Philosophy of Mathematics

Joel David Hamkins and Ruizhi Yang

School of Philosophy

International Summer Session, Fudan University

Summer 2025

Setup of the Course

Peoples

- Main lecturer:

  Joel David Hamkins, John Cardinal O'Hara Professor of Logic at the University of Notre Dame

  - Research interests: set theory, particularly with forcing and large cardinals, philosophy of the mathematics, philosophical logic, etc.

Setup of the Course

Peoples

- Main lecturer:

  Joel David Hamkins, John Cardinal O'Hara Professor of Logic at the University of Notre Dame

  - Author of the book *Lectures on the Philosophy of Mathematics* (The MIT Press, 2021 and 上海人民出版社, 2025)

Setup of the Course

Peoples

- Main lecturer:

  Joel David Hamkins, John Cardinal O'Hara Professor of Logic at the University of Notre Dame

  - He is active on MathOverflow. He has earned the top-rated reputation score.
  - He will start delivering lectures from July 7.

Setup of the Course

Peoples

- Lecturer (for the first three meetings):

  Ruizhi Yang 杨睿之, Associate Professor at the School of Philosophy, Fudan University.

  Email: yangruizhi@fudan.edu.cn

- Teaching assistant:

  Boxiang Zeng 曾柏翔, 24210160034@m.fudan.edu.cn

Setup of the Course

Prerequisites

We do NOT intend to assume any particular background in philosophy or mathematics. However, some experience with mathematics will surely be helpful, and some familiarity with logic will be great.

Setup of the Course

Grading and Evaluation

- In-class discussion 40%: Students are encouraged to participate by asking questions and presenting their views.
- Final presentation 60%: The presentation should reflect the readings and the student's reflections.
    - Time: July 24 (and maybe July 22)
    - Format: TBA

Setup of the Course

Reference

- Joel David Hamkins, *Lectures on the Philosophy of Mathematics*, The MIT Press, 2021.
- Øystein Linnebo, *Philosophy of Mathematics*, Princeton University Press, 2017
- Stewart Shapiro, *Thinking about Mathematics: The Philosophy of Mathematics*, Oxford University Press, 2000.
- Jean van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press, 1967.
- Paul Benacerraf and Hilary Putnam (ed.), *Philosophy of Mathematics: Selected Readings*, Cambridge University Press, 1984.

Setup of the Course

Course website

- https://logic.fudan.edu.cn/event2025/jdh
- https://jdh.hamkins.org/

Setup of the Course

Scan to join the wechat group

Plan for the first three meetings

- General introduction to philosophy of mathematics
  - The philosophical challenges from mathematics
  - The search for a foundation of mathematics
- (If time permits) Topic that Joel might not cover
  - Type theory, proof assistants, and AI

Mathematics as universal character

... yet no one has attempted a language or characteristic which includes at once both the arts of discovery and judgement, that is, one whose signs and characters serve the same purpose that arithmetical signs serve for numbers, and algebraic signs for quantities taken abstractly.

Leibniz, *Zur allgemeinen Charakteristik*

Mathematics as universal character

My intention was not to represent an abstract logic in formulas, but to express a content through written signs in a more precise and clear way than it is possible to do through words. In fact, what I wanted to create was not a mere calculus ratiocinator but a lingua characterica in Leibniz's sense.

Frege, 1882

Mathematics as universal character

Thus it happens that our entire present-day culture, insofar as it rests on intellectual insight into and harnessing of nature, is founded on mathematics. Already, Galileo said: Only he can understand nature who has learned the language and signs by which it speaks to us; but this language is mathematics and its signs are mathematical figures.

Hilbert, 1930

Mathematics as universal character

A hard-to-rebut claim

All rigorous content can be expressed using the language of mathematics.

Example

- Arithmetic: everything finite
- Analysis and Topology: continuity
- Logic: truth, proof, and knowledge

Mathematics as universal character

A hard-to-rebut claim

That which cannot be expressed using the language of mathematics is only that content which is itself not rigorous.

Example

- Arithmetic: everything finite
- Analysis and Topology: continuity
- Logic: truth, proof, and knowledge

Mathematics as universal character

A hard-to-rebut claim

That which cannot be expressed using the language of mathematics is only that content which is itself not rigorous.

Example

- Arithmetic: everything finite
- Analysis and Topology: continuity
- Logic: truth, proof, and knowledge

Mathematics as universal character

This is a universal proposition, which cannot be proven by examples. What makes us believe it is true?

- Is it an empirical (*a posteriori*) truth?
- Is it analytic?

Compare it with the claim: mathematical truth is necessary.

The request for a foundation

The first formulations of the calculus were not even mathematically rigorous. An inexact, semi-physical formulation was the only one available for over a hundred and fifty years after Newton! ... The development was as confused and ambiguous as can be, and its relation to empiricism was certainly not according to our present (or Euclid's) ideas of abstraction and rigour.

(Von Neumann, *The Mathematician*)

The request for a foundation

After deserting for a time the old Euclidean standards of rigour, ... In arithmetic, ... it has been the tradition to reason less strictly than in geometry, ... The discovery of higher analysis only served to confirm this tendency; for considerable, almost insuperable, difficulties stood in the way of any rigorous treatment of these subjects. ... in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough.

(Frege, *Die Grundlagen der Arithmetik*)

The request for a foundation

- The request came from not only philosophers, but also (mainly) from mathematicians
- Historical crises in mathematics, such as controversies surrounding infinitesimals, the discovery of irrational numbers, the introduction of complex numbers, and the development of non-Euclidean geometry, reflect not only gaps in logical rigor but also questions about the nature of mathematical objects themselves.

Foundation of Mathematics

Frege's definition of natural numbers

The number which belongs to the concept F is the extension of the concept "being equinumerous to the concept F"

(Frege, *Die Grundlagen der Arithmetik*)

Foundation of Mathematics

Frege's definition of natural numbers

0 is the number which belongs to the concept "not identical with itself".

1 is the number which belongs to the concept "identical with 0"

(Frege, *Die Grundlagen der Arithmetik*)

Foundation of Mathematics

The integers are usually defined to be the differences between two natural numbers.

unordered and ordered pair

- For unordered pair (pair set), $\{a, b\} = \{b, a\}$
- For ordered pair, $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$

Foundation of Mathematics

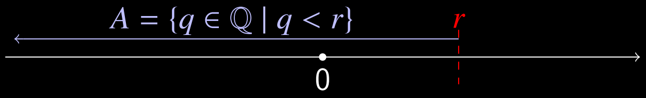
- Let $\mathbb{N}$ be the set of all natural numbers. We define the Cartesian product $\mathbb{N} \times \mathbb{N} = \{(n, m) : n, m \in \mathbb{N}\}$.
- Then we define an equivalent relation on $\mathbb{N} \times \mathbb{N}$: $(n_1, m_1) \sim (n_2, m_2)$ if and only if $n_1 + m_2 = n_2 + m_1$
- We define the equivalent class (represented by (n, m)), namely $[(n, m)]_\sim = \{(n', m') \in \mathbb{N} \times \mathbb{N} : (n', m') \sim (n, n)\}$. These are the integers! For example, -1 is $[(0, 1)]$.

Foundation of Mathematics

How to define the rationals in the same manner?

Foundation of Mathematics

Dedekind: The reals are sets of rationals



A Dedekind cut is a set of rationals which is bounded to the "right" and closed to the "left". $\sqrt{2}$ is the set $\{q \in \mathbb{Q} : q^2 < 2$ or $q < 0\}$

Foundation of Mathematics

We have successfully defined many mathematical objects to be sets or classes. But

- Russell's paradox: Let $R = \{x : x \notin x\}$. Do we have $R \in R$?
- Russell's simple type theory:
  - Each mathematical object is of some type. For example, Type 0 consists of individuals, Type 1 consists of properties of individuals, Type 2 consists of properties of Type 1 objects, etc. Membership relation is between objects of Type n and Type $n+1$, so $x \in x$ is not legitimate.

Foundation of Mathematics

We have successfully defined many mathematical objects to be sets or classes. But

- Russell's paradox: Let $R = \{x : x \notin x\}$. Do we have $R \in R$?
- Axiomatic set theory:
  - Axioms of comprehension: given a formula $\varphi(x)$, for each set X, there is a set $\{x \in X : \varphi(x)\}$

Set Theory as Foundation

- The language $\mathcal{L}$ of set theory: $\{\in\}$
- The theory ZFC of set theory:
    - Axiom of extensionality and foundation
    - Axiom of pairing, union, power set, infinity, separation, replacement, and choice

Set Theory as Foundation

The ZFC axioms for set theory

- Extensionality, Foundation
- Pairing, Union, Power set, Infinite
- Separation (Comprehension), Replacement
- Choice

Set Theory as Foundation

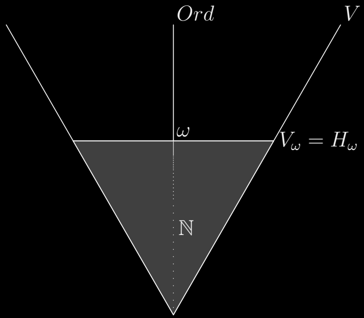
- von Neumann ordinals:
  - $0 = \emptyset,\ 1 = \{0\} = \{\emptyset\},\ 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}\ \ldots$
  - $\omega = \{0, 1, 2, \ldots\},\ \omega + 1 = \omega \cup \{\omega\},$

    $\omega + \omega = \omega \cup \{\omega + 1, \omega + 2, \ldots\} = \bigcup\{\omega + n : n \in \omega\},\ \ldots$

Set Theory as Foundation

- The cumulative hierarchy" of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

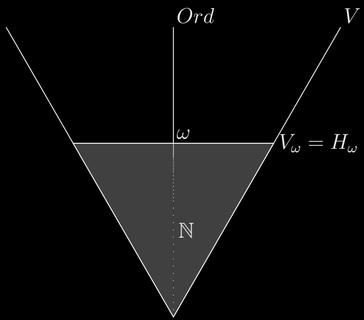
    $\vdots$



Set Theory as Foundation

- The universe of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

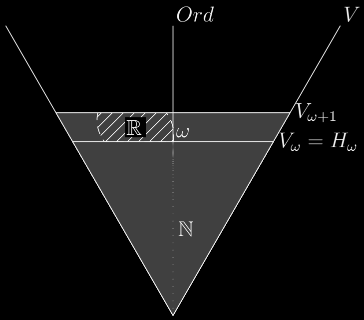
  $\vdots$



Set Theory as Foundation

- The universe of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

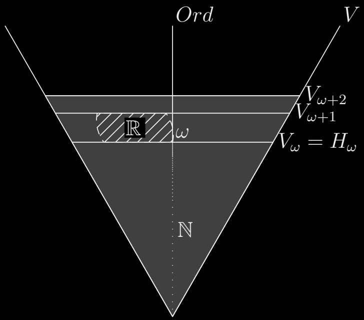
  $\vdots$



Set Theory as Foundation

- The universe of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

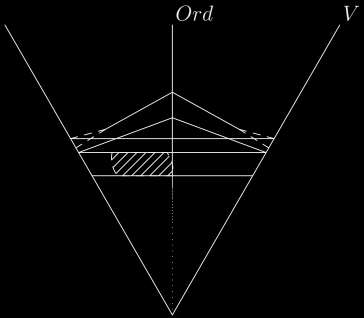
  $\vdots$



Set Theory as Foundation

- The universe of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

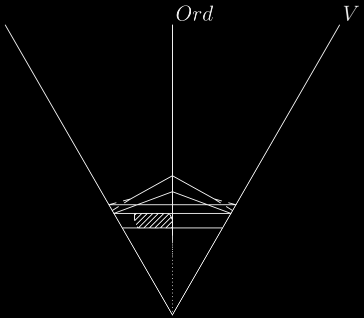
    $\vdots$



Set Theory as Foundation

- The universe of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

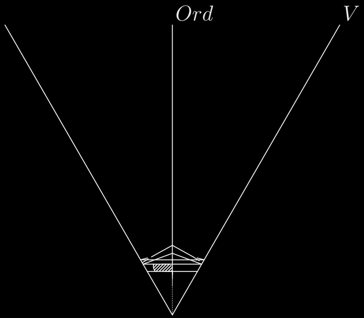
  $\vdots$



Set Theory as Foundation

- The universe of sets:
  - $V_0 = \emptyset$
  - $V_{\alpha+1} = P(V_\alpha)$
  - $V_\omega = \bigcup_{n<\omega} V_n$

  $\vdots$



Foundation of Mathematics

Compared to our confidence in mathematics, our victory in the search for the foundations of mathematics is even more surprising.

All mathematics can be expressed in set theory.

Foundation of Mathematics

Compared to our confidence in mathematics, our victory in the search for the foundations of mathematics is even more surprising.

All mathematics can be expressed in the language of set theory.

Foundation of Mathematics

All mathematics can be expressed in set theory.

- It is a universal proposition, but stronger.
- It is harder to say that it is an analytical truth,
- or empirical

Foundation of Mathematics

Hilbert's Program

- Axiomatization for all classical mathematics
- Prove it is complete
- Prove it is consistent
- Achieve these using only finitary mathematics

Foundation of Mathematics

Gödel's Incompleteness Theorem

Any axiomatizable theory T, which is consistent and has enough expressive power, must be

- incomplete: there is a σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$
- incapable of proving its own consistency: $T \nvdash \mathrm{Con}(T)$

Interpretability Power

Definition

Let T_1 and T_2 be theories in language $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We say T_1 is interpretable in T_2, if there is an effective function π translating each $\mathcal{L}_1$ sentence into a $\mathcal{L}_2$ sentence such that for each $\mathcal{L}_1$ sentence φ,

$$T_1 \vdash \varphi \;\Rightarrow\; T_2 \vdash \pi(\varphi)$$

Interpretability Power

Example

- Let φ be a sentence in the language $\mathcal{L}_{\mathsf{A}} = \{0, S, +, \cdot, \leq\}$ for arithmetic. We have

$$\mathsf{PA} \vdash \varphi \;\Rightarrow\; \mathsf{ZFC} \vdash \text{``}(\omega, S_\omega, +_\omega, \cdot_\omega, \leq_\omega) \models \varphi\text{''}$$

- With Ackermann coding: $\alpha(x) = \sum_{y \in x} 2^{\alpha(y)}$, the theory of hereditarily finite sets can be translated into arithmetic. For example, $x \in y$ is translated to $\left\lfloor \frac{x}{2^y} \right\rfloor \bmod 2 = 1$

Interpretability Power

Large Cardinal

- ω is a large cardinal above all finite ordinals, so that we have

  ZFC $\models \mathrm{Con}(\mathrm{PA})$

- Inaccessible cardinal: not accessible from smaller ordinals by the set-theoretical operations

  If κ is inaccessible, then $V_\kappa \models$ ZFC. Thus, ZFC+ there exists an inaccessible $\models \mathrm{Con}(\mathrm{ZFC})$

Interpretability Power

Large Cardinals

- Mahlo cardinal
- Weakly compact cardinal
- Measurable cardinal
- Strong cardinal
- Supercompact cardinal

  $\vdots$
- $0 = 1$

Other candidates for the foundation

- Strict finitism (PRA)
- Constructive mathematics
  - Intuitionistic (HA)
  - Martin-Löf type theory
- predicative mathematics (ATR)
- Category theory

Roughly ordered by interpretability power

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

> *We had not perceived the sharp concept of mechanical procedures before Turing, who brought us to the right perspective.*
>
> Gödel (Wang, *A Logical Journey*)

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

> *We had not perceived the sharp concept of mechanical procedures before Turing, who brought us to the right perspective.*
>
> Gödel (Wang, *A Logical Journey*)

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

> *We had not perceived the sharp concept of mechanical procedures before Turing, who brought us to the right perspective.*

Gödel (Wang, *A Logical Journey*)

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

Turing's argument is so compelling that we now define the expressive power of an artificial language through the notion of Turing completeness — the ability to simulate a Turing machine.

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

> *the "**computable" numbers** include all numbers which would naturally be regarded as computable. All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically.*
>
> (Turing, 1937)

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

> *the "computable" numbers include all numbers which would naturally be regarded as computable. All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically.*
>
> (Turing, 1937)

The concept of computability

Turing (1937): Everything computable can be computed by a Turing machine

The three kinds of argument

- A direct appeal to intuition.
- A proof of the equivalence of two definitions
- Giving examples of large classes of numbers which are computable.

The success we enjoy

We achieve the following agreement

- Mechanical procedures is characterized by Turing machine
- Axiomatic set theory can serve as the foundation of mathematics
    - Sets are the extension of concepts
    - Some other candidates of foundation of mathematics are interpretable in set theory, and many can interpret quite a lot of set theory
    - Many mathematical theories is interpretable in set theory

The success we enjoy

Axiomatic set theory serves as a scalable foundation of mathematics. From strict finitism to large cardinal axioms, we have a hierarchy of theories ordered by interpretability power all the way towards increasingly completeness and inconsistency.

- We can choose the interpretability power as we need
- The risk of inconsistency seems to be contained in the hierarchy

The success we enjoy

- The trust we enjoy
  - Although there is no way to effectively discover a proof, verifying a proof is effective.
  - To verify proofs more efficiently, new mathematical foundations (such as proof assistants Coq, Isabelle, Lean) have been developed. Their reliability and completeness ultimately reduce to set theory.
- Philosophy of mathematics is no longer in vogue.

Two challenges

- The "real" independent statements
  - Continuum hypothesis (CH)
  - Inner model: $L \models \text{ZFC} + 2^{\aleph_0} = \aleph_1$
  - Forcing extension: $L[G] \models \text{ZFC} + 2^{\aleph_0} = \aleph_{256}$
- Frege's Caesar Problem: Is Zero Julius Caesar?

Set-theoretic Multiverse View

- The Multiverse view is proposed in opposition to the Universe view.
- Universe View: Set theory is about THE universe of all sets
- Multiverse View: There are many different universes of sets, and different concepts of set underlying them. Therefore, the multiversists hold different views on test problems such as CH compared to universists.

Set-theoretic Multiverse View

This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set theoretic to us.

(Hamkins, 2012)

Anti Set-theoretic foundationalism

Benacerraf's identification problem

Must the numbers be the von Neumann ordinals? Why not the Zermelo ordinals?

$$0 = \emptyset,\ 1 = \{0\} = \{\emptyset\},\ 2 = \{1\} = \{\{0\}\}\ \ldots$$

Anti Set-theoretic foundationalism

Structuralism

- Mathematical objects are no more than positions in structures.
- Mathematical truths are truths in structures.

  There is no absolute mathematical truth; neither arithmetic truths nor set-theoretic truths are absolute.

Anti Set-theoretic foundationalism

The other purported foundational role for set theory that seems to me spurious is what might be called the Metaphysical Insight. The thought here is that the set-theoretic reduction of a given mathematical object to a given set actually reveals the true metaphysical identity that object enjoyed all along.

(Maddy, 2017)

A unified foundation is still needed

Every universe of sets thinks its own arithmetic structure is standard. But under the multiverse view (which we cannot refute), every universe of sets is considered non-standard by a larger and better universe of sets.

Theorem (Hamkins and Y. 2013)

Assume ZFC is consistent. There exist models M_1 and M_2 of ZFC such that the defined arithmetic structures are isomorphic, but their arithmetic truths are different.

Foundation as coding system

- A background theory should serve as a coding system, rather than a metaphysical base.
- While arithmetic is a good example, its power of interpretability is limited.
- Set theory offers scalable interpretability power, but its inherent structure remains unclear.

Transfinite Arithmetic

- A natural next step is to generalize arithmetic to the transfinite, a theory of the structure $(\mathsf{ON}, +, \cdot, 0, \omega, \ldots)$
- It is hard to find an natural and arithmetical operation O on the ordinals such that the interpretability power of the theory of $(\mathsf{ON}, +, \cdot, 0, \omega, O)$ is not strictly below that of ZFC.

Transfinite Arithmetic

It is not impossible if we drop the requirements of being naturals and arithmetical.

Since there is a definable global well-ordering $<_L$ on L or other L-like inner models, ..., namely, we can define $\alpha E \beta$ if and only if the αth (with respect to $<_L$) element is a member of the βth.

Then the theory of the structure $(\mathrm{ON}, +, \cdot, 0, \omega, E)$ surely interprets, for example, ZFC + $V = L$.

Transfinite Arithmetic

- The opponent may say that it is just set theory in disguise
- However, if a theory of transfinite arithmetic can serve as a foundation, it must somehow interpret set theory. Moreover, since the ordinals are well-ordered, the set theory it interprets must possess a global well-ordering..

Transfinite Arithmetic

- Given the extra restriction (bearing the global well-ordering), and that its coding style might not as natural as set theory, are there any notable benefits?
  - Provide an independent justification for the inner model and W. Hugh Woodin's Ultimate-L program.
  - Inspire some new research

The Inner Model Program

- Gödel's theorem indicates that there is no purely mathematical proof of consistency of mathematics
- The inner model program aims to find well-structured inner models for large cardinal axioms. For instance, $L[U] \vDash$ there exists a measurable cardinal, thereby providing empirical evidence for the consistency of large cardinals.

The Inner Model Program

- Large cardinal axioms combined with statements asserting that $V =$ the corresponding inner model (e.g., $V = L[U]$), can lead to theories that are effectively complete.
- Defining inner models for stronger large cardinals tends to be significantly more intricate

The Inner Model Program

- Woodin's theorem: An inner model for a supercompact cardinal would also accommodate all known stronger large cardinals. This leads to the Ultimate-L program.
- An arithmetic-style foundation with global well-ordering, transparent structure, and scalable interpretability power is arguably equivalent to the theory of Ultimate-L+ large cardinals.

Next

Type theory and Lean