A Pragmatistic View on Philosophy of Mathematics

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Abstract

In this paper, the author will introduce a pragmatistic view on the philosophy of mathematics. The idea is to measure the positiveness of philosophical doctrines by measuring their consequence on mathematical practice. To defend the idea, the author will argue that the characters of mathematics make it a special field that the principle of pragmatism can be applied naturally. Some cases will be studied in the following section to give a vision of how philosophical standpoints impact on mathematical research. To maximized the understanding, some basic knowledge in classical logic and set theory is presumed.

Key Words: philosophy of mathematics   pragmatism   set theory

1 Introduction

Pragmatism is a philosophical movement firstly promoted by William James (1842 - 1910) and Charles Sanders Peirce (1839 - 1914). The basic rule of pragmatism, namely, identified the meaning of concept, statement or opinion with their ``practical consequences'', is ubiquitous in everyday-life reasoning. Accordingly pragmatism is heavily attacked on its truth theory, ethics, ontology and epistemology problems. The author will not try to defend pragmatism in general, rather, a certain application of pragmatistic method will be exhibited, while the author hopes the reader will agree that the application is natural, coherent and effective.

The introduce of pragmatism is to resolve dilemmas. The initiators of pragmatism believe that disputes in philosophy are caused by ambiguities of the concepts, and can be settled by clarifying the meanings of the concepts *practically*. In this article, the author wants to keep pragmatistic standpoint at a weak level. Thus pragmatistic method, which known as the ``mediating philosophy'' should be restricted to the cases when seriously irresoluble disputes or dilemmas rises.

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In philosophy of mathematics, pragmatistic method has already been introduced to deal with cases of such kind.

Kurt Gödel has suggested a methodology to pick out the best statement (whose truth can not be determined mathematically) from the candidates of axioms, yet there is no intrinsic justification for such choice.

> ... even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its ``success''. Success here means fruitfulness in consequences, in particular in ``verifiable'' consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. ... There might exists axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. [6, 264]

This methodology as is known as Gödel's *external justification*. It is no doubt a pragmatistic process to determine mathematical truth by empirically studying on its success and consequences. This suggestion has been heavily discussed and criticized in the community of researchers of foundation of mathematics.

However, in this article, pragmatistic method will be applied to settle conflicts among philosophical standpoints, e.g. set theory realism, constructivism, formalism, and multiverse view, rather than the choice of new axioms for mathematics.

2 Mathematical Achievement

A core notion of pragmatism is ``consequence''. In this section, we will clarify what are the *consequences* that we are really concerning about in the realm of mathematics, and what properties of them which makes it more appropriate to apply pragmatistic treatment.

2.1 Mathematics gives knowledge

Philosophers, as Hume, Kant and Frege, classified truths or judgements into *analytic / synthetic*. Briefly, a truth is analytic if it is logically true; a truth is synthetic if it is true for some reason other than logic. More detailed discussion, can be find in [27]. Since people can possess different views of logic, the exact extension of ``analytic'' can be different. For example, from Kant's view, i.e. Aristotelian logic is all, ``$5 + 7 = 12$'' is synthetic, while Russell must claim it is analytic because for Russell, simple type theory is logic. I do not want to stick myself on the endless debate of whether mathematical truth is analytic or synthetic.

Usually, a proposition is analytically true only because of its structure, e.g. ``$0 = 0$'', ``$\neg(p \wedge \neg p)$'', and ``$\{p, q\} \vdash p$''. It is a temptation to think analytic truths as above do not provide new knowledge.

Most mathematics could be formulated in the language of set theory. It is in the sense that those mathematical notion are defined in the language of set theory, and the mathematical theorems can be considered as theorems proved by the axioms of set theory. Although people may possess different views on set theory and even on logic, it is clear that all results made by mathematicians are in the form

$$\Lambda \vdash_L \Sigma \quad ,$$

where Λ is a fragment of some variation of set theory, $\vdash_L$ is the notion of provability in a logic for mathematics[1], and Σ is a simply definable set of formulas. All of them can be regard as analytical truths or schemas of analytical truth. Does it mean mathematicians do not produce knowledge at all?

My answer is yes, mathematics does give knowledge. The outcome that mathematicians present to us are analytic truths, or in another words, facts of logic. But a discovery of logic fact is, in some sense, a second-ordered fact or historical fact. I am not saying that the knowledge of mathematics is just the history of mathematics. The point is that if the mathematical truths are there, which is a view of platonism, then the discovery of them is just like a discovery of a principles in physics, which gives knowledge with no doubt; else if mathematics is a mental construction, it provides the knowledge of the inventions. It is clear immediately to every one who understand it that $5 + 7 = 12$ follows from the axioms of addition. But no one knew it before Gödel's discovery in 1931 that

$$\text{ZFC} \vdash \text{Con}(\text{ZFC}) \to \text{Con}\left(\text{ZFC} + \neg\,\text{Con}(\text{ZFC})\right). \tag{2.1}$$

Now we *know* (2.1) is an analytic truth. But before the knowledge that Gödel brought to humans, people were still seeking a proof of Con(ZFC) from ZFC, which is now known to be impossible if ZFC is consistent.[2]

Hence mathematics does give knowledge. and the knowledge does have effects on humans' behavior.

2.2 ``Absoluteness'' of mathematical achievement

Before continuing our journey, recall that we are restricting our discussion within the realm of mathematics. That is we only concern about mathematical facts and the discovery of such facts. I will argue in the following paragraphs that the discovery of a mathematical proof is absolutely *good* for mathematics.

As we have seen, the mathematical achievements are the discoveries of new proofs, which give us a clearer map of the universe of mathematical statements. I would like to claim that such an achievement, when it has been done, can not be falsified anymore, and it will play a more or less but permanently positive role in the realm of mathematics.

A *proof* is a sequence of formula where each formula is either an assumption or follows from previous formulas via certain logic rules. It can be checked *effectively*[3] whether a sequence is a valid proof or not. Once it passes the suspicion, the sequence is there and will never disappear.

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[1]Practically, in this article, we only concern two kind of mathematical logic, namely classical logic and the intuititionism

[2]This is how the original Hilbert's Program went to its end. However, technically, it is still possible to prove Con(ZFC) from ZFC or even PA, because we do not even know they are consistent or not. Thus the end of Hilbert's Program is not solo a mathematical corollary, rather it is also a consequence of people's philosophical believe that PA or even ZFC is consistent. Here we encounter a simple case of how philosophical opinion can affect the research in mathematics practically.

[3]Here I refer to the notion in computability theory and neglect the physical limitation of humans' memory and life.

It is possible that the significants of a mathematical achievement be revalued even dramatically from a more advanced view of foundation of mathematics. But the value can never be drop down to zero or negative, though it maybe very close to zero. For example, Playfair's axiom, i.e. ``given a line and a point not on it, there is at most one parallel to the given line through the point", is provable in Euclidean geometry. But it is not a theorem in the *absolute geometry*, namely the geometry based on axioms without Euclid's fifth postulate, say σ_5. However, it is provable in absolute geometry that ``Euclid's fifth postulate implies Playfair's axiom". The latter proof can be transformed from the former one effectively, namely we can just replace each formula φ in the former proof by $\sigma_5 \to \varphi$.

To explore the phenomenon more specifically, we start with some definitions.

Definition 2.1. An *axiomatic mathematical system* is a pair (Λ, L) where Λ is a computably enumerable set of axioms (sentences in a formal language compatible with L), and L is a logic consisting valid formulas as logical axioms and rules for inference.

A *proof* (or a *deduction*) respected to an axiomatic mathematical system (Λ, L) is described as above where the assumption consists Λ and the logical axioms

Definition 2.2. A *transform* of proofs from (Λ_0, L_0) to (Λ_1, L_1) is a function π, for each sequence of formula D, which is a deduction respect to (Λ_0, L_0), $\pi(D)$ is a deduction respect to (Λ_1, L_1).

Mathematicians may have different opinion on how strong the axioms of mathematics should be, and may even disagree on the ground logic. Nevertheless, they respect each others' work because proofs in one system can usually be transformed to a proof in the system they like. Let's consider the following cases.

Case one: two mathematicians agree on the ground logic, e.g. the classical first order logic, and they are concerning about the same subject, so they are using the same language, but they do not agree with each other on the axioms of the subject.

The the relationship between Euclid geometry and absolute geometry is one of the instants of the case. A more fundamental case is that set theorists may have different opinion on the legality of the axioms of set theory, e.g. the axiom of choice.[1] In these cases, we can transform a proof of a statement into a proof of a conditional version of the statement.

Given set of axioms Λ_0 and Λ_1 in the same language. Let D be a proof of φ from Λ_0. Note that each proof is finite, and so there are only finite many assumptions used in a proof. We can assume the finite set $\Sigma \subseteq \Lambda_0$ contains all sentences used in D, and let $\widehat{\Sigma}$ be the conjunction of all sentences in Σ, which is a well-formed formula. Now we define $\pi(D)$ such that each formula ψ in D is replaced by $\widehat{\Sigma} \to \psi$. It is a manner of routine to check that $\pi(D)$ is a proof of $\widehat{\Sigma} \to \varphi$ even from $\emptyset$.

Case two: two mathematicians are interested in the same subject, but do not agree on the ground logic. In this case, other than classical logic, we only concern the intuitionistic logic because it is the only non-classical logic that has been seriously concerned by the community of mathematicians as the ground logic.

Classical logic is the strengthen of intuitionistic logic by adding ``$\neg\neg\varphi \to \varphi$". It is clearly that every proof by intuitionistic logic is a proof by classical logic. Therefore we consider the reverse direction.

[1] The axiom of choice (AC) is stated as ``for each family $F$ of nonempty sets, there exists a function $f$ whose domain is $F$ and for each $X \in F$, $f(X) \in X$". That is the function $f$ ``choose" one element in X for each set X in F.

Gödel [4] and Gentzen [2] showed independently that proofs by classical logic can be transformed to be a proof by intuitionistic logic. The so-called double negation translation is inductively defined as follows:

(i) $\varphi^N = \neg\neg\varphi$ for non-negated atomic formula φ,

(ii) $(\varphi \wedge \psi)^N = \varphi^N \wedge \psi^N$,

(iii) $(\varphi \vee \psi)^N = \neg(\neg\varphi^N \wedge \neg\psi^N)$,

(iv) $(\varphi \to \psi)^N = \varphi^N \to \psi^N$,

(v) $\big(\forall x\varphi(x)\big)^N = \forall x\varphi(x)^N$,

(vi) $\big(\exists x\varphi(x)\big)^N = \neg\forall x\neg\varphi(x)^N$.

And we have the theorem:

Theorem 2.3. $\Lambda \vdash_C \varphi$ if and only if $\Lambda^N \vdash_I \varphi^N$.

The $\vdash_C$ and $\vdash_I$ above indicate provability relation based on classical logic and intuitionistic logic respectively, and Λ^N is the set for all formulas translated from formulas in Λ by double-negation as defined above. See [23] or [22] for details.

Case three: One mathematician works in the framework of set theory, while the other only works in the world of nature numbers. For convenience, we assume they both work under the classical logic.

The language for set theory have a binary relation symbol $\in$ as its only parameter to indicate the membership relation between sets, while the language of number theory contains symbols $<, S, +, \cdot, 0$ for common indication.

As noted before, the structure $\mathfrak{N} = (\mathbb{N}, <, S, +, \cdot, 0)$ is defined in set theory, and every theorem of a true (as proved in set theory) number theory can be translated into a theorem of a typical set theory. Again, we consider the non-trivial direction: to transform a proof in set theory to a proof in number theory.

We further assume the set theorists works under ZFC, while the number theorists works under PA or even Q, which is PA minus the axioms of exponentiation[1] and the axioms of induction. The reason I choose Q is that it is the minimal system of axioms of arithmetic, which is still interesting enough. The following theorem can give such an impression.

Theorem 2.4. Let R be a n-placed relation on $\mathbb{N}$ defined by φ, then R is *computable*[2] if and only if R is determined in Q, i.e. for natural numbers $x_1, \dots, x_n$, if $\mathfrak{N} \models R(x_1, \dots, x_n)$, then $Q \vdash \varphi(x_1, \dots, x_n)$, and if $\mathfrak{N} \not\models R(x_1, \dots, x_n)$, then $Q \vdash \neg\varphi(x_1, \dots, x_n)$.

Actually Q is the weakest natural axiomatization, which determines computable relations.

By Gödel's coding, the statements concerning syntactic facts (of countable language) can be effectively translated into statements of natural numbers. For example, ``D is a proof of φ from Σ'', where Σ is a definable (in $\mathfrak{N}$) set of formulas, can be translated as (2.2). Note that the set of well-formed formulas and logically valid formulas, and the rules (functions) of deduction are all definable (and computable)

[1] However, the exponentiation operation can be represented in Q.

[2] By church's thesis, an n-placed relation of natural numbers is *computable* if and only if there exists an algorithm such that for each n-nary sequence, it is decided in finite steps whether the sequence is in the relation or not.

(2.2) D is a number coding a finite sequence $\langle e_1, \ldots, e_n \rangle$, e.g. $D = p_1^{e_1+1} \cdot p_2^{e_2+1} \cdot \cdots \cdot p_n^{e_n+1}$; and each e_i codes a well-formed formula, i.e. a finite sequence of numbers, where each number in the sequence corresponding with a symbol in the language; furthermore each formula coded by e_i is either logically valid or in Σ, or deducted by certain rules from the previous formulas.

in $\mathfrak{N}$. We will use $\#\varphi$, $\#D$ to indicate the Gödel's number of a formula or a deduction respectively, and $\ulcorner\Phi\urcorner$ to indicate the translation of a syntactic statement (in the language of set theory) Φ into the language of number theory. It can be proved (from ZFC) that if Φ is true, then $\mathfrak{N} \models \ulcorner\Phi\urcorner$.

If Σ is a computable set of formulas, then the binary relation

$$\rho^{\Sigma}(x, y) = \ulcorner x \text{ is a proof of } y \text{ from } \Sigma\urcorner$$

is computable as we can check it in finite steps. If a formula φ is provable from Σ, then we have $\mathfrak{N} \models \exists x \rho^{\Sigma}(x, \#\varphi)$. With theorem 2.2, it is easy to see

Theorem 2.5. Suppose Σ is a computable set of formula in a countable language. For each formula φ of the language,

$$\Sigma \vdash \varphi \Rightarrow Q \vdash \ulcorner\Sigma \vdash \varphi\urcorner.$$

The proof of Theorem 2.5 actually gives a effective method to transform every proof from Σ into a proof of the fact in Q.

As far as I know, there is no set theorist works under intuitionistic logic. And a number theorist working under the intuitionistic logic can understand the work of a set theorist by first transforming the proof into a proof in number theory using classical logic, and then further transforming it into a proof using intuitionistic logic.

Therefore, as we have seen, once a discovery of mathematica proof is achieved, it will illuminate the realm of mathematics, no matter which philosophical position people may stand on. Although the light might be considerably faint in some area, i.e. the translated result may be weak and lack of nature meanings, yet its value will remain at least positive. It is in the above sense that I claim the achievements are absolute.

Since the achievements of mathematics, i.e. the discoveries of mathematical proofs are absolutely *good*, they can be treated as the dead ends of the chains of practical consequences. Thus one major difficulty of pragmatism, say the vagueness of the notion ``consequence'', no long exists in the realm of mathematics. Therefore, I claim that to measure the positiveness of doctrines of philosophy of mathematics by their practical consequences is feasible.

3 Philosophical impacts on mathematical practice

As we argued above, it is appropriate to analyze the practice in mathematics from a pragmatistic point of view. Yet, we still have to show that philosophy does have impacts on mathematical practice. To argue for the existential statement, I am supposed to give some specific evidences as follows.

3.1 Philosophy gives motivations

Mathematicians' motivations come from all kinds of sources, for example the requirements for applications to the real world, e.g. physics or economics, or purely seeking for self-generalization and beauty. Although many working mathematicians are not even aware of philosophical issues, philosophy does play an crucial role in motivating the research of the fundamental problems of mathematics.

Long in the history, mathematics enjoyed the title of the most rigorous and objective intellectual activity of human beings. Immanuel Kant, in his representative work, even regard mathematics as the model of his ideal metaphysics. However, the foundation of mathematics was shaken by new techniques such as integral and new discoveries such as Russell's paradox. The research on the foundation of mathematics began blooming from late nineteenth century motivated by the philosophical requirement that mathematics should be rigorous and free from contradiction.

Under the slogan, several proposals have been put forward. Among them, logicism, formalism, and intuitionism are the big three. Yet all of the three programs have failed in some senses. However, the legacies are sumptuous. Frege, who was the initiator of logicism, was also the founder of the classical predicate logic. Russell's paradox buried Frege's project, but from its ruin rose the axiomatic set theory, which is the most successful resolution for the foundation of mathematics. Hilbert's program on finding a finitary foundation for mathematics was destroyed by Gödel's incompleteness theorems. Although the theorem is a negative answer to Hilbert's program, it lighted up a vast area of mathematics, and encouraged people to explore the new world of unprovability. Inspired by Brouwer's distinguishing interpretation of mathematical truth, the adherents of intuitionism has developed a splendid garden of intuitionistic mathematics based on the former mentioned intuitionistic logic. By studying the intuitionistic version of arithmetic and analysis, some subtle differences between notions, which are provable to be equivalent in classical mathematics, become visible.

However, I would like to leave the following space for more frontier programs.

3.1.1 Gödel's program

Gödel 's First Incompleteness Theorem was proved by Kurt Gödel before 1931, which states that

Theorem 3.1 (Gödel's First Incompleteness Theorem)**.** On a not too trivial subject of mathematics, e.g. $\mathfrak{N} = (\mathbb{N}, <, +, \cdot, 0)$, every computable set of axioms is incomplete unless it is inconsistent.

By ``incomplete'', we mean there exists a statement concerning the subject, while neither itself or its negation can be proved from the set of axioms. We also say, the statement is *independent* from the axioms.

Two typical examples of independent sentences of ZFC are the consistency of ZFC itself and Cantor's continuum hypothesis (CH)[1]. The former one is given by Gödel's second incompleteness theorem, while the independence of CH is finally proved by Kurt Gödel and Paul Cohen in 1963.

As a realist, Kurt Gödel would think: ``... a proof of the undecidability... would by no means solve the problem.'' A mathematical statement ``must be either true or false.'' ``... its undecidability from the axioms being assumed today can only

[1]The continuum hypothesis says that each uncountable set of reals has the same cardinality of the whole continuum.

mean that these axioms do not contain a complete description of that reality."[6, 260] Hence Gödel proposed that new axioms should be discovered to determine those still undecidable mathematical statement, e.g. CH. This is so called *Gödel's Program* which affect deeply the research in set theory until now.

If we need some additional axioms to decide whether CH or not CH, why don't just make CH or ¬CH an axiom? They are surely the weakest statement we can find to settle themselves, and so they are the safest as hypothesists in some sense. However, the solution may be accepted by a formalist, but not a realist. The realists suggest that each mathematical statement has an objective truth value, hence it can not be decided arbitrarily. Thus a *justification* is required for candidates of new axioms.

However, a candidate of a new axiom can not be justified mathematically since it is expected to be independent from the current axioms. Therefore we can only justify a candidate via a philosophical argument. Gödel has identified two kinds of philosophical argument, intrinsic and extrinsic. An *intrinsic* argument based on unfolding ``the content of the concept of set". [6, 264]. An *extrinsic* argument is given by an induction of the candidate's ``fruitfulness in consequences, in particular, ... consequences demonstrable without the new axiom, whose proofs with the help of new axiom... are considerably simpler and easier to discover...".[6, 265].

The theatrical situation is that to propose a philosophical justification one must provide enough mathematical facts as support. To demonstrate the observation, we will take a short tour into the theory of large cardinals and inner models.

The axioms of large cardinals (or higher infinity) are considered to have strong intrinsic and extrinsic argument. The axiom of infinite states that a property of the set theory universe, namely infinite, can be reflected into a initial segment of the universe, i.e. V_ω. On the other hand, it is to say that the property of infinite is not enough to illustrate the magnificence of the whole set theory universe. The large cardinal axioms are the stronger versions of the axioms of infinity. They extend the intuition of the first infinite by stating that the set theory universe have but not only have the property of inaccessibility, compactness, measurebility, etc. Because the large cardinal axioms are compatible with people's intuition that the operation ``set of" can always be iterated (see [6, 260]), they are considered by many set theorists to be the canonical instinctively justified new axioms for set theory.

The development of the large cardinal theory revealed an interesting phenomenon: all those naturally formalized large cardinal axioms forms a well-ordering under the relation of consistency strength, and nearly all of the interesting independent statement in set theory can be showed equiconsistent[1] with some large cardinal axioms. This gives us a systematically resolution of the consistency problem. Namely, we can now confidently claim that ZFC is consistent, because Con(ZFC) is provable from ZFC+ ``there exists an inaccessible cardinal". And nearly all reasonable extension of ZFC can be proved consistent by assuming ZFC + LCA or ZFC + Con(ZFC + LCA),[2]where LCA is some large cardinal axioms. Therefore the hierarchy of large cardinals plays a role as the benchmark of extensions of ZFC.

Even more, the large cardinal axioms also solved natural independent problems in arithmetic or analysis directly. For example,

[1]A theory T_0 is consistent relative to T_1, or say $T_0 \leq T_1$, if we can prove (from base theory such as RCA_0, ZF and ZFC) that $\mathrm{Con}(T_1) \to \mathrm{Con}(T_0)$. We say T_0 and T_1 are equiconsistent, or $T_0 \equiv T_1$, if $T_0 \leq T_1$ and $T_1 \leq T_0$.

[2]In the former case, the consistency strength of ZFC + LCA is strictly greater then the extension, while in the latter case, the consistency strength of ZFC + LCA is greater or equal to that of the extension by the Gödel's second incompleteness theorem.

Theorem 3.2 (Shelah-Woodin[17])**.** If there exists infinitely many Woodin cardinals, then every projective set of reals is Lebesgue measurable.

Woodin cardinal is a large cardinal property, and projectiveness is a property of sets of reals, which defines a certain level of complexity. I will return to the example and provide more background in subsection 3.2.

Those resolutions are so elegant that lots of set theorists accept large cardinal axioms as canonical extension of ZFC, except that they might be inconsistent and they are not omniscience.

Those large cardinal axioms are lined up by consistency strength, which means we can not hope to form a even relative consistency proof of a stronger large cardinal axioms from a weaker one. Thus the higher we step on the hierarchy, the closer we are to the inconsistency. And the danger is no joke.

Theorem 3.3 (Kunen[11])**.** Assuming ZFC, there is no *Reinhardt cardinal*, i.e. there is no non-trivial elementary embedding $j : V \mapsto V$.

The Reinhardt cardinal is suggested by William Nelson Reinhardt. It is the critical point of a non-trivial elementary embedding from the whole universe to itself. It is the strongest large cardinal have ever been defined. Unfortunately, it is inconsistent with ZFC.

Even we accept all of large cardinal axioms that have been defined and not yet been proved inconsistent, they still cannot solve all independent problems. There exists some problems who can not be determined by any of the large cardinal axioms. CH is one of them. To explore even more about large cardinals and continuum hypothesis, I shall introduce an important large cardinal property.

Definition 3.4. A cardinal κ is *measurable* if there exists a non-principal κ-complete ultrafilter on U on $P(\kappa)$.

Gödel has proved in 1938 that if all sets are constructible, i.e. $V = L$, then CH is true. However,

Theorem 3.5 (Scott[15])**.** If there exists a measurable cardinal, then $V \neq L$.

Because large cardinal axioms clearly receive more support based on the facts mentioned previously, people tend to believe that $V = L$ is fault.

In 1963, Paul Cohen invented a powerful tool, *forcing*, to build a model extending a given model of ZFC where the negation of CH holds. He was granted the Fields medal for this contribution. The idea is to ``add'' $> \aleph_1$ many generic (new) reals into the new model (and preserve the cardinals). Similarly, one can also ``add'' a mapping from $\aleph_1$ onto $\mathbb{R}$ (yet preserve $\mathbb{R}$), which would collapse the cardinality of continuum to $\aleph_1$, i.e. witness CH holds in that model. Therefore, the truth of CH can be flipped throughout the forcing extensions. Moreover, those forcings usually preserves large cardinal properties:

Theorem 3.6 (Lévy-Solovay[12])**.** Let κ be a measurable cardinal, U be a non-principal κ-complete ultrafilter on $P(\kappa)$, and assume $\mathbb{P}$ is a forcing notion of size less than κ. Then the filter generated from $\check{U}$,

$$U' = \left\{\dot{A}^G \mid (\exists p \in \mathbb{P}) p \Vdash [\dot{A} \subseteq \check{\kappa} \wedge (\exists x \in \check{U})(x \subseteq \dot{A})]\right\}.$$

is a non-principal κ-complete ultrafilter in the forcing extension by $\mathbb{P}$. Thus κ remains to be a measurable cardinal in the forcing extension.

Theorem 3.6 tells that the axioms of large cardinals as measurable will not settle the problems which can be flipped via forcings with small sized forcing notions. Thus if we want to settle statements like CH, or even more ambitiously, if we want to reach to a ``effectively complete'' set theory (its theorem will not be easily changed by forcing), we have to find other approaches to strengthen the set theory.

The inner model program has been suggested to give evidences of the consistency of large cardinal axioms. Surely, we cannot prove the consistency of large cardinal axioms. However if a large cardinal can live in a more transparent inner model of the universe, and if it is inconsistent, we can find the witness easily in that inner model. Luckily, recent research showed the inner model program may also provide candidates of axioms extending ZFC + LCA, which may decide the truth value of CH.

Inner models are those parameter definable subclasses of the universe, which are transitive and satisfying ZFC. The class of constructive set L is a canonical inner model, and it is provably the minimal one. $V = L$ settles lots of independent results. However, as we have mentioned, $V = L$ is not compatible with large cardinals. Thus people tend to believe $V \neq L$. The *inner model program* is to find canonical inner models which is compatible with large cardinals.

A cardinal κ can not be measurable in L, because the universe L is too thin to contain the ultrafilter U who witnesses the measurability of κ. Thus the idea is to put U into the constructible universe. And it works for measurable cardinal.

Theorem 3.7. Let κ be a measurable cardinal witnessed by U. Then $\bar{U} = L[U] \cap U$ is a non-principal κ-complete ultrafilter in $L[U]$.

Thus κ is preserved as a measurable cardinal in $L[U]$. However, in this case, κ is the only measurable cardinal in $L[U]$. It becomes much harder even to move one step higher.

Definition 3.8. A cardinal κ is an λ-*strong cardinal* if there exists an elementary embedding $j : V \mapsto M$ with critique point κ, i.e. κ is the first ordinal such that $j(\kappa) > \kappa$, and $V_\lambda \subseteq M$.

κ is *strong* if it is λ-strong for all λ.

It can be show that a measurable cardinal κ is $(\kappa + 1)$-strong, and if κ is a $(\kappa + 2)$-strong cardinal, then there are κ many measurable cardinals below it. The property of strongness can be witnessed by *extenders*, systems of ultrafilters, as measurable can be witnessed by an ultrafilter. For example, κ is λ-strong if and only if there is a $(\kappa, |V_\lambda|^+)$-extender E such that $V_\lambda \subseteq \text{Ult}(V, E)$ and $j_E(\kappa) > \lambda$. Note that $\text{Ult}(V, E)$ and j_E are definable from E.

To find an inner model for λ-strong cardinal where $\lambda > \kappa + 1$, a natural first attempt is to check $L[E]$ where E witnesses the strongness of κ in V. However, it can be proved that $L[E] = L[U]$ where $U = E_{\{\kappa\}}$. Thus there can be no more than one measurable cardinal in $L[E]$, and no λ-strong cardinal in $L[E]$.

Mitchell (1974) showed that $L[\mathcal{U}]$ can contain many measurable cardinals where $\mathcal{U}$ is a sequence of ultrafilters with a coherent condition. This technique can be generalized to construct an inner model $L[\mathcal{E}]$ for a strong cardinal where $\mathcal{E}$ is a proper class and a coherent sequence of extenders. The technique developed by Mitchell and Steel for the inner model program was also applied in other area, say to prove the consistency of determinacy, which we will discuss in the next subsection.

Currently, the inner model program has climbed up to Woodin cardinal (the set of strong cardinals in V_δ is unbounded if δ is a Woodin cardinal), and superstrong cardinal with a plausible assumption. In 2010, Woodin [26] observed that if we can find an inner model for supercompact cardinal (way beyond strong, Woodin, and superstrong), than it is an ultimate inner model for effectively arbitrary large cardinals. And it is likely that CH would hold in the ultimate L-like model. The above results constitute just a small core of the mathematical consequences of Gödel's program. The techniques developed in the program have innumerable applications, which are impossible to be listed here.

3.1.2 Friedman's program

In many informal discussions, Friedman expressed that he can not have a visualization of a strong theory, say ZFC plus some large cardinal axiom, unless it has consequences in ``concrete'' mathematics.[1] Thus he joked on himself, ``I am always wearing a hat'', which means he will not accept arbitrarily strong systems, but the systems that he can visualize.

Harvey Friedman has proposed a program to find simple (Π^0_2 or even Π^0_1) and natural (not meta-mathematical) arithmetical statements that require strong systems to settle. Strong systems can be visualized if it gives convincing insight on such a concrete independent mathematical problem.

Friedman's ideas remind people of Hilbert's program, which intended to legalize the strong systems by providing them a finitary consistency proof. Hilbert's program demands a finitistic or concrete proof for each theorem of the classic mathematics, which including abstract mathematics such as set theory; while Friedman's program, on the contrary, involving statements of concrete mathematics yet requiring a proofs from the abstract. However, they share the presupposition that finitary or concrete mathematics such as a weak fragment of first order arithmetic is intuitionistic, visualizable and safe.[2] And they will accept the rests of mathematics if they are proved to be able to inherit some of these properties. It is clear that Friedman's position is much weaker than Hilbert's, which is arguably hopeless by Gödel's Theorems. Hilbert wanted to justify all classic mathematics once for all, while Friedman is only trying to climb as high as possible. I call this an *open-minded constructivism*. Bill Tait summarized: ``the point of Friedman's program'' is ``to justify the introduction of cardinals by their low-down (combinatorial or whatever) consequences.''[21]

Reverse Mathematics is proposed by Friedman (1975) [1], which is a section of the general Friedman's program. The background of reverse mathematics is the phenomenon that the subsystems of second-order arithmetic having fundamental significance are linearly ordered in the consistency strength relation. This coincides the phenomenon in the large cardinal theory. Stephen Simpson [18] provided a whole picture of the benchmarks of the mathematical systems in the hierarchy of consistency strength as in Figure 3.1.

The reverse mathematics program focus on the subsystems of the full second-order arithmetic. Among them, RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0 are the big five, which are particularly interesting because each of them is arguably corresponding to a certain level of constructivism. Thus another motivation of

[1] Friedman using *concrete* to denote finite structures, finitely generated structures, discrete strucures (countably infinite), continuous and piecewise continuous functions between Polish (complete separable metric) spaces, and Borel measurable functions between Polish spaces. In the contrast, abstract denotes sets, large cardinals, etc.

[2] These weak systems of mathematics are safer from the perspective of mathematical constructivism, which I shall talk more in the next section.

strong:

$0 = 1$ (Reinhardt cardinal)
I0
$\vdots$
ω-huge
$\vdots$
supercompact cardinal
superstrong cardinal
Woodin cardinal
strong cardinal
measurable cardinal
$\vdots$
ZFC
ZC
simple type theory

medium:

Z_2 (second-order arithmetic)
$\vdots$
Π^1_2-CA_0 (Π^1_2 comprehension)
Π^1_1-CA_0 (Π^1_1 comprehension)
ATR_0 (arithmetical transfinite recursion)
ACA_0 (arithmetical comprehension)

weak:

WKL_0 (weak König's lemma)
RCA_0 (recursive comprehension)
PRA_0 (primitive recursive arithmetic)
EFA (elementary function arithmetic)
$\vdots$
Q

Figure 3.1: Benchmarks in the Consistency Strength Hierarchy

reverse mathematics is to draw a picture of how much one can expect to preserve under a certain philosophical doctrine.

Basically, reverse mathematics checks the strength of important mathematical theorems. Given a proved theorem φ (or theorem scheme), adherents of the program try to find a system T_1 in the hierarchy so that T_1 proves φ, and prove that T_1 is the weakest such system by showing that φ is actually equivalent to T_1 from the perspective of a weak base system T_0, i.e.

$$T_0 \cup \{\varphi\} \vdash T_1 \tag{3.1}$$

Therefore reverse mathematics does produce new theorems. Simpson in his survey paper [18] listed the major achievements from the program of reverse mathematics. These results together drew a real map of the whole world of mathematics (people can have only seen the top-down way without reverse mathematics).

3.1.3 Multiverse view

In 2011 [7], Joel Hamkins proposed a ``provocative" philosophical view of set theory to deal with the independence phenomenon. It is the *multiverse view*. The traditional realism on set theory is classified as the *universe view* in contrast to the multiverse view. Universists think all *sets* exist, and there is a true comprehension of the concept of set, while the multiverse view is claimed to be a second order realism, i.e the universes of sets exist, and correspondingly, there are many set concepts, and it is impossible to decide which one is the truth (although they allow some preference among the other concepts). From the prospect of multiverse view, problems as CH have already been settled in the sense that we are so familiar with universes where CH is hold or not and we understand so thoroughly that how it can be satisfied or falsified. And because of the situation, a universists' dream solution, namely finding a self-evident or justifiable principle, which proves CH or its negation, will never come. Therefore, the fundamental task for set theorists is no longer searching for new axioms, but to understand all those different set theory universes and the relationship among them. We will have a glimpse of what the multiversists have achieved guided by the philosophical innovation.

The key point here is that the set theory universes can be considered as if they exist. Therefore, as the axioms of set theory expressing the closure properties of the set theory universe, we would expect a formal description of the richness of the multiverse. Hamkins [7] found it is possible to have a first order formalization of the multiverse axioms:

Definition 3.9 (Multiverse Axioms, Hamkins)**.**

(i) *Realizability Principle*. For each model M of ZFC in the multiverse, if N is an inner model of M from the prospective of M, then N is in the multiverse.

(ii) *Forcing Extension Principle*. For any model M of ZFC in the multiverse and any forcing notion $\mathbb{P}$ in M, the forcing extension $M[G]$, where G is M-generic for $\mathbb{P}$, is in the multiverse.

(iii) *Reflection Axiom*. For any model M of ZFC in the multiverse, there exists a ``taller" model N of ZFC in the multiverse such that M can be embedded into a initial segment of N.

(iv) *Countability Principle*. For each model M of ZFC in the multiverse, there exists a model N of ZFC in the multiverse, such that from the prospective of N, M is countable.

(v) *Well-foundedness Mirage*. For each model M of ZFC in the multiverse, there exists a model N of ZFC in the multiverse, such that from the prospective of N, M is ill-founded.

(vi) *Reverse Embedding Axiom*. For any model M_1 of ZFC in the multiverse and any elementary embedding $j : M_1 \mapsto M_2$ in M_1, there exists a model M_0 in the multiverse and an elementary embedding $i : M_0 \mapsto M_1$ in M_0 such that $j = i(i)$.

Further more, Gitman and Hamkins have proved the above formalization is coherent.

Theorem 3.10 (Gitman-Hamkins[3])**.** If ZFC is consistent, then the class of all countable computably saturated models of ZFC satisfies all the multiverse axioms.

The trick in the proof is that each countable saturated model of ZFC has a nonstandard natural number (hence ill-founded), and contains an isomorphism copy (observed from outside) of itself, who it thinks is countable and has a nonstandard natural number. The proof is no doubt a valid proof from ZFC, which provided a definable subclass of the universe having the desired closure properties. Thus this is definitely an interesting result even from a universe view, yet the motivation behind is clearly the multiverse view of set theory.

Another interesting achievement inspired by the multiverse view is the modal logic of forcing. Modal logic is based on the language of classical logic with some additional symbol, say $\Diamond$ or $\Box$ for modality, e.g. possibility, knowledge, or tense. A typical model for modal logic is a relation model (W, R), where W is the set of ``possible worlds" and $R$ is the accessibility relation on $W$. Thus ``φ is possible", i.e. $\Diamond\varphi$ can be interpreted as: there is a possible world which is accessible from the current world (have the relation R), and φ holds there. It is natural for multiversists to treat the set theory multiverse with the forcing extension relation as a modal logic model.

We can define the language of modal logic of forcing as the language of set theory equipped with a single modality $\Box$, and the well-formed formula is inductively defined as usual except that we do not allow the quantifiers to range over modality.

The interpretation of the modality in the language is defined as follow.

Definition 3.11. Given a set theory formula φ. $\Box\varphi$ is true from the perspective of the universe M, if and only if for each forcing notion $\mathbb{P}$ of M, $M[G] \models \varphi$ for every M-generic filter G over $\mathbb{P}$ (or we say $M^{\mathbb{P}} \Vdash \varphi$). $\Diamond\varphi$ is true from the perspective of M, if there exists a forcing notion $\mathbb{P}$ of M such that $M^{\mathbb{P}} \Vdash \varphi$.

Hamkins and Löme have showed that the the modal logic axioms system S4.2 is valid for multiverse of ZFC models.

Theorem 3.12 (Hamkins-Löme[8])**.** For each set theory formula φ. The following are hold from the perspective of a ZFC model.

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Dual	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
T	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
.2	$\Diamond\Box\varphi \to \Box\Diamond\varphi$

The proof is quite straight forward. K is valid because forcing respect implication. Dual holds because a statement φ is not forceable, i.e. there is no forcing notion $\mathbb{P}$ such that $M^{\mathbb{P}} \Vdash \varphi$, if and only if $\neg\varphi$ holds in every forcing extensions. Note that we interpret $\Diamond\varphi$ as ``there exists a forcing notion $\mathbb{P}$, and *for any* generical filter G over $\mathbb{P}$, $M[G] \vDash \varphi$", so Dual is not completely trivial. T is valid because there can be a forcing notion $\mathbb{P}$ containing a strongest condition p, then the generic filter $G = \{q \in \mathbb{P} \mid q \text{ is compatible with } p\}$ is definable from $(\mathbb{P}, p)$, hence in the ground model. We know that a generic extension $M[G]$ is the smallest extension of the ground model M containing G. Thus if $G \in M$, then $M[G] = M$, which is said to be a trivial forcing extension. Formula 4 is given by the fact that any two step iteration of forcing can be achieved by a single forcing, see [9, 267]. And .2 is given by the product forcing.

Theorem 3.13 (The Product Forcing Theorem[19])**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be two notions of forcing in M. If $G_1 \subseteq \mathbb{P}$ is generic over M and $G_2 \subseteq \mathbb{Q}$ is generic over $M[G_1]$, then G_1 is generic over $M[G_2]$, and $M[G_1][G_2] = M[G_2][G_1]$.

Let $\mathbb{P}$ be a forcing notion of M such that for each M-generic filter G_1 over $\mathbb{P}$, we have for all forcing notion $\mathbb{Q}$ of $M[G_1]$ and all $M[G_1]$-generic filter G_2 over $\mathbb{P}$, $M[G_1][G_2] \vDash \varphi$. Then for any forcing notion $\mathbb{Q}$ of M and any M-generic filter G_2 over $\mathbb{Q}$, $\mathbb{P}$ is a forcing notion of $M[G_2]$ (and also of M), and for all $M[G_2]$-generic filter G_1 over $\mathbb{P}$, $M[G_2][G_1] = M[G_1][G_2]$, and thus satisfies φ.

Moreover, Hamkins and Löme have argued informally that S4.2 is the best one can have among the usual axiom systems of modal logic. For example,

$$5: \qquad \Diamond\Box\varphi \to \varphi$$

can be violated because there are set theory statement, e.g. ``ω_1^L is countable", which can be forced once forever yet false in the ground model. And

$$M: \qquad \Box\Diamond\varphi \to \Diamond\Box\varphi$$

is not valid because statement as CH can be forced to be true or false from any ground model. Statements like ``ω_1^L is countable" are called buttons, which can be pushed down and not reversible; while statements like CH are switches, which can be turned on and off. These researches lead to an interesting subject exploring the structure of the generic multiverse.

3.2 Philosophy makes conjectures

When we are working on mathematics, it is much better if there is a target ahead. That is why doing exercises is always much easier than doing research, when we do not even know the statement we are trying on is provable on or not. Nevertheless, working mathematicians always set targets by themselves. They make *conjectures*.

Conjectures on mathematics come from various sources. In finite cases, mathematical statements may have physical witnesses, and people have strong intuitions on those mathematical objects. Thus conjectures can be based on experiences and intuitions. However, when we come to the realm of infinite, the intuition fades away desperately. Infinite objects can behave crazily, which can be way beyond our imagination. In these cases, only philosophy together with the verified knowledge of mathematics can give a mathematician some inspiration. The rest of the subsection is to exhibit a significant case in set theory, which demonstrates that philosophy helps mathematicians make conjectures.

The story concerns the proofs of the axioms of determinacy. Some axioms of determinacy are expected to be the missing true statement in set theory, e.g. the axiom of projective determinacy. And eventually they are justified by being proved from large cardinal axioms. We will see how philosophy plays a role in discovering the theorems as:

Theorem 3.14 (Martin-Steel[13])**.** Suppose there exist infinite many Woodin cardinals, then every projective set is determined.

The axiom of choice (AC) asserts that every set can be well ordered. It has been attacked continuously throughout the history of set theory because of its lacking of intuition and the consequences that apparently contradicts the common sense. One typical example is the *Banach-Tarski paradox*: A solid ball in 3-dimensional space can be decomposed into finitely many pieces, and by rotations and translations, these pieces can be put together to be two identical copies of the original ball. However, the axiom of choice is so powerful in proving theorems, most mathematicians choose to tolerate it.

The construct of Banach-Tarski paradox is a generalization of Giuseppe Vitali's proof of the existence of non-measurable sets.[1] Actually the pieces found in Banach-Tarski paradox must be non-measurable. A natural idea is that people should never encounter such ``paradox'' in actual mathematical practice. Some regularity properties is proposed, e.g. Lebesgue measurable, the property of Baire[2], and the perfect set property[3]. And they are supposed to be possessed by all ``simple'' sets.

In descriptive set theory, the complexity of sets of reals[4] are represented by the Borel hierarchy and the projective hierarchy. The family of all open set is defined to be $\mathbf{\Sigma}^0_1$. A set A is $\mathbf{\Pi}^0_\alpha$ if the complement of A is $\mathbf{\Sigma}^0_\alpha$. For $\alpha > 1$, A is $\mathbf{\Sigma}^0_\alpha$ if there exists $\langle A_i : i < \omega \rangle$ such that each A_i is $\mathbf{\Pi}^0_{\alpha_i}$ for some $\alpha_i < \alpha$ and $A = \bigcup_{i<\omega} A_i$. $\mathbf{\Delta}^0_\alpha$ is defined to be the intersection of $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$. The Borel hierarchy is graphed as Figure 3.2, where the arrows indicate the relation of inclusion. We say a set A

$$\begin{array}{ccccccccccccc}
 & & \mathbf{\Sigma}^0_1 & & & & \mathbf{\Sigma}^0_2 & \cdots & & & \mathbf{\Sigma}^0_\alpha & & & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & & & \nearrow & & \searrow & & & \\
\mathbf{\Delta}^0_1 & & & & \mathbf{\Delta}^0_2 & & & \cdots & \mathbf{\Delta}^0_\alpha & & & & \mathbf{\Delta}^0_{\alpha+1} & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & & & \searrow & & \nearrow & & & \\
 & & \mathbf{\Pi}^0_1 & & & & \mathbf{\Pi}^0_2 & \cdots & & & \mathbf{\Pi}^0_\alpha & & & & \cdots
\end{array}$$

Figure 3.2: The Borel hierarchy

is *Borel* if it is in the Borel hierarchy, equivalently the family of Borel sets is the smallest who contains all open sets and closed under complement and countable union.

[1]We say a subset A of reals is *Lebesgue measurable* if for all $\varepsilon > 0$ there is a closed $C \subseteq A$ and an open $U \supseteq A$ such that the outer measure $\mu^*(U \setminus C) < \varepsilon$.

[2]A set $A \subseteq \mathbb{R}$ has the *property of Baire* if it differs from a open set by a meager set (a countable union of nowhere dense sets).

[3]We say a set $A \subseteq \mathbb{R}$ has the *perfect set property* if A is either countable or contains a nonempty perfect subset. In other words, it can not be an evidence of $\neg$CH.

[4]Descriptive set theorist always think of the Baire space $\mathcal{N} = \omega^\omega$ as the space of reals. Both Baire space and the standard space of the real line are Polish space.

Analytic sets are continuous images of Borel sets (actually enough to be continuous image of $\mathcal{N}$). Suslin showed that there are analytic sets which are not Borel.

Projective hierarchy goes even beyond. The family of analytic sets is declared to be $\mathbf{\Sigma}^1_1$. Sets in $\mathbf{\Pi}^1_n$ are complements of $\mathbf{\Sigma}^1_n$ sets. $\mathbf{\Sigma}^1_{n+1}$ sets are projections of $\mathbf{\Pi}^1_n$ sets. The structure of projective hierarchy has a similar picture of the Borel hierarchy[1] except that it has length ω.

It was proved in the early years of descriptive set theory (mostly in 1930) that all Borel sets and even analytic sets are Lebesgue measurable, have the property of Baire and the perfect set property. The proofs are all made in ZFC. It was natural to ask whether the projective sets also possess the properties of regularity. However, no progress was made until 1938, when Gödel developed the constructive universe L, in which there exists a non-measurable $\mathbf{\Sigma}^1_2$ set.[2] In other words, Gödel showed

$$\text{Con(ZFC)} \rightarrow \text{Con(ZFC} + \neg\mathbf{\Sigma}^1_2\text{-Measurable)} \tag{3.2}$$

Therefore, it is technically impossible to find a proof of the regularity of all projective sets within ZFC as long as ZFC is consistent. Here comes a familiar situation, the consistency of continuum hypothesis. People who think mathematics is all about proving theorem in ZFC may have no interest in seeking for a ``proof'' of the regularity of projective any more.[3] However, as dealing with the continuum hypothesis, realists will continue to search for a ``proof'' or ``refutation'' of the regularity beyond analytic.

Since projective sets are considered to be naturally constructed, in the sense that they are definable sets of reals using reals numbers as parameter. Although the regularity of projective sets can not be proved within ZFC, Set theory realists tended to believe they are true, and kept trying to find some self-evident axioms to prove them. Thus they conjectured

Conjecture 3.15. There will be a justifiable extension of ZFC, which proves the regularity of projective sets.

In 1962, the axiom of determinacy (AD) was introduced by Hugo Steinhaus and Jan Mycielski[20]. The concept of determinacy is based on the *infinite games*: Given a set of reals $A \subseteq \mathcal{N}$, two players, I and II play on natural numbers in turn

$$\begin{array}{lcccccc} \text{I:} & a_0 & & a_2 & & \dots & \\ \text{II:} & & a_1 & & a_3 & & \dots \end{array}$$

A *play* is an infinite sequence $\langle a_i : i < \omega \rangle$ enumerated by I and II. I wins the game G_A if the play is in A, otherwise II wins. A *strategy* is a function $f : \omega^{<\omega} \mapsto \omega$ telling player which move to take at each stage. A play r is *consistent* with the strategy f for player I if for each $n < \omega$, $r(2n+1) = f(r{\restriction}2n)$, which means the play is a possible result if I follows the strategy f. For player II, we interchange the even and the odd. A strategy f is a *winning strategy* for I if all play r consistent with it on the I's part are in A; similarly for player II. A game is determined if either player has a winning strategy. The axiom of determinacy states that for every set of reals A, the game G_A is determined. Soon after AD being proposed, it was found that nearly all problem in descriptive set theory can be settle by assuming

[1]Note that the projective hierarchy is essentially the same with relativized analytical hierarchy.

[2]In L, the set of reals is well ordered by a $\mathbf{\Sigma}^1_2$ binary relation, and a non-measurable Vitali set derived from such well-ordering has the same complexity.

[3]In [16], Saharon Shelah took the value of the independence results to be ``The Rubble Removal'', i.e. they leave the ``strong candidates for theorems of ZFC''.

AD. Particularly, assuming AD, all sets of reals have the Baire property (Mazur-Banach); every set of reals is Lebesgue measurable (Mycielski-Świerczkowski); and every uncountable set of reals containing a perfect set, so CH holds (Morton Davis).

The bad news is that AD had already be proved to be inconsistent with AC (Gale-Stewart). Since set theorists prefer AC, AD cannot be a candidate of new axioms. Nonetheless, the good news is that all of the proofs above are ``local'', e.g. the proof of AD implies all sets are Lebesgue measurable can be transferred to prove that if all $\mathbf{\Sigma}^1_n$ sets are determined, then all $\mathbf{\Sigma}^1_{n+1}$ are Lebesgue measurable. Therefore, if we assume that all projective sets are determined (PD), then all projective sets satisfied the properties of regularity, which is exactly what people desired. However, the axioms of determinacy hardly received any intrinsic justification from philosophy. Although, PD is pretty fruitful in its consequences, people still expected it can be followed by some more plausible statement.

It is reported that in 1964, Solovay discovered

$$\text{ZFC} + \text{PD} \vdash \text{Con}(\text{ZFC} + \text{ there exists a measurable cardinal}).$$

This shows PD has some relationship with large cardinal axioms, and it is not provable merely from the existence of a measurable cardinal. Nonetheless Solovay made the following conjecture in the late 1960's (see [10, 378]).

Conjecture 3.16. ZFC together with some large cardinal axioms proves the projective determinacy, so the regularity properties of projective sets.

Again, the large cardinal axioms came as the most promising solution. They are philosophically well-justified (see the Gödel's program above) and the most canonical extensions of ZFC. This time, not as the case of the continuum hypothesis, large cardinal do settle the problem. But the discovery of the fact is by no means straightforward.

The first progress in this program came in 1969 when Donald A. Martin showed that every analytic ($\mathbf{\Pi}^1_1$) set is determined assuming there exists a measurable cardinal in the universe. This gives an evidence that we were on the right way. However, there was no progress for the next nine years until in 1978, Martin made only one move up by assuming a very strong large cardinal axioms. He proved $\mathbf{\Pi}^1_2$ determinacy from the statement asserting the existence of an ω-huge cardinal. Finally, in 1984, Woodin introduced I0, and proved PD from it. I0 was the strongest (which had not been proved inconsistent) large cardinal axioms at that time. This result completed the program in the weakest sense. Since the assumption, I0, is just below the edge of the known inconsistency, people could not help doubting on its consistency, needless to say on its truth.

In 1988 [24], Woodin surprisingly declared that if there exists a supercompact cardinal, then every projective sets are Lebesgue measurable. Recall that $\mathbf{\Sigma}^1_n$-Determinacy implies $\mathbf{\Sigma}^1_{n+1}$-Lebesgue measurable. It was reasonable to expect that supercompact cardinal would also gives the determinacy of projective sets. Actually, Woodin (with Shelah) successfully strengthen the result by relating the projective measurability with the existence of Woodin cardinals. This is Theorem 3.2. Finally, theorem 3.14 is proved by Martin and Steel in 1988 [13].

Since the inner model program has successfully covered Woodin cardinals. People had much more confidence on this weaker large cardinal axiom. And because of the intrinsic argument for the large cardinal axioms and the fruitfulness of the consequences of large cardinal and determinacy axioms, it is accepted by many set theorists that PD is indeed a missing truth of set theory.

Let us make a short review of the whole story, and see how philosophy played a role. Conjecture 3.15 set the goal for the whole program. The idea behind the conjecture is that first, people did not want to lose AC, rather they believe they can live safely with it; second, although PD cannot be proved within ZFC, realists insisted it must have a truth value, and they believe the value is positive, and this preference requires a justification. Note that the tolerance of AC is backed Gödel's extrinsic argument, and the significance of the conjecture can only be recognized by an adherent of Gödel's realism.

The pattern is even clearer on conjecture 3.16. It is nature for people who follow Gödel's doctrine and take the large cardinal axioms as canonical extension of ZFC to make such a conjecture. Actually, now we know that PD is provable from many infinite combinatorial statements, e.g. Martin's Maximum. However, until now, there is no known direct proof of these theorem. The proofs always involving models of Woodin cardinals axioms. See [25, 575]. Thus if it is not the luck that set theorists made the conjecture 3.16, a proof of PD may be still concealing itself from us.

3.3 Philosophy sets barriers

It is definitely not the case that every philosophical idealogy does good to all mathematical practice. Recall the case we have just considered. A typical formalist concentrating on theorems of ZFC will regard (3.2) as a solution, and she may find no interests in making a conjecture like 3.15 (see the discussion below (3.2)). As for a constructivist, the large cardinals and the full set theory universe are fictional. The attempt to justify PD by proving it from large cardinal axioms is ridiculous. Thus conjecture 3.16 can never be proposed by a constructivist. In other words, the philosophical standpoint became a barrier for formalists or constructivists to discover the proofs.

In the following text, I will provide another demonstration of how philosophical preconception blocks people's genius. It is about Russell's ramified type theory and Gödel's constructible universe L. I shall demonstrate that the structure of L is nothing but the generalization of ramified hierarchy to arbitrary ordinal numbers.

As we know, Russell's type theory serves as a solution of the foundation of mathematics. There are two versions of type theory, namely the simple type theory and the ramified type theory. It was pointed out by Ramsey and other scholars that the simple type theory is sufficient to serve as a consistent foundation of mathematics, while the only reason for Russell to propose the much more complicated ramified type theory is to eliminate the semantic paradoxes, e.g. ``the first indefinable ordinal''. Let us take a look at the structure of the simple types and the ramified types.[1]

In type theory (no matter simple or ramified), the argument of a property (or propositional function) can only range over a certain type. The individuals form the first type, and all properties of things in the nth type constitute the $(n+1)$th type. The structure of simple type theory is pictured in Figure 3.3.

The ramified type theory further classifies properties of each types into orders. The first-order (predicative) properties of type s are properties of things in type s whose definition (function) containing only quantifiers ranging over type s and parameters in type s. The second-order properties are also the properties of things in type s, but the definition can involve a reference to the first-order properties. It

[1] For convenience, we only concern the unary case.

Type 0:	a_1	a_2	a_3	$\dots$
Type 1:	P_1	P_2	P_3	$\dots$
Type 2:	$\mathscr{P}_1$	$\mathscr{P}_2$	$\mathscr{P}_3$	$\dots$
		$\vdots$		

Figure 3.3: hierarchy of simple types

is forbidden to refer to all properties (functions) of a type. Orders and types of ramified type theory can be defined inductively as follow.

Definition 3.17. (i) *Types*: Type 0 is the universe of individuals. For each n, type $n+1$ is the collection of all jth-order properties of things in type i for all i, j such that $i+j=n$;

(ii) *Orders*: φ is the $(m+1)$th-order property of things in type n if its free variables range over type n and the bound variables range over no higher than type $n+m$. We say the property is *predicative* when $m=0$.

The structure of ramified type theory looks like:

Type 0:	individuals[1]
Type 1:	1st-order properties of Type 0
Type 2:	2nd-order properties of Type 0 1st-order properties of Type 1
Type 3:	3rd-order properties of Type 0 2nd-order properties of Type 1 1st-order properties of Type 2
$\vdots$	$\vdots$

Figure 3.4: hierarchy of ramified types

Ramified type theory is notorious for the axiom of reducibility, which says: ``Every propositional function is equivalent, for all its values, to some predicative function."[14, 167] This is to say if we concern only the extensions, all properties of a certain type have already appeared in the first order. The axiom is proposed to preserve enough induction principle so that the classical mathematics can be represented in the ramified type theory.

Now let us turn to the constructible universe.

Definition 3.18. Let $L_\alpha(\omega)$ be defined inductively.[2]

(i) $L_0(\omega)=\omega$,

(ii) $L_{\alpha+1}(\omega)=\mathcal{D}(L_\alpha(\omega))$,

(iii) For limit ordinal α, $L_\alpha(\omega)=\bigcup_{\beta<\alpha} L_\beta(\omega)$.

[1]Russell introduced an axiom of infinite asserting that there are infinite many individuals.

The *constructible universe* $L = \bigcup_{\alpha \in Ord} L_\alpha(\omega)$.

Informally, $\mathcal{D}(L_\alpha(\omega))$ is the family of all sets of the form $\{x \in L_\alpha(\omega) \mid L_\alpha(\omega) \vDash \varphi[x, a]\}$ where φ is a set theory formula and a is a parameter in $L_\alpha(\omega)$. Therefore $L_{\alpha+1}(\omega)$ contains all definable subsets of $L_\alpha(\omega)$ using parameters in $L_\alpha(\omega)$.

It is not hard to see if we concern only the extensions of the properties and allow the types to be downward compatible (which is pretty reasonable), then the hierarchy of types coincides with that of constructible sets in the finite orders. The only difference is that the hierarchy of ramified types is not defined for limit ordinal orders. If we regard the constructible hierarchy as the natural extension, i.e. we extend the ramified types to transfinite orders, an interesting consequence is that we can show the axiom of reducibility in the strong sense is false, while in a weak version is provable from ZFC.

Definition 3.19. We say $A \in L$ can be *constructed within* $\beta + 1$ *steps* if and only if $A \in L_{\alpha+\beta+1}(\omega)$ where α is the least such that $A \subseteq L_\alpha(\omega)$.

The strong version of reducibility in the context of constructible sets is stated as

(3.3) Every constructible sets can be constructed within 1 step.

It is refutable in ZFC. But we can weaken the assertion by allow sets to be constructed before the step of the next cardinals rather than the next ordinals:

(3.4) For each $X \in L$, if $X \subseteq L_\alpha(\omega)$ for some α, then $X \in L_{(\alpha^+)^L}(\omega)$.

This weaker formalization is provable in ZFC. It should be remarked that (3.4) is also a key lemma for proving GCH from $V = L$.

Therefore, as Gödel admitted, ```constructible' sets are defined to be those sets which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders."[5] The generalization of the hierarchy of ramified types to transfinite orders is straightforward. The only barrier for Russell to go one step ahead can only be his Predicativism standpoint. In his theory, to refer to all properties of things in a certain type is not allowed, but it is necessary in the definition for limit cases.

Predicativism gave Russell the inspiration of the ramified type theory. However, it also prevent people to extend the construction so as to find more brilliant result such as the relative consistency of AC, ¬PD, and GCH.

4 Conclusion

In the paper, I have proposed a new perspective on the doctrines of philosophy of mathematics, namely a pragmatistic view. And I have argued that the foundation of mathematics is a suitable field to adopt the methodology of pragmatism because of the nature of mathematical practice. The key for applying such a methodology is to analyze the practical consequences of each philosophical doctrines if it is held by a working mathematician. Therefore, I have to at least show that philosophy

[2] We take the first step to be ω to fit with Russell's construction of ramified types, which starts with infinite many individuals. Since all hereditarily finite sets are definable without parameters, $L_1(\omega) = L_{\omega+1}$, where the hierarchy of L_α's begins with $L_0 = \emptyset$. Thus we have $L = \bigcup_{\alpha \in Ord} L_\alpha(\omega) = \bigcup_{\alpha \in Ord} L_\alpha$.

did impact on many researches in mathematics. My demonstration is based on several case studies. All the cases come from the frontier research on the foundation of mathematics, and the mathematicians we encountered are those who also have definite philosophical standpoint.

This paper serves as a pathfinder for a more ambitious program. The ultimate question is *which philosophy of mathematics is better*. I have argued in this paper that this is a meaningful question and pointed out the fact that standpoints on philosophy of mathematics can do good or bad on one's mathematical practice. And I have established the main methodology for the program, namely pragmatistic analysis based on case study. To push forward, more cases should be discovered and studied, and more delicate standards should be developed to measure the positiveness of a philosophical doctrine, say, if one is doing good somewhere while also doing bad elsewhere.

References

[1] Harvey Friedman. Some systems of second order arithmetic and their use. In *the International Congress of Mathematicians (Vancouver, B. C., 1974)*, pages 235--242, Montreal, 1975. Canadian Mathematical Congress.

[2] Gerhard Gentzen. Über das Verhältnis zwischen intuitionistischer und klassischer Arithmetik. *Archiv für mathematische Logik und Grundlagenforschung*, 16:119--132, 1974.

[3] Victoria Gitman and Joel David Hamkins. A natural model of the multiverse axioms. *Notre Dame Journal of Formal Logic*, 51(4):475--484, 2010.

[4] Kurt Gödel. Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines mathematisches Kolloquiums*, 4:34--38, 1933.

[5] Kurt Gödel. The consistency of the axiom of choice and of the generalized continuum hypothesis. In *Kurt Gödel: Collected Works: Volume II Publications 1938-1974*, pages 26--27. Oxford University Press, 1938.

[6] Kurt Gödel. What is Cantor's continuum problem? In *Kurt Gödel: Collected Works: Volume II Publications 1938-1974*, pages 254--270. Oxford University Press, 1964.

[7] Joel David Hamkins. The Set-Theoretic Multiverse. *Submitted*, pages 1--35, 2011.

[8] Joel David Hamkins and Benedikt Löme. The modal logic of forcing. *Transactions of the American Mathematical Society*, pages 1793--1817, 2008.

[9] Thomas Jech. *Set Theory*. Springer, Berlin, 3rd millen edition, 2002.

[10] A Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer, Berlin Heidelberg, 2nd edition, 2003.

[11] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *The Journal of Symbolic Logic*, 36(3):407--413, 1971.

[12] A Levy and Robert M Solovay. Measurable cardinals and the continuum hypothesis. *Israel Journal of Mathematics*, pages 234--248, 1967.

[13] Donald A Martin and John R Steel. Projective determinacy. *Proceedings of the National Academy of Sciences of the United States of America*, 85(18):6582--6586, September 1988.

[14] Bertrand Russell. Mathematical logic as based on the theory of types. In *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, pages 150--182. Harvard University Press, New York, 1908.

[15] D. S. Scott. Measurable cardinals and constructible sets. *Bull. Acad. Pol. Sci.*, 9:521--524, 1961.

[16] Saharon Shelah. Logical dreams. *Bulletin of The American Mathematical Society*, 40(2):203--228, 2003.

[17] Saharon Shelah and W. Hugh Woodin. Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable. *Israel Journal of Mathematics*, 70(3):381--394, 1990.

[18] Stephen G Simpson. The Gödel hierarchy and reverse mathematics. In *Kurt Gödel: Essays for his Centennial*, pages 109--127. Cambridge University Press, 2010.

[19] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92(1):1--56, 1970.

[20] Hugo Steinhaus and Jan Mycielski. A mathematical axiom contradicting the axiom of choice. *Bulletin de l'Académie Polonaise des Sciences*, 10:1--3, 1962.

[21] Bill Tait. Friedman's independence results, an epochal F.O.M. advance (post on F. O. M. mailing list), 1998.

[22] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction*. Elsevier Science Publishers B.V, Amsterdan, 1988.

[23] Anne S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. Springer, New York, 1973.

[24] W. Hugh Woodin. Supercompact cardinals, sets of reals, and weakly homogeneous trees. *Proceedings of the National Academy of Sciences of the United States of America*, 85(18):6587--91, September 1988.

[25] W. Hugh Woodin. The Continuum Hypothesis, Part I. *Notices of the American Mathematical Society*, (June/July):567--576, 2001.

[26] W Hugh Woodin. Suitable Extender Models I*. *Journal of Mathematical Logic*, 10(1-2(2010)):101--339, 2011.

[27] Ruizhi Yang. Kant and Gödel on objectivity of mathematical truth. *Philosophical Trends*, (7):75--80, 2009.