Forcing and Its Philosophy

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Outline

1. The Idea of Forcing
2. The Notion of Forcing
3. Forcing and Combinatorics
4. Forcing and Modal Logic

Independence

Let us briefly recall what it means for a sentence φ to be independent of ZFC.

Syntactics and Semantics

- From a syntactical point of view it means neither φ nor its negation is provable from ZFC;
- From a semantical point of view it means that there are models of ZFC in which φ holds and some in which φ fails.

Independent

In sum, φ is independent of ZFC if and only if φ as well as its negation is consistent with ZFC(i.e.,$ZFC + \varphi$ as well as $ZFC + \varphi$ has a model).

Consistency

In order to prove that a given sentence φ is consistent with ZFC, we have to show that $ZFC + \varphi$ is consistent–tacitly assuming the consistency of ZFC.

Two Approaches

- One could apply the Compactness Theorem and show that whenever $ZFC^* \subseteq ZFC$ is a finite set of axioms, then $ZFC^* + \varphi$ has a model(i.e., $ZFC^* + \varphi$ is consistent);
- Starting from a model of ZFC, one could construct directly a model of $ZFC + \varphi$.

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Current Forcing Notation

- In the 'countable transitive model' approach, we refer to M as the ground model,and M[G] as the generic extension;
- In the 'forcing over the universe' approach, we refer to V as the ground model, and V[G] as the generic extension.

How to extend models?

Next we present a general technique, called forcing, for extending models of ZFC.

The main ingredients

- The main ingredients to construct such an extention are a model V of ZFC(e.g., V = L), a partially ordered set $\mathbb{P} = (P, \leq)$ contained in V, as well as a special subset G of P which will not belong to V.
- The extended model V[G] will then consist of all sets which can be "described" or "named" in V, where the "naming" depends on the set G.

The main task

The main task will be to prove that V[G] is a model of ZFC as well as to decide(within V) whether a given statement is true or false in a certain extension V[G].

People living in universe

- To get an idea of how this is done, think for a moment that there are people living in V.
- For these people, V is the unique set-theoretic universe which contains all sets.

The key point

Now, the key point is that for any statement, these people are able to compute whether the statement is true or false in a particular extension V[G], even though they have almost no information about the set G(in fact, they would actually deny the existence of such a set).

Expositions of Forcing

- The method of forcing was introduced by Cohen[4,5], although the treatment in terms of posets and filters(or ideals) is due to Solovay.
- Modern expositions of forcing owe a lot to the exposition in Shoenfield's [6].

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Jensen' work

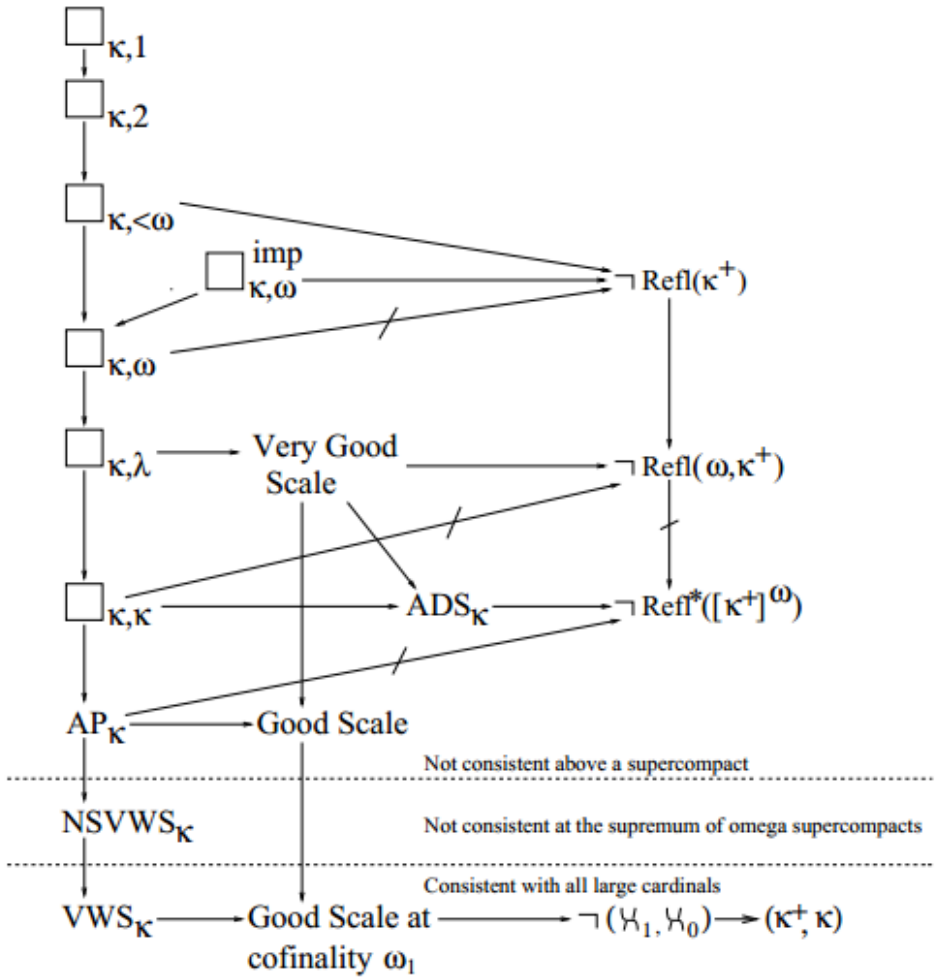
- Around 1970, Jensen discovered that Godel's Axiom of Constructibility V = L implies $\neg SH$ (there exists a Suslin tree).
- He then defined a simple set-theoretic principle, $\diamondsuit$, and proved that $V = L \rightarrow \diamondsuit$ and $\diamondsuit \rightarrow \neg SH$.
- This $\diamondsuit$ now has many other applications besides the Suslin problem.

Solovay's work

- Solovay showed that $V = L \rightarrow KH$ (there exist a Kurepa tree).
- This proof led to the stronger principle $\diamondsuit^+$ and proofs that $V = L \rightarrow \diamondsuit^+$ and $\diamondsuit^+ \rightarrow KH$.

Forcing and diamond principle

- Using forcing, one can show that $GCH + \diamondsuit$ does not imply KH.
- Also, GCH does not imply $\neg SH$; this was first proved by Jensen (see [7]), and later, using a different method, by Shelah [8].


κ,1
κ,2
κ,<ω
imp κ,ω
¬Refl(κ+)
κ,ω
κ,λ
Very Good Scale
¬Refl(ω,κ+)
κ,κ
ADSκ
¬Refl*([κ+]ω)
APκ
Good Scale
Not consistent above a supercompact
NSVWSκ
Not consistent at the supremum of omega supercompacts
Consistent with all large cardinals
VWSκ
Good Scale at cofinality ω1
¬(ℵ1,ℵ0)→(κ+,κ)
κ strong limit of cofinality ω

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Models of set theory and Kripke model

- Because the ground model V has some access via names and the forcing relation to the objects and truths of the forcing relation V[G], there are clear affinities between forcing and modal logic.
- One might even imagine the vast collection of all models of set theory, related by forcing, as an enormous Kripke model.

Modal operators

- Accordingly, we define that a statement of set theory φ is forceable or possible if φ holds in some forcing extension, and φ is necessary if it holds in all forcing extensions.
- The modal notations $\Diamond\ \varphi$ and $\Box\ \varphi$ express, respectively, that φ is possible or necessary.

Eliminable

- The forcing modal operators are eliminable in the language of set theory, because their meaning can be expressed in the usual language of set theory by means of the forcing relation or Boolean values.

- $\Diamond\ \varphi$ simply means that there is some partial order $\mathbb{P}$ and condition $p\ \in\ \mathbb{P}$ such that $p\ \Vdash_{\mathbb{P}}\ \varphi$, and $\Box\ \varphi$ means that for all partial orders $\mathbb{P}$ and $p\ \in\ \mathbb{P}$ we have $p\ \Vdash_{\mathbb{P}}\ \varphi$.

Main definition 1

Main Definition 1. A Modal assertion $\varphi(q_0, ..., q_n)$ is a valid principle of forcing if for all sentences ψ_i in the language of set theory, $\varphi(\psi_0, ..., \psi_n)$ holds under the forcing interpretation of $\Diamond$ and $\Box$.

Main Question 2

Main Definition 2. What are the valid principles of forcing?

$$
\begin{aligned}
\mathsf{S5} &= \mathsf{S4} + 5 \\
\mathsf{S4W5} &= \mathsf{S4} + \mathrm{W5} \\
\mathsf{S4.3} &= \mathsf{S4} + .3 \\
\mathsf{S4.2.1} &= \mathsf{S4} + .2 + \mathrm{M} \\
\mathsf{S4.2} &= \mathsf{S4} + .2 \\
\mathsf{S4.1} &= \mathsf{S4} + \mathrm{M} \\
\mathsf{S4} &= \mathsf{K4} + \mathrm{S} \\
\mathsf{Dm.2} &= \mathsf{S4.2} + \mathrm{Dm} \\
\mathsf{Dm} &= \mathsf{S4} + \mathrm{Dm} \\
\mathsf{Grz} &= \mathsf{K} + \mathrm{Grz} = \mathsf{S4} + \mathrm{Grz} \\
\mathsf{GL} &= \mathsf{K4} + \text{Löb} \\
\mathsf{K4H} &= \mathsf{K4} + \mathrm{H} \\
\mathsf{K4} &= \mathsf{K} + 4 \\
\mathsf{K} &= \mathsf{K} + \mathrm{Dual}
\end{aligned}
$$

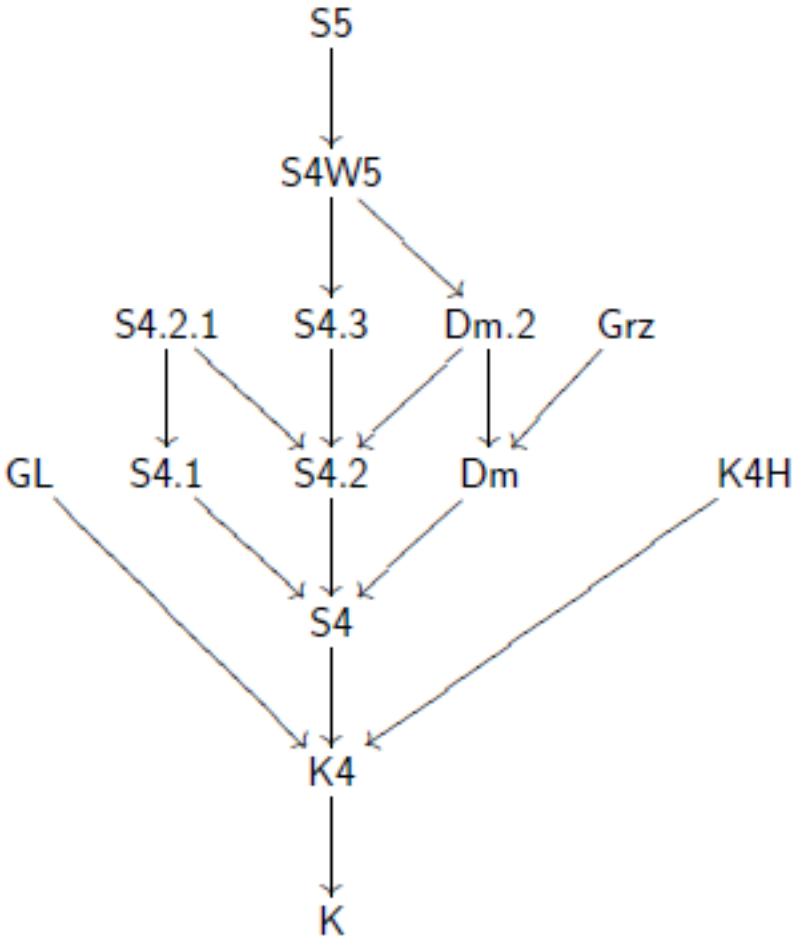

FIGURE 1. Some common modal theories

Theorem 3

Theorem 3. Every assertion in the modal theory S4.2 is a ZFC-provable principle of forcing.

Main Theorem 4

Main Theorem 4. If ZFC is consistent, then the ZFC-provable principles of forcing are exactly those in the modal theory S4.2.

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THANK YOU!