx \overset{\text{def}}{=} K_nxx，C_nx \overset{\text{def}}{=} \exists yz(R_ny \wedge R_nz \wedge \exists u{:}\, H_n(P_nzi_nu \wedge S_nyux))。$$

由于$0^* = 0, 1^* = 1$，由归纳法，对任意$n \in \mathbf{N}$，$(n+1)^* = n+1$，因此$\mathbf{N} \subseteq \mathbf{R}$；设

$n \in \mathbf{N}$，若$n + x = 0$，两边取共轭得$n + x^* = 0$，于是$x = x^*$，因此$\mathbf{Z} \subseteq \mathbf{R}$；设$m \in \mathbf{Z}, n \in \mathbf{N}^+$，若$xn = m$，两边取共轭得$nx^* = x^*n = m$，于是$x = x^*$，因此$\mathbf{Q} \subseteq \mathbf{R}$；据此可同理证明$Q_n \subseteq R_n$。又因为每个非实数的四元数形如$s + x\mathrm{i} + y\mathrm{j} + z(\mathrm{ij})$，其共轭为$s - x\mathrm{i} - y\mathrm{j} - z(\mathrm{ij})$，故在之后的证明中无需去验证构造的映射是否保持$K_n$。由于$\mathrm{HQ}(\mathscr{V}_n), \mathrm{CA} \vdash_2 \forall xy\colon H_n(L_n xy \leftrightarrow R_n x \wedge R_n y \wedge \exists zu\colon R_n(P_n zzu \wedge S_n xuy))$，所以在之后的证明中只要验证了构造的映射保持$S_n, P_n$就能保持$L_n$。（参见[11], p. 56，习题 2.2.23）

1° 令$\eta_N(F) \stackrel{\text{def}}{=} Fo_1o_2 \wedge \forall x_1y_1\colon N_1 \forall x_2y_2\colon N_2(S_1x_1e_1y_1 \wedge S_2x_2e_2y_2 \wedge Fx_1x_2 \wedge Fy_1y_2 \to Fz_1z_2)$，存在$I_N xy :\Leftrightarrow \forall F(\eta_N(F) \to Fxy)$。容易证明$\mathrm{HQ}, \mathrm{CA} \vdash_2 \mathrm{PA}_2$，由 Button & Walsh[12], pp. 243-245（定理 10.2），$I_N\colon N_1 \cong N_2$。

2° 令$\eta_Z(F) \stackrel{\text{def}}{=} I_N \subseteq F \wedge \forall x_1\colon Z_1 \forall y_1\colon N_1 \forall x_2\colon Z_2 \forall y_2\colon N_2(S_1y_1x_1o_1 \wedge S_1y_1x_1o_1 \wedge I_N y_1y_2 \to Fx_1x_2)$，存在$I_Z xy :\Leftrightarrow \forall F(\eta_Z(F) \to Fxy)$。由于整数的加法和乘法都归约为自然数的加法和乘法、取自然数的相反数，故$I_Z$显然保持$S_n, P_n$。存在$Gxy :\Leftrightarrow I_Z xy \wedge Z_1 x \wedge Z_2 y$，由于$Z_n \supseteq N_n$且对“取相反数”封闭，易得$\eta_Z(G)$，进而$I_Z$与$G$共外延。接着欲证明$\forall x\colon Z_1 \exists!\, y I_Z xy$：存在$Axy :\Leftrightarrow I_Z xy \wedge (N_1 x \to I_N xy) \wedge (\neg N_1 x \to \forall z(I_Z xz \to z = y))$，由于$I_N$是$N_1$与$N_2$间的双射，且每个整数只有唯一的相反数，故有$\forall x\colon Z_1 \exists!\, y Axy$，还因为非零自然数的相反数不是自然数（0的相反数是0），综上有$\eta_Z(A)$，进而$I_Z$与$A$共外延。同理可证$\forall y\colon Z_2 \exists!\, x I_Z xy$。以上就证明了$I_Z\colon Z_1 \cong Z_2$。

3° 令$\eta_Q(F) \stackrel{\text{def}}{=} I_Z \subseteq F \wedge \forall x_1\colon Q_1 \forall y_1z_1\colon Z_1 \forall x_2\colon Q_2 \forall y_2z_2\colon Z_2(y_1 \neq o_1 \wedge y_2 \neq o_2 \wedge P_1x_1y_1z_1 \wedge P_2x_2y_2z_2 \wedge I_Z y_1y_2 \wedge I_Z z_1z_2 \to Fx_1x_2)$，存在$I_Q xy :\Leftrightarrow \forall F(\eta_Q(F) \to Fxy)$。由于每个有理数都形如$pq^{-1}(p, q \in \mathbf{Z})$，$p_1q_1^{-1} + p_2q_2^{-1} = (p_1q_2 + p_2q_1)(q_1q_2)^{-1}$，$(p_1q_1^{-1})(p_2q_2^{-1}) = (p_1p_2)(q_1q_2)^{-1}$，归结为了整数的加法、乘法、除法，故$I_Q$显然保持$S_n, P_n$。鉴于$Q_n$对四则运算封闭，且整数的除法是二元函数，按照情形 2°中的方法可证$I_Q$是$Q_1$与$Q_2$间的双射，从而得$I_Q\colon Q_1 \cong Q_2$。

4° 令$\eta_R(F) \stackrel{\text{def}}{=} I_Q \subseteq F \wedge \forall x_1\colon R_1 \forall x_2\colon R_2(\neg Q_1x_1 \wedge x_2 = \sup I_Q\left[L_1\hat{y}_1x_1\right] \to Fx_1x_2)$，其中$I_Q\left[L_1\hat{y}_1x_1\right]y_2 \stackrel{\text{def}}{=} \exists y_1(Q_1y_1 \wedge L_1y_1x_1 \wedge I_Q y_1y_2)$，

$x_2 = \sup I_Q\left[L_1\hat{y}_1x_1\right] \stackrel{\text{def}}{=} \forall y_2\big(I_Q\left[L_1\hat{y}_1x_1\right]y_2 \to L_2y_2x_2 \vee y_2 = x_2\big) \wedge \forall z_2\colon R_2(L_2z_2x_2 \to \exists y_2(I_Q\left[L_1\hat{y}_1x_1\right]y_2 \wedge L_2z_2y_2)$。

由$\mathrm{HQ}', \mathrm{CA} \vdash_2$Archimedes 性质，$\forall y_1\colon Q_1(L_1y_1x_1 \to (\exists z_1\colon Q_1)L_1x_1z_1)$，由$I_Q\colon Q_1 \cong Q_2$，$\exists z_2\colon Q_2(I_Q z_1z_2 \wedge \forall y_2(I_Q\left[L_1\hat{y}_1x_1\right]y_2 \to L_2y_2z_2))$，因此$\sup I_Q\left[L_1\hat{y}_1x_1\right]$存在。由CA，存在$I_R xy :\Leftrightarrow \forall F(\eta_R(F) \to Fxy)$。

存在$Gxy :\Leftrightarrow I_R xy \wedge R_1 x \wedge R_2 y$，由于$R_n \supseteq Q_n$且对“取上确界”封闭，易得$\eta_R(G)$，进而$I_R$与$G$共外延。

根据基本的数学分析知识（见[11], p. 55，习题 2.2.15），实数的确界原理等价于 Dedekind 分割原理（“$\mathrm{HQ}', \mathrm{CA} \vdash_2$”中可证），则每个实数$r$对应分割$D_r \stackrel{\text{def}}{=} \{q \in \mathbf{Q} \mid q \leq r\}$。而$r + r' = \sup D_r + \sup D_{r'} = \sup\{\, q + q' \mid q \in D_r, q' \in D_{r'} \,\} = \sup D_{r+r'}$；

$-\sup D_r = \inf\{\,-q \mid q \in D_r\,\} = -r = \sup D_{-r}$；设$r, r' \geq 0$，$rr' = \sup D_r \cdot \sup D_{r'} = \sup\{\,qq' \mid 0 \leq q \in D_r, 0 \leq q' \in D_{r'}\,\} = \sup D_{rr'}$，$(-r)r' = -(rr')$，$(-r)(-r') = rr'$。据此易证$I_R$保持$S_n, P_n$。

下面证$\forall x\colon R_1 \exists!\, y I_R xy$：存在$Axy :\Leftrightarrow I_R xy \wedge (Q_1 x \to I_Q xy) \wedge (\neg Q_1 x \to \forall z(I_R xz \to z = y))$，由于$I_Q$是$Q_1$与$Q_2$间的双射，且有上界有理数的上确界是唯一的，故有$\forall x\colon R_1 \exists!\, y Axy$，还因为$I_Q$保持$L_n$，故若$I_Q xy$，则$y = \sup\{v \mid I_Q uv \wedge L_1 ux\}$，综上有$\eta_R(A)$，进而$I_R$与$A$共外延。同理可证$\forall y\colon R_2 \exists!\, x I_R xy$。

以上就证明了$I_R\colon R_1 \cong R_2$。（此情形的证明思路参考[13], p. 84）

5° 令$\eta_C(F) \overset{\text{def}}{=} I_R \subseteq F \wedge F i_1 i_2 \wedge \forall x_1 y_1\colon R_1 \forall z_1\colon C_1 \forall x_2 y_2\colon R_2 \forall z_2\colon C_2 (I_R x_1 y_2 \wedge I_R x_1 x_2 \wedge \exists u_1\colon H_1(P_1 y_1 i_1 u_1 \wedge S_1 x_1 u_1 z_1) \wedge \exists u_2\colon H_2(P_2 y_2 i_2 u_2 \wedge S_2 x_2 u_2 z_2) \to F z_1 z_2)$，存在$I_C xy :\Leftrightarrow \forall F(\eta_C(F) \to Fxy)$。由于每个复数都可唯一表示为$a + b\mathrm{i}(a, b \in \mathbf{R})$，依情形 3° 中的方法可证$I_C\colon C_1 \cong C_2$。

6° 令$\eta_H(F) \overset{\text{def}}{=} I_C \subseteq F \wedge F j_1 j_2 \wedge \forall x_1 y_1\colon C_1 \forall z_1\colon H_1 \forall x_2 y_2\colon C_2 \forall z_2\colon H_2 (I_C x_1 y_2 \wedge I_C x_1 x_2 \wedge \exists u_1\colon H_1(P_1 y_1 j_1 u_1 \wedge S_1 x_1 u_1 z_1) \wedge \exists u_2\colon H_2(P_2 y_2 j_2 u_2 \wedge S_2 x_2 u_2 z_2) \to F z_1 z_2)$，存在$I_H xy :\Leftrightarrow \forall F(\eta_H(F) \to Fxy)$。由于每个四元数都形如$c + d\mathrm{j}(c, d \in \mathbf{C})$，与情形 5° 同理可证$I_H\colon H_1 \cong H_2$。

综上就证明了$\mathrm{ISOM}(I_H, \mathscr{V}_1, \mathscr{V}_2)$。∎

Väänänen[8], p. 98 认为内在范畴性表明了如下事实：任给一个CA的 Henkin 模型$\langle M, \mathscr{K}\rangle$（其中$\mathscr{K}(\sigma)$为变元$v^\sigma$的取值范围），欲将其膨胀为HQ′的模型$\langle M, \mathscr{K}, \mathscr{H}'^M\rangle$，在同构的意义上只有唯一的膨胀方式。而范畴性不过是取CA的满Henkin模型（即$\sigma$为$n$元谓词的类型时，$\mathscr{K}(\sigma) = \wp(M^n)$；$\sigma$为$n$元函数的类型时，$\mathscr{K}(\sigma) = M^{M^n}$）时的特殊情形。如此我们就在二阶逻辑的一个递归可公理化片段内证明了HQ是四元数理论的正确公理化。

内在范畴性定理本质地确证了 Hilbert“公理确定的结构关系先于对象本体”（点线面可代之以桌椅啤酒杯）的形式主义主张，支持了 Putnam“对象虽相对于概念框架存在，但理论的客观性并不会丧失”的内在主义观点。并且这个定理是纯句法、纯逻辑地证明的，不涉及任何语义上溯，让我们真正免于回答四元数在本体论意义上是什么的问题，而只需聚焦四元数存在所依凭的概念框架。

## 4 古典数学的确定性

设$\langle M_1, \mathscr{K}_1, \mathscr{H}'^{M_1}\rangle$和$\langle M_2, \mathscr{K}_2, \mathscr{H}'^{M_2}\rangle$是HQ′的两个模型，内在范畴性并不能保证$\mathscr{H}'^{M_1}$与$\mathscr{H}'^{M_2}$同构甚至初等等价，HQ′的非标准 Henkin 模型始终是避免不了的。鉴于此 Button & Walsh[12], p. 232 认为内在范畴性本身的意义比较模糊，其最大价值在于导出内在不容忍性：

**系理.**（HQ′的内在不容忍性）设$\phi^H(\mathscr{V})$是自由变元只有$\mathscr{V}$且无函数变元并相对化到$H$的任意$\mathscr{L}_2^{\varnothing}$-公式，则在有穷元数学中可证

$$\mathrm{CA} \vdash_2 \forall \mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \phi^H(\mathscr{V})) \vee \forall \mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \neg\phi^H(\mathscr{V}))$$

**证明：**只需要施归纳于$\phi^H(\mathscr{V}, \vec{x}, \vec{X})$的结构去证内同构引理：

$$\mathrm{CA} \vdash_2 \forall \mathscr{V}_1 \mathscr{V}_2(\mathrm{ISO}(I, \mathscr{V}_1, \mathscr{V}_2) \to \forall \vec{x}, \vec{X}\colon H_1(\phi^H(\mathscr{V}_1, \vec{x}, \vec{X}) \leftrightarrow \phi^H(\mathscr{V}_2, \overrightarrow{I`x}, \overrightarrow{I``X}))$$

其中$I`x = (\iota y)Ixy$，$\forall y_1 \ldots y_n((I``X)y_1 \ldots y_n \leftrightarrow \exists x_1 \ldots x_n(Xx_1 \ldots x_n \wedge Ix_1y_1 \wedge \ldots \wedge Ix_ny_n))$。Button & Walsh[12], p. 245-246（定理 10.3）实质上已经证明了此结果。∎

不容忍性本是模糊性理论中的术语，之所以在此使用，是因为 Button[14], p. 170 将HQ($\mathscr{V}$)诠释为超赋值理论中的精确化方式，于是内在不容忍性便断言：$\phi^H(\mathscr{V})$要么超真，要么超假。鉴于超真/超假在模糊性理论中对应“确定成立/确定不成立”，Button 又将$\forall \mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \phi^H(\mathscr{V}))$读作“$\phi$是确定的”，从而内在不容忍性表达了古典数学是确定的。

内在不容忍性并未比“对任意$\mathscr{L}_2^{\mathscr{H}}$语句$\phi$都有，$\mathrm{HQ}' \vDash \phi$或$\mathrm{HQ}' \vDash \neg\phi$”提供更多信息，其进步性只体现在，它和内在范畴性一样是免于模型论的纯逻辑结论（并且“$\mathrm{CA} \vdash_2$”的一致性不会强于$\mathrm{PA}_1$），无需牵扯二阶语义特别是与“任意子集”有关的问题。但是通常认为的确定性是在能行可靠的二阶演算D中有$\mathrm{HQ}' \vdash_{\mathrm{D}} \phi$或$\mathrm{HQ}' \vdash_{\mathrm{D}} \neg\phi$，即必须有能行手段判定$\phi$是否成立，但Gödel定理说一定存在独立于HQ′的语句。然而这种独立性并不是HQ′公理不足造成的，而是因为D的公理不足。为了统一此处涉及的完全性概念，我们将“$\mathrm{HQ}' \vdash_{\mathrm{D}}$”换为与之等价的“$\mathrm{HQ}', \Delta \vdash_2$”（$\Delta$可判定且$\vDash \Delta$）。不完全现象不是由于HQ′未能在同构的意义上捕获标准四元数结构，实是因为逻辑公理集$\Delta$不足以将模型固定为完整的 Henkin 结构。Kreisel[15], pp. 150-151 早就指出需要区分独立性产生的原因，例如连续统假设CH和 Fraenkel 替换公理Repl均独立于二阶 Zermelo 系统“$\mathrm{Z}_2, \mathrm{CA} \vdash_2$”；但是$\mathrm{Z}_2 \vDash \mathrm{CH}$或$\mathrm{Z}_2 \vDash \neg\mathrm{CH}$，$\mathrm{Z}_2 \nvDash \mathrm{Repl}$且$\mathrm{Z}_2 \nvDash \neg\mathrm{Repl}$，这是因为$\mathrm{Z}_2$的公理足以捕获累积分层结构$V_{\omega+\omega}$，但不够捕获更高的$V_\kappa$（$\kappa$是第一个不可达基数）。

在这个意义上，内在主义确实可以解决 Button 命名的 Skolem-Gödel 二律背反——“我们的数学概念具有完美的精确性。然而这些完美精确的数学概念需要通过一个形式理论完全获取和呈现，而该形式理论又要通过一个算法可检查的证明系统来理解，因此它是不完全的。”[14], p. 166——在适当强度的二阶演算中，算法可检查的形式系统HQ′“精确捕获了四元数概念”与“不完全”二者并存，不存在矛盾。内在不容忍性所表达的确定性更多是在前者的意义上言说的，诚如 Maddy & Väänänen[16], p. 47 所言，Button & Walsh 并非试图确立那些独立的数学陈述的确定性，而是试图消除人们认为其不确定性的倾向。

A. Bacon 的一项新近工作（[17]，定理 C.2）在高阶模态逻辑中建立了实数理论的内在必然不容忍性，而此结果可以原封不动地简单推广到四元数理论。

考虑非逻辑符号集为$\mathscr{H}'$的三阶模态语言$\mathscr{L}_{3\Box}^{\mathscr{H}'}$。该语言的变元有个体变元、命题变元、n 元谓词变元、一元二阶谓词变元。项为$t ::= x \mid 0 \mid 1 \mid \mathrm{i} \mid \mathrm{j}$，公式为

$$\phi ::= \alpha \mid Xt_1 \ldots t_n \mid \Xi X \mid t_1 = t_2 \mid \mathrm{H}t \mid \mathrm{S}t_1t_2t_3 \mid \mathrm{P}t_1t_2t_3 \mid \mathrm{K}t_1t_2 \mid t_1 < t_2 \mid \neg\phi$$
$$\mid (\phi_1 \to \phi_2) \mid \forall x\phi \mid \forall X\phi \mid \forall \Xi\phi \mid \forall \alpha\phi \mid \Box\,\phi$$

$\mathscr{L}_{3\Box}^{\mathscr{H}'}$的形式演算TOM由如下公理和规则构成：(1) 命题逻辑重言式；(2) 模态逻辑的 K、T、5 公理；(3) $\vdash_{\text{TOM}} \forall v^{\sigma}\phi \to \phi(u^{\sigma}/v^{\sigma})$；(4) $\vdash_{\text{TOM}} \forall xy(x = y \leftrightarrow \forall X(Xx \leftrightarrow Xy)$；(5) 若$v^{\sigma}$不在$\phi$中自由，则由$\vdash_{\text{TOM}} \phi \to \psi$可得$\vdash_{\text{TOM}} \phi \to \forall v^{\sigma}\psi$；(6) 由$\vdash_{\text{TOM}} \phi$可得$\vdash_{\text{TOM}} \Box\phi$；(7) $\vdash_{\text{TOM}} \exists x(x = t)$；(8) CA：$\vdash_{\text{TOM}} \exists v^{\sigma\to\tau}\forall\vec{s}(v^{\sigma\to\tau}\vec{s} \leftrightarrow \phi)$，其中$v^{\sigma\to\tau}$不在$\phi$中出现，$\vec{s}$是合适的变元序列，特别地，当$v$是命题变元（即类型为$\tau$）时，$\vec{s}$为空序列。

由模态谓词逻辑的基本知识可知 Barcan 公式及其逆$\forall v^{\sigma}\Box \leftrightarrow \Box\forall v^{\sigma}$，必然相同$\forall xy(x = y \to \Box\, x = y)$和必然相异公式$\forall xy(x \neq y \to \Box\, x \neq y)$都是TOM的可证式。

内在必然不容忍性建立在如下形而上学假设上

$$\Box\text{AW}：\Box\exists\alpha(\alpha \wedge \forall\beta(\Box(\alpha \to \beta) \vee \Box(\alpha \to \neg\beta)))，$$

其直观意思是，必然存在一个现实世界@，即见证了$\Box$AW的那个命题。Wittgenstein 说，世界是事实的总体，而事实又是诸事态的存在，事态的存在和不存在即是实在，全部实在即是世界。（[18], 1.1, 2, 2.06, 2.063）而事态的存在与不存在是用原子公式和其否定来表达，而满足$\forall\beta(\Box(\alpha \to \beta) \vee \Box(\alpha \to \neg\beta))$的那个命题$\alpha$确定了它所严格蕴涵的全部实在，如果还有$\Diamond\alpha$成立，那么用$\alpha$来表征一个可能世界是合 Wittgenstein 思想的。而@本身成立，因此表征的是现实世界。对此思想进行拓展，还可以用谓词表征一个可能世界：

$$\text{World}(w^{\sigma\to\tau}) \overset{\text{def}}{=} \Diamond\exists\vec{s}w^{\sigma\to\tau}\vec{s} \wedge \forall u^{\sigma\to\tau}(\Box\forall\vec{s}(w^{\sigma\to\tau}\vec{s} \to u^{\sigma\to\tau}\vec{s}) \vee \Box\forall\vec{s}(w^{\sigma\to\tau}\vec{s} \to \neg u^{\sigma\to\tau}\vec{s}))$$

当$\text{World}(w^{\sigma\to\tau})$时，$\Box\forall\vec{s}(w^{\sigma\to\tau}\vec{s} \to u^{\sigma\to\tau}\vec{s})$的作用相当于$w^{\sigma\to\tau} \vDash u^{\sigma\to\tau}$。

$\Box$AW算是一个较合理的假设，引入它的目的是导出如下两个必要的引理。

**引理**（[19], pp. 131, 135，命题 2.6, 2.10）. 在TOM中$\Box$AW蕴涵如下两个句子：

$\Box\text{RC}：\Box\forall X\exists Y(\Box\forall Z(\forall\vec{x}(Y\vec{x} \to \Box Z\vec{x}) \leftrightarrow \Box\forall\vec{x}(Y\vec{x} \to Z\vec{x})) \wedge \forall\vec{x}(X\vec{x} \leftrightarrow Y\vec{x}))$

$\text{LP}：\forall v^{\sigma\to\tau}(\Diamond\exists\vec{s}v^{\sigma\to\tau}\vec{s} \to \exists w^{\sigma\to\tau}(\text{World}(w^{\sigma\to\tau}) \wedge \Box\forall\vec{s}(w^{\sigma\to\tau}\vec{s} \to v^{\sigma\to\tau}\vec{s})))$

**证明**：$\Box$RC：任给$X$，由CA可取满足$\forall\vec{x}(Y\vec{x} \leftrightarrow \Box(@ \to X\vec{x}))$的$Y$，验证即可。

LP：设$\Diamond\exists\vec{s}v^{\sigma\to\tau}\vec{s}$，由 Barcan 公式，$\exists\vec{s}\Diamond v^{\sigma\to\tau}\vec{s}$。接着使用存在消去推理，取这样存在的$\vec{s}$，由CA取满足$\forall\vec{x}(w^{\sigma\to\tau}\vec{x} \leftrightarrow @ \wedge \vec{x} = \vec{s})$的$w^{\sigma\to\tau}$，易验证$w^{\sigma\to\tau}$符合要求。于是由$\Diamond(v^{\sigma\to\tau}\vec{s} \wedge \text{AW})$得到了$\Diamond\exists w^{\sigma\to\tau}(\text{World}(w^{\sigma\to\tau}) \wedge \Box\forall\vec{s}(w^{\sigma\to\tau}\vec{s} \to v^{\sigma\to\tau}\vec{s}))$，由 Barcan 公式，$\exists w^{\sigma\to\tau}(\Diamond\text{World}(w^{\sigma\to\tau}) \wedge \Diamond\Box\forall\vec{s}(w^{\sigma\to\tau}\vec{s} \to v^{\sigma\to\tau}\vec{s}))$，公式中的第二个$\Diamond$由 5 公理消去，第一个$\Diamond$则因 Barcan 公式易证$\Diamond\text{World}(w^{\sigma\to\tau}) \to \text{World}(w^{\sigma\to\tau})$。■

$\Box$RC称为刚性概括公理，它断言每个谓词都必然有与之共外延的刚性谓词。谓词$X$是刚性的，即$\text{Rigid}(X) \overset{\text{def}}{=} \Box\forall Z(\forall\vec{x}(Y\vec{x} \to \Box Z\vec{x}) \leftrightarrow \Box\forall\vec{x}(Y\vec{x} \to Z\vec{x}))$，由 T 公理，$\text{Rigid}(X) \to \Box\forall\vec{x}(X\vec{x} \leftrightarrow \Box X\vec{x})$，故所谓刚性即谓词的外延在所有可能世界始终相同（试类比 Barcan 公式）。LP 即 Leibniz 可能性条件：可能就是在某个可能世界上成立。

**定理**.（HQ′的内在必然不容忍性）设$\varphi(\mathscr{V})$是自由变元只有$\mathscr{V}$且无命题变元、二阶谓词变元、模态词并相对化到$H$的任意$\mathscr{L}_{3\Box}^{\varnothing}$-公式，则

$$\Box\text{AW} \vdash_{\text{TOM}} \Box\forall\mathscr{V}(\text{HQ}(\mathscr{V}) \to \varphi(\mathscr{V})) \vee \Box\forall\mathscr{V}(\text{HQ}(\mathscr{V}) \to \neg\varphi(\mathscr{V}))$$

**证明：** 首先申明，易知对于无二阶谓词变元和模态词的公式$\phi$，外延原则对其成立，即如果$\forall x(Xx \leftrightarrow Yx)$，那么$\phi \leftrightarrow \phi(Y/X)$。
反证法。假设$\Diamond\exists\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \wedge \neg\varphi(\mathscr{V})) \wedge \Diamond\exists\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \wedge \varphi(\mathscr{V}))$。则有$\Diamond\exists\mathscr{V}\mathrm{HQ}(\mathscr{V})$。下将证明$\Diamond\exists\mathscr{V}\mathrm{HQ}(\mathscr{V}) \to \exists\mathscr{V}\mathrm{HQ}(\mathscr{V})$，等价于证$\forall\mathscr{V}\neg\mathrm{HQ}(\mathscr{V}) \to \Box\forall\mathscr{V}\neg\mathrm{HQ}(\mathscr{V})$。由$\Box$RC，不妨只考虑那些刚性的$\mathscr{V}$。逐条检查HQ′的公理，由$\forall xy(x \neq y \to \Box x \neq y)$和$\mathscr{V}$中元素是刚性的（即谓词的外延在所有可能世界相同）易知HQ′的(I)(II)(III)组公理符合欲证命题；唯一非平凡的只有实数的确界原理，此证明过程参见（[17], Appx. pp. 18-19，引理 C.5）。
于是存在刚性的$\mathscr{V}$使得$\mathrm{HQ}(\mathscr{V})$成立。下使用存在消去推理。任取一个这样的$\mathscr{V}$，由于它刚性，不妨就用常项集$\mathscr{H}'$指代之。称$\varphi(\mathscr{H}')$为分析公式，施归纳于分析公式$\varphi$的结构同时证明：$\forall\vec{X}:\mathrm{H}\forall\vec{x}:\mathrm{H}(\varphi \to \Box\varphi) \wedge \forall\vec{X}:\mathrm{H}\forall\vec{x}:\mathrm{H}(\neg\varphi \to \Box\neg\varphi)$。
1° 当$\varphi$为原子公式时，由$\mathscr{H}'$中的谓词都刚性，$\Box$RC得必然存在与$X$共外延的刚性谓词变元，必然相同和必然相异公式知，命题成立。
2° 当$\varphi$为布尔公式时，由归纳假设和一些基本的重言式以及 K 公理易得。
3° 当$\varphi$形如$\forall x(\mathrm{H}x \to \psi)$时，由归纳假设得$\forall x(\mathrm{H}x \to \Box\psi)$，H刚性则$\Box\forall x(\mathrm{H}x \to \psi)$；若$\neg\forall x(\mathrm{H}x \to \psi)$，即$\exists x(\mathrm{H}x \wedge \neg\psi)$，由归纳假设和H刚性，$\exists x(\Box\mathrm{H}x \wedge \Box\neg\psi)$，进而可得$\Box\neg\forall x(\mathrm{H}x \to \psi)$。
4° 将$\forall x_1 \ldots x_n(Xx_1 \ldots x_n \to \mathrm{H}x_1 \wedge \ldots \wedge \mathrm{H}x_n)$简记为$X \subseteq \mathrm{H}^n$。当$\varphi$形如$\forall X(X \subseteq \mathrm{H}^n \to \psi)$时，去证$\varphi$成立时$\Box\varphi$也成立。若不然假设有$\Diamond\exists X(X \subseteq \mathrm{H}^n \wedge \neg\psi)$，则由$\Box$RC，对任意$X$都必然存在刚性的$X'$与$X$共外延，于是有$\Diamond\exists X(X \subseteq \mathrm{H}^n \wedge \mathrm{Rigid}(X) \wedge \neg\psi)$。由CA，存在$\Xi$使得$\forall X(\Xi X \leftrightarrow X \subseteq \mathrm{H}^n \wedge \mathrm{Rigid}(X) \wedge \neg\psi)$，由LP，存在$\Upsilon$使得$\mathrm{World}(\Upsilon) \wedge \Box\forall Y(\Upsilon Y \to \Xi Y)$，于是有$\Box\forall Y(\Upsilon Y \to Y \subseteq \mathrm{H}^n \wedge \mathrm{Rigid}(Y))$。由 CA 和$\Box$RC，存在刚性的$G$使得$\forall x_1 \ldots x_n(Gx_1 \ldots x_n \leftrightarrow \mathrm{H}x_1 \wedge \ldots \wedge \mathrm{H}x_n \wedge \Box\forall Y(\Upsilon Y \to Yx_1 \ldots x_n))$。由（[17], Appx. p. 19，引理 C.6）有$\Box\forall Y(\Upsilon Y \to \forall\vec{x}(Y\vec{x} \leftrightarrow G\vec{x}))$。于是有$\Box\forall Y(\Upsilon Y \to \Xi Y \wedge \forall\vec{x}(Y\vec{x} \leftrightarrow G\vec{x}))$，又因$\mathrm{World}(\Upsilon)$有$\Diamond\exists Y\Upsilon Y$，得$\Diamond\exists Y(\Xi Y \wedge \forall\vec{x}(Y\vec{x} \leftrightarrow G\vec{x}))$，最终借助 T 公理和 Barcan 公式得$\exists Y(Y \subseteq \mathrm{H}^n \wedge \mathrm{Rigid}(Y) \wedge \Diamond\neg\psi)$。但是由$\varphi$和归纳假设有$\forall X(X \subseteq \mathrm{H}^n \wedge \mathrm{Rigid}(X) \to \Box\psi)$，矛盾！
同理可证$\neg\varphi \to \Box\neg\varphi$。引理证毕！
由上述引理得$\mathrm{HQ}' \to \Box\mathrm{HQ}'$。由内在范畴性定理和内同构引理得

$$\vdash_{\mathrm{TOM}}\Box\forall\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \wedge \mathrm{HQ}' \to (\varphi \leftrightarrow \varphi(\mathscr{V}))$$

进而有$\Box\mathrm{HQ}' \wedge \Box\varphi \to \Box\forall\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \varphi(\mathscr{V}))$，而又有$\mathrm{HQ}' \wedge (\varphi \vee \neg\varphi) \to \Box\mathrm{HQ}' \wedge (\Box\varphi \vee \Box\neg\varphi)$，于是$\mathrm{HQ}' \wedge (\varphi \vee \neg\varphi) \to \Box\forall\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \varphi(\mathscr{V})) \vee \Box\forall\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \neg\varphi(\mathscr{V}))$，与$\exists\mathscr{V}\mathrm{HQ}(\mathscr{V})$一道使用存在消去规则，立得$\Box\forall\mathscr{V}\big(\mathrm{HQ}(\mathscr{V}) \to \varphi(\mathscr{V})\big) \vee \Box\forall\mathscr{V}(\mathrm{HQ}(\mathscr{V}) \to \neg\varphi(\mathscr{V}))$，与假设矛盾！∎

内在必然不容忍性进一步表明古典数学具有必然的确定性。但这个加强的结果对内在主义的哲学主张并没有太多增益，因为这个定理实质上假设了每个四元数彼此必然不同的以及数学谓词都是刚性后容易预见的结果。

## 5 古典数学的真理性

Button[14], p. 171 特别指出，内在范畴性及内在（必然）容忍性有意义的前提是，$\exists\mathscr{V}HQ(\mathscr{V})$成立。用Hilbert的观点来说，要保证HQ是一致的，谈论HQ的范畴性才有数学价值。据 Gödel 定理，在纯逻辑中证明$\exists\mathscr{V}HQ(\mathscr{V})$是不可能的，并且只可能在一致性强度大于HQ的形式系统中才能证明$\exists\mathscr{V}HQ(\mathscr{V})$；而据Russell & Whitehead在《数学原理》中的工作，假设无穷公理的前提下，在高于二阶的逻辑中才能证明$\exists\mathscr{V}HQ(\mathscr{V})$。既如此，直接承认HQ才是最经济的选择。

那么HQ何以为真？约定使然！Quine 曾说：

> 一门科学的发展程度越低，其术语就越倾向于建立在未经批判的相互理解假设之上。随着严谨性的提高，这一基础会逐渐被定义的引入所取代。为这些定义所构建的相互关系获得了分析性原理的地位；曾经被视为关于世界的理论，如今被重新解读为语言约定。因此，从理论到约定的某种转变，是任何科学在逻辑基础方面取得进步的一个附随现象。[20], p. 329

起初人们从计数活动中抽象出自然数概念，从度量活动及对连续时空的直观中抽象出实数概念，又从力学中抽象出了矢量概念，Hamilton 创造性地发明了四元数统合了这些概念。但彼时的四元数依旧只是存于直观中的非形式概念，Hamilton是在对四元数的直观理解下，推导并记录下了四元数理应满足的诸多性质。所谓公理化乃是一个“马后炮”的过程，HQ精心挑选了那些被既有数学实践公认为反映了四元数本性的语句，试图将那些关于四元数的证明进行精确化处理。范畴性论证无非表明HQ确实精确完整地描述了数学家直观中的四元数概念。正是在这个意义上HQ反过来定义了什么是四元数，至此我们可以剥离四元数与外在世界的关系，在原则上可以完全脱离那有些飘渺的直观去研究四元数，这就是 Quine 所说的理论在逻辑基础方面取得了巨大进步，亦是 Kreisel[15]所讲的“非形式严格”的成功案例。HQ是约定的，因而是分析的，从而整个古典数学都是分析性真理；但同时这种约定又不是任意的，它是依循数学实践的发现，而非随意的发明。

约定论最大的困难是不完全定理，但在第 4 节中我们已经回应了这一问题，不完全不是HQ导致的，而是逻辑造成的。结合内在主义的观点我们可以宣布，HQ配合带完整概括公理的二阶演算能自足地为古典数学奠定坚实的基础。

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# Axiomation of Classical Mathematics and Its Determinacy

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**Abstract**

This paper presents a second-order axiomatic theory HQ for quaternions. It proves that HQ possesses internal categoricity and internal intolerance within second-order calculus incorporating the full comprehension axiom. Furthermore, within a modal third-order logic system TOM, under the premise of "it is necessary that there is an actual world," it demonstrates that HQ exhibits internal necessary intolerance. Employing the methodology of the axiomatic school, this paper establishes the foundations of classical mathematics (as opposed to modern mathematics) upon HQ. By leveraging internalism, it overcomes the Skolem-Gödel antinomy, thereby substantiating the certainty of classical mathematics. Additionally, it employs conventionalism to argue for the truth of classical mathematics.

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