Philosophy of Mathematics

Joel David Hamkins and Ruizhi Yang

School of Philosophy

International Summer Session, Fudan University

Summer 2025

Setup of the Course

Peoples

- Main lecturer:

  Joel David Hamkins, John Cardinal O'Hara Professor of Logic at the University of Notre Dame

  - Research interests: set theory, particularly with forcing and large cardinals, philosophy of the mathematics, philosophical logic, etc.

Setup of the Course

Peoples

- Main lecturer:

  Joel David Hamkins, John Cardinal O'Hara Professor of Logic at the University of Notre Dame

  - Author of the book *Lectures on the Philosophy of Mathematics* (The MIT Press, 2021 and 上海人民出版社, 2025)

Setup of the Course

Peoples

- Main lecturer:

  Joel David Hamkins, John Cardinal O'Hara Professor of Logic at the University of Notre Dame

  - He is active on MathOverflow. He has earned the top-rated reputation score.
  - He will start delivering lectures from July 7.

Setup of the Course

Peoples

- Lecturer (for the first three meetings):

  Ruizhi Yang 杨睿之, Associate Professor at the School of Philosophy, Fudan University.

  Email: yangruizhi@fudan.edu.cn

- Teaching assistant:

  Boxiang Zeng 曾柏翔, 24210160034@m.fudan.edu.cn

Setup of the Course

Prerequisites

We do NOT intend to assume any particular background in philosophy or mathematics. However, some experience with mathematics will surely be helpful, and some familiarity with logic will be great.

Setup of the Course

Grading and Evaluation

- In-class discussion 40%: Students are encouraged to participate by asking questions and presenting their views.
- Final presentation 60%: The presentation should reflect the readings and the student's reflections.
    - Time: July 24 (and maybe also July 22)
    - Format: TBA

Reference

- Joel David Hamkins, *Lectures on the Philosophy of Mathematics*, The MIT Press, 2021.
- Øystein Linnebo, *Philosophy of Mathematics*, Princeton University Press, 2017
- Stewart Shapiro, *Thinking about Mathematics: The Philosophy of Mathematics*, Oxford University Press, 2000.
- Jean van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press, 1967.
- Paul Benacerraf and Hilary Putnam (ed.), *Philosophy of Mathematics: Selected Readings*, Cambridge University Press, 1984.

Setup of the Course

Course website

- https://logic.fudan.edu.cn/event2025/jdh
- https://jdh.hamkins.org/

Plan for the first three meetings

- General introduction to philosophy of mathematics
  - The philosophical challenges from mathematics
  - The search for a foundation of mathematics
- (If time permits) Topic that Joel might not cover
  - Type theory, proof assistants, and AI

The success of mathematics

Mathematics has long enjoyed a reputation as the "The queen of the sciences". (Carl Friedrich Gauss) Optimism about mathematics reached its peak in David Hilbert's celebrated speech in 1930.

The success of mathematics

David Hilbert's 1930 Radio Address

The tool implementing the mediation between theory and practice, between thought and observation, is mathematics. Mathematics builds the connecting bridges and is constantly enhancing their capabilities. Therefore it happens that our entire contemporary culture, in so far as it rests on intellectual penetration and utilization of nature, finds its foundations in mathematics.

The success of mathematics

David Hilbert's 1930 Radio Address

Already some time ago Galileo said "Only one who has learned the language and signs in which nature speaks to us can understand nature."

This language however is mathematics, and these signs are the figures of mathematics.

Kant remarked "I maintain that, in any particular natural science, genuine scientific content can be found only in so far as mathematics is contained therein."

The success of mathematics

David Hilbert's 1930 Radio Address

In fact we do not have command of a scientific theory until we have peeled away and fully revealed the mathematical kernel. Without mathematics, modern astronomy and physics would be impossible. The theoretical parts of these sciences almost dissolve into branches of mathematics. Mathematics owes its prestige, to the extent that it has any among the general public, to these sciences along with their numerous broader applications. Although all mathematicians have denied it, the applications serve as the measure of worth of mathematics.

The success of mathematics

David Hilbert's 1930 Radio Address

Gauss speaks of the magical attraction which made number theory the favorite science of the first mathematician—not to mention the inexhaustible richness of number theory which far surpasses that of any other field of mathematics.

Kronecker compares number theorists with the lotus eaters, who, once they started eating this food, could not let go of it.

The success of mathematics

David Hilbert's 1930 Radio Address

The great mathematician Poincare once sharply disagreed with Tolstoy's declaration that the proposition "science for the sake of science" would be silly.

The achievements of industry for example would not have seen the light of the world if only applied people had existed and if uninterested fools had failed to promote these achievements.

The success of mathematics

David Hilbert's 1930 Radio Address

The honor of the human spirit, so said the famous Konigsburg mathematician Jacobi, is the only goal of all science. We ought not believe those who today, with a philosophical air and reflective tone, prophesy the decline of culture, and are pleased with themselves in their own ignorance. For us there is no ignorance, especially not, in my opinion, for the natural sciences.

Instead of this silly ignorance, on the contrary let our fate be: "We must know, we will know".

The success of mathematics

In summary of Hilbert's view in 1930

- Mathematics has universal applicability.
- The measure of worth of mathematics is in itself
- Optimism on mathematical knowledge

Things might not be so when we dig more deeply into the nature of mathematics.

The Platonistic view of mathematics

This is probably the most popular and naive view of mathematics.

- Mathematical knowledge is a priori
- Mathematical truths are necessary
- Mathematics concerns abstract objects

The Platonistic view of mathematics

By “naive,” I do not mean that it is wrong, but rather that it requires deeper investigation. None of these features is uncontroversial. In fact, the very fact that mathematics is so different from the ordinary empirical sciences, yet remains so “rock solid,” makes it philosophically puzzling and therefore a challenge.

The Platonistic view of mathematics

Mathematical knowledge is a priori. Namely, it doesn't rely on sense experience or on experimentation, but on reflection alone.

The Platonistic view of mathematics

Socrates: And that is the line which the learned call the diagonal. ... you ... are prepared to affirm that the double space is the square of the diagonal? (Plato, *Meno*)

The Platonistic view of mathematics

The story of the slave boy is meant to establish two points.

- Mathematical concepts are innate; that is, they are not acquired but form part of the mind's inborn endowment.
- Mathematical truths are a priori and can be known without relying on experience for one's justification.
  Objection: The slave boy relies on experience in order to understand Socrates' questions.

The Platonistic view of mathematics

A real challenge: How can we possess innate mathematical concepts and a priori mathematical knowledge?

> *The soul, then, as being immortal, and having been born again many times, and having seen all things that exist, whether in this world or in the world below, has knowledge of them all.*
>
> (Plato, *Meno*)

The Platonistic view of mathematics

A real challenge: How can we possess innate mathematical concepts and a priori mathematical knowledge?

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The Platonistic view of mathematics

- Althought this explanation has little appeal today. At least, Plato realized that an explanation is needed.
- We are not doing much better today.

The Platonistic view of mathematics

Gödel once said (reported by Hao Wang):

> *I conjecture that some physical organ is necessary to make the handling of abstract impressions (as opposed to sense impressions) possible, ... Such a sensory organ must be closely related to the neural center for language. But we simply do not know enough now.*

(Wang, *A Logical Journey: From Gödel to Philosophy*)

The Platonistic view of mathematics

Mathematical truths are necessary, in the sense that things could not have been otherwise. It is therefore safe to appeal to mathematical truths when reasoning not only about how the world actually is but also about how it would have been had things been otherwise.

The Platonistic view of mathematics

- When we speak of necessity in the empirical world, we often mean that something is true across space and time.
- Necessity as a modality is usually defined to be universally valid, or true in all possible worlds.
- When we speak of necessity in mathematics, what do we mean by "all possible worlds".

The Platonistic view of mathematics

The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?

(Frege 1884, §14)

The Platonistic view of mathematics

Since such truths can freely be appealed to throughout our counterfactual reasoning, it follows that these truths are counterfactually independent of us humans, and all other intelligent life for that matter. That is, had there been no intelligent life, these truths would still have remained the same. ... Pure mathematics is in this respect very different from humdrum contingent truths... Had intelligent life never existed, there would have been no laws, contracts, or marriages... (Linnebo 2017)

The Platonistic view of mathematics

Mathematics concerns abstract objects such as numbers, functions, etc, that are not located in space or time, and that don't participate in causal relationships. The latter are concrete objects.

The Platonistic view of mathematics

"Mathematics concerns abstract objects" can be divided into two separate claims.

- Object realism: There are mathematical objects.
- Abstractness: Mathematical objects are abstract.

The Platonistic view of mathematics

Frege's defence of object realism

Consider the following sentences:

- Evelyn is prim. (Evelyn 是拘谨的。)
- Eleven is prime.

The two sentences have the same logical structure, namely a simple predication based on a proper name, which refers to an object, and a predicate, which ascribes some property to this object. It is true only if the proper name successfully refers to an object.

The Platonistic view of mathematics

The claim that mathematical objects are abstract has been less controversial ... If possible, our philosophical account of mathematics should avoid claims that would render our ordinary mathematical practice misguided or inadequate. But if mathematical objects had spatiotemporal location, then our ordinary mathematical practice would be misguided and inadequate. We would then expect mathematicians to take a professional interest in the location of their objects, just as zoologists are interested in the location of animals. (Linnebo 2017)

The Platonistic view of mathematics

The platonist conception of mathematics does not stop with the claim that there are abstract mathematical objects. It adds a claim about the robust reality of these objects:

Reality: Mathematical objects are at least as real as ordinary physical objects.

The Platonistic view of mathematics

Plato: Mathematical objects have a higher degree of reality.

Allegory of the cave

People only know reality as shadows of the real things they see interacting on a wall

The Platonistic view of mathematics

Frege: Mathematical truth are discovered, not invented.

> *Just as the geographer does not create a sea when he draws borderlines and says: the part of the water surface bordered by these lines I will call Yellow Sea, so too the mathematician cannot properly create anything by his definitions.*

(Frege, *Grundgesetze*, I, xiii)

The Platonistic view of mathematics

It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions

(Gödel, *Russell's mathematical logic*)

The Platonistic view of mathematics

The point is even enshrined in U.S. patent law, which permits one to patent inventions but not mathematical truths or laws of nature.

(Linnebo, 2017)

The Integration Challenge

Assume we take mathematical language and practice more or less at face value and accept some version of the platonistic conception. A philosophical challenge arises. Can we make sense of a science that works in this way? Can we explain how human beings in a seemingly a priori way acquire knowledge of necessary truths concerned with abstract objects? As we have seen, this challenge has been with us since Plato's Meno.

(Linnebo, 2017)

The Integration Challenge

- "take mathematical language ... at face value". For example, when we assert that there are infinitely many prime numbers. This suggests what we called object realism.
- If knowledge (as is the case with most empirical knowledge) requires a causal connection between the knower and the known, how can we know about abstract objects?

The Integration Challenge

Two sets of questions

- Metaphysical: What is mathematics about? Is it really concerned with abstract objects?
- Epistemological: How do mathematicians and others with some degree of mathematical competence arrive at their mathematical belief? How do mathematicians settle on their first principles (or axioms), and how do they use these to prove mathematical results (or theorems)?

The Integration Challenge

The integration challenge is to integrate the metaphysics of mathematics (namely, what mathematics is about) with its epistemology (namely, how we form our mathematical beliefs). To give an account of how it is that our ways of forming mathematical beliefs are responsive to what mathematics is about, why is it not just a happy accident that our mathematical beliefs tend to be true?

The Integration Challenge

How to take this challenge? Suggestions by Linnebo:

- The challenge is not external to mathematics.

  For example, we need to listen to what mathematics itself has to say about numbers and sets.

The Integration Challenge

How to take this challenge? Suggestions by Linnebo:

- No prejudice against mathematics.

  For example, to insist on a causal connection between the subject matter of mathematics and mathematicians' beliefs

The Integration Challenge

How to take this challenge? Suggestions by Linnebo:

- We are allowed to use mathematics. "We are presupposing that our perceptual beliefs are reliable in order to explain why they are reliable." This does not make it trivial.

Philosophy-first vs philosophy-last

Linnebo's suggestions reflect a reconsideration of the Cartesian "first philosophy" , or the philosophy-first principle. This orientation suggested that philosophy precedes practice in some deep metaphysical sense. At the fundamental level, philosophy determines practice.

Concerning mathematics, the revisionists demand revisions in mathematics based on their philosophical view. For example, the intuitionists and predicativists.

Philosophy-first vs philosophy-last

Perhaps most philosophers, such as Linnebo, reject philosophy-first. An extreme opposite position is called philosophy-last-if-at-all principle, the thesis that philosophy is irrelevant to mathematics. Many mathematicians are not in the least interested in philosophy. This view roots back to the Vienna Circle. For example, Rudolf Carnap might claim that philosophical questions concerning the real existence of mathematical objects are mere 'pseudo-questions'.

Philosophy-first vs philosophy-last

More philosophers are in between. An anti-revisionist may suggests the job of the philosophers is to give a coherent account of mathematics.

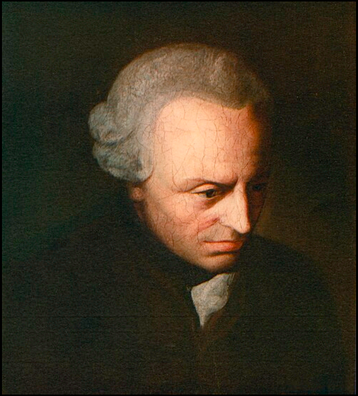
Yet, some mathematicians claim that philosophy can set the direction of mathematical research.For example, Gödel claimed that his realism was an important factor in the discovery of both the completeness of first-order logic and the consistency of the continuum hypothesis, which are direct generalizations of Skolem's and Russell's work, respectively.

Philosophy-first vs philosophy-last

You are encouraged to reflect on the philosophy-first and philosophy-last debate as we discuss the brief history of the search for the foundations of mathematics (perhaps) in the next meeting.

Now, let's return to the integration challenge.

Kant's view on mathematics



Immanuel Kant (1724–1804) was one of the most influential figures in Western philosophy. He is best known for his work in epistemology, metaphysics, and ethics.

Kant's view on mathematics

Kant's distinctions between *a priori* and *a posteriori*, and analytic and synthetic knowledge

- *a priori* vs *a posteriori*: independent of / dependent on experience

  Note that Kant made a distinction between cognition that "commences with experience" (all cognitions are) and that "arises from experience".

Kant's view on mathematics

Kant's distinctions between *a priori* and *a posteriori*, and analytic and synthetic knowledge

- analytic vs synthetic: a judgment ["every A is B"] as analytic if "the predicate B belongs to the subject A as something that is (covertly) contained in this concept A", and synthetic otherwise.

  For example, Kant thought "Bodies are extended" is analytic, whereas "Bodies are heavy" is synthetic.

Kant's view on mathematics

Kant's distinctions between *a priori* and *a posteriori*, and analytic and synthetic knowledge

- analytic vs synthetic:

  What if a judgment is not in a simple subject-predicate form? Another characterization that Kant gives: those that are based on the principle of contradiction. So we can generalize the definition of analytic to all logical truths. (Logic in Kant's day was almost Aristotelian logic.) This was adopted by Frege.

Kant's view on mathematics

Kant: Mathematical knowledge is a priori and synthetic. For Kant, mathematical knowledge serve as an example of how this type of knowledge is possible.

Kant's view on mathematics

Why mathematical knowledge is synthetic? Consider the judgment that $7 + 5 = 12$.

> *[t]he concept of twelve is by no means already thought merely by my thinking of that unification of seven and five*

(Kant, *Critique of Pure Reason*, B15)

Kant's view on mathematics

Why mathematical knowledge is synthetic? Consider the judgment that $7 + 5 = 12$.

We must go beyond the concepts involved and bring in the aid of intuition to represent these concepts, for example, by producing the relevant numbers of fingers or points. So arithmetical truths are not grounded in facts about conceptual containment but are "ampliative" and thus synthetic. The case of geometry is analogous.

Kant's view on mathematics

Why mathematical knowledge can also be a priori?

Kant's Copernican turn

If intuition has to conform to the constitution of the objects, then I do not see how we can know anything of them a priori; but if the object ... conforms to the constitution of our faculty of intuition, then I can very well represent this possibility to myself.

(Kant, *Critique of Pure Reason*, Bxvii)

Kant's view on mathematics

Why mathematical knowledge can also be a priori?

Kant's Copernican turn

That is, we need to reverse the usual order of epistemic conformity to explain how arithmetical and geometrical truths can be simultaneously synthetic and a priori. The claim is thus that, in order to explain how mathematics is possible, we need to adopt Kant's transcendental idealism, which results from this reversed order of conformity.

Kant's view on mathematics

Kant's solution to the integration challenge in short:

While we cannot speak about the reality of mathematical objects (they may be "thing in itself"), mathematical knowledge has content and is therefore synthetic. This content arises from our faculty of intuition. Mathematical knowledge is a priori because the empirical world conforms to our intuition, not the other way around. It is necessary because the world can only be understood through this form

The options by Kant's conception

	analytic	**synthetic**
a priori	Frege	Plato, Gödel (pre-Copernican) Kant, Brouwer (Copernican)
a posteriori	**X**	Mill, Quine

The options by Kant's conception

Some views fall outside of this table because they deny that mathematics is all about truth.

For example, the formalists suggest that mathematics, unlike most other sciences, operates with a standard of correctness that is less demanding than truth. Many formalists compare mathematics to a game.

The options by Kant's conception

Some views fall outside of this table because they deny that mathematics is all about truth.
The formalists deny that mathematical sentences are meaningful and propose instead to understand mathematics as the activity of proving pure formal theorems from purely formal axioms.

The options by Kant's conception

Some views fall outside of this table because they deny that mathematics is all about truth.

Moreover, some fictionalists view mathematics as a useful fiction, where the standard of correctness isn't literal truth but truth according to some fiction.

Next time

The search for a foundation of mathematics