The World of Large Cardinals

V. Large Cardinals Beyond Choice and the HOD Conjecture. Rethinking the large cardinal hierarchy.

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Exacting and Ultraexacting cardinals as Structural Reflection principles

Given a first-order language $\mathcal{L}$ containing a unary predicate symbol $\dot{P}$, we say that an $\mathcal{L}$-structure A has type $\langle \mu, \lambda \rangle$ if the universe of A has rank λ and $\dot{P}^A$ has rank μ.

Exacting Structural Reflection

The Exacting Structural Reflection principle for $\mathcal{C}$ at a cardinal λ ($XSR_{\mathcal{C}}(\lambda)$) asserts that there is a cardinal $\mu < \lambda$ with the property that for some structure B in $\mathcal{C}$ of type $\langle \mu, \lambda \rangle$, if there is any, there exists a structure A in $\mathcal{C}$ of type $\langle \nu, \lambda \rangle$, for some $\nu < \mu$, and an elementary embedding of A into B.

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For a limit ordinal λ and a function $f : V_\lambda \to V_\lambda$, a square root of f is a function $r : V_\lambda \to V_\lambda$ with $r^+(r) = f$, where $r^+(r) = \bigcup\{r(r \cap V_\alpha) : \alpha < \lambda\}$.

Example: If $j : V_\lambda \to V_\lambda$ is an I3 embedding, then j is a square root of j^2.

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Theorem

A cardinal λ is exacting if and only if the principle $XSR_{\mathcal{C}}(\lambda)$ holds for all definable, with parameter λ, classes $\mathcal{C}$ of $\mathcal{L}$-structures).

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A cardinal λ is ultraexacting if and only if the principle $UXSR_{\mathcal{C}}(\lambda)$ holds for all ordinal definable classes $\mathcal{C}$ of $\mathcal{L}$-structures.

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A cardinal λ is ultraexacting if and only if the principle $UXSR_{\mathcal{C}}(\lambda)$ holds for all ordinal definable classes $\mathcal{C}$ of $\mathcal{L}$-structures.

Let $\mathcal{L}$ be the first-order language that extends the language of set theory by a unary function symbol $\dot{f}$, a binary relation symbol $\dot{E}$, and a unary predicate symbol $\dot{P}$.

Let $\mathcal{U}$ to be the class of $\mathcal{L}$-structures A of the form

$$\langle V_\lambda, \in, \dot{f}^A, \dot{E}^A, \dot{P}^A \rangle$$

where

- λ is a limit cardinal, and
- If ζ is the least cardinal in $C^{(2)}$ greater than λ, then there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, \dot{f}^A\} \subseteq X$ and a bijection $\tau : X \to V_\lambda$ with $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$ and

  $$x \in y \iff \tau(x) \; \dot{E}^A \; \tau(y)$$

  for all $x, y \in X$.

Note that $\mathcal{U}$ is a Δ_3 class.

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Lemma

If $\lambda \in C^{(1)}$ *and* $UXSR_{\mathcal{U}}(\lambda)$ *holds, then* λ *is ultraexacting.*

Proof

Let $f : V_\lambda \to V_\lambda$ and $\mu < \lambda$ witness that $UXSR_{\mathcal{U}}(\lambda)$ holds.

Let ζ be the least cardinal in $C^{(2)}$ greater than λ, let Y be an elementary submodel of V_ζ of cardinality λ with $V_\lambda \cup \{\lambda, f\} \subseteq Y$, and let $\pi : Y \to V_\lambda$ be a bijection with $\pi(\lambda) = \langle 0, 0\rangle$, $\pi(x) = \langle 1, x\rangle$ for all $x \in V_\lambda$.

Then there is an $\mathcal{L}$-structure B extending $\langle V_\lambda, \in\rangle$ with $\dot{f}^B = f$, $\dot{E}^B = \{\langle \pi(x), \pi(y)\rangle : x, y \in Y,\ x \in y\}$ and $\dot{P}^B = \mu$.

It follows that B is an element of $\mathcal{U}$ of type $\langle \mu, \lambda\rangle$.

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Proof continued

Hence, there is a structure A in $\mathcal{U}$ of type $\langle \nu, \lambda \rangle$, with $\nu < \mu$, and a square root $r : V_\lambda \to V_\lambda$ of f that is an elementary embedding of A into B.

Notice that $r(\nu) = \mu$. Also, we have that $r(\langle m, x \rangle) = \langle m, r(x) \rangle$ holds for all $x \in V_\lambda$ and $m < \omega$.

Since $A \in \mathcal{U}$, let X be an elementary submodel of V_ζ with $V_\lambda \cup \{\lambda, \dot{f}^A\} \subseteq X$, and let $\tau : X \to V_\lambda$ be a bijection such that $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$, and

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Proof continued

Now define

$$j \;:=\; \pi^{-1} \circ r \circ \tau : X \to V_\zeta.$$

We claim that j is an ultraexact embedding at λ.

First note that

$$j(\lambda) = (\pi^{-1} \circ r \circ \tau)(\lambda) = (\pi^{-1} \circ r)(\langle 0, 0\rangle) = \pi^{-1}(\langle 0, 0\rangle) = \lambda$$

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Further, j is an elementary embedding: for every $a \in X$ and every formula $\varphi(x)$ in the language of set theory,

$$X \models \varphi(x) \quad \text{iff} \quad \langle V_\lambda, \dot{E}^A\rangle \models \varphi(\tau(a)) \quad \text{iff} \quad \langle V_\lambda, \dot{E}^B\rangle \models \varphi(r(\tau(a))) \quad \text{iff}$$

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Proof continued

Claim: $j \restriction V_\lambda = \dot{f}^A$.

Why? For a contradiction, pick $m < \omega$ with $j \restriction V_{\lambda_m} \neq \dot{f}^A \restriction V_{\lambda_m}$, where $\langle \lambda_m : m < \omega \rangle$ is the critical sequence of r.

Elementarity then implies that

$$r(r \restriction V_{\lambda_m}) = r(j \restriction V_{\lambda_m}) \neq r(\dot{f}^A \restriction V_{\lambda_m}) = \dot{f}^B \restriction V_{\lambda_{m+1}} = f \restriction V_{\lambda_{m+1}}.$$

But since r is a square root of f, we also have that

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Lemma

If λ is an ultraexacting cardinal, then for every OD class $\mathcal{C}$ of $\mathcal{L}$-structures, the principle $UXSR_{\mathcal{C}}(\lambda)$ holds.

Proof

Let $\mathcal{C}$ be an ordinal definable class of $\mathcal{L}$-structures, and let $\mathcal{C}_{\lambda+1} = \mathcal{C} \cap V_{\lambda+1}$.

Thus, $\mathcal{C}_{\lambda+1}$ is an element of $V_{\lambda+2}$ that is ordinal-definable with λ as an additional parameter.

Let $j : (V_{\lambda+1}, \mathcal{C}_{\lambda+1}) \to (V_{\lambda+1}, \mathcal{C}_{\lambda+1})$ be an elementary embedding with critical point, κ, less than λ.

Let $\mu = j(\kappa)$, and let $f := j \restriction V_\lambda : V_\lambda \to V_\lambda$. We claim that $UXSR_{\mathcal{C}}(\lambda)$ holds, witnessed by f and μ.

So suppose $A \in \mathcal{C}_{\lambda+1}$ is a structure of type $\langle \mu, \lambda \rangle$.

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Let $j : (V_{\lambda+1}, \mathcal{C}_{\lambda+1}) \to (V_{\lambda+1}, \mathcal{C}_{\lambda+1})$ be an elementary embedding with critical point, κ, less than λ.

Let $\mu = j(\kappa)$, and let $f := j \restriction V_\lambda : V_\lambda \to V_\lambda$. We claim that $UXSR_{\mathcal{C}}(\lambda)$ holds, witnessed by f and μ.

So suppose $A \in \mathcal{C}_{\lambda+1}$ is a structure of type $\langle \mu, \lambda \rangle$.

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Then the elementarity of j implies that $j(A) \in \mathcal{C}_{\lambda+1}$, and the restriction map $j \restriction A : A \to j(A)$ is an elementary embedding.

Notice that $j \restriction A$ is the restriction to A of the function $j \restriction V_\lambda$, which is a square root of $j(f) = j^2$.

Thus, we have shown that:

"There is a structure (namely A) in $\mathcal{C}_{\lambda+1}$ of type $\langle j(\kappa), \lambda \rangle$ with an elementary embedding from that structure into $j(A)$ which is the restriction of a function that is a square root of $j(f)$"

Pulling back this statement via j^{-1}, we conclude that there is a structure B in $\mathcal{C}_{\lambda+1}$ of type $\langle \kappa, \lambda \rangle$ with an elementary embedding from B into A which is the restriction of a function that is a square root of f.

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Ultraexacting cardinals as *SR* principles

Theorem

A cardinal λ *is ultraexacting if and only if the principle* $UXSR_{\mathcal{C}}(\lambda)$ *holds for all ordinal definable classes* $\mathcal{C}$ *(equivalently, for the particular* Δ_3 *class used in the proof of Lemma 3) of* $\mathcal{L}$*-structures.*

Exacting SR

Exacting *SR*

Let $\mathcal{L}$ be a first-order language with a unary predicate symbol $\dot{P}$.

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures,

Exacting Structural Reflection

The Exact ingStructural Reflection principle for $\mathcal{C}$ at a cardinal λ ($XSR_{\mathcal{C}}(\lambda)$) asserts that there is a cardinal $\mu < \lambda$ with the property that for some structure B in $\mathcal{C}$ of type $\langle \mu, \lambda \rangle$, if there is any, there exists a structure A in $\mathcal{C}$ of type $\langle \nu, \lambda \rangle$, for some $\nu < \mu$, and an elementary embedding of A into B.

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Let $\mathcal{L}$ be the first-order language that extends the language of set theory by a a binary relation symbol $\dot{E}$ and a unary predicate symbol $\dot{P}$. Define $\mathcal{E}$ to be the class of $\mathcal{L}$-structures A of the form

$$\langle V_\lambda, \in, \dot{E}^A, \dot{P}^A \rangle$$

where

- λ is a limit cardinal, and
- If ζ is the least cardinal in $C^{(2)}$ greater than λ, then there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$ and a bijection $\tau : X \to V_\lambda$ with $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$ and

  $$x \in y \iff \tau(x) \ \dot{E}^A \ \tau(y)$$

  for all $x, y \in X$.

It is easily seen that $\mathcal{E}$ is a Δ_3 class.

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It is easily seen that $\mathcal{E}$ is a Δ_3 class.

Lemma

If $\lambda \in C^{(1)}$ *and* $XSR_{\mathcal{E}}(\lambda)$ *holds, then* λ *is exacting.*

Proof

Let $\mu < \lambda$ witness $XSR_{\mathcal{E}}(\lambda)$.

Let ζ be the least cardinal in $C^{(2)}$ greater than λ, let Y be an elementary submodel of V_ζ of cardinality λ with $V_\lambda \cup \{\lambda\} \subseteq Y$, and let $\pi : Y \to V_\lambda$ be a bijection with $\pi(\lambda) = \langle 0, 0\rangle$, $\pi(x) = \langle 1, x\rangle$ for all $x \in V_\lambda$.

Then there is an $\mathcal{L}$-structure B extending $\langle V_\lambda, \in\rangle$ with $\dot{E}^B = \{\langle \pi(x), \pi(y)\rangle$ $x, y \in Y,\ x \in y\}$ and $\dot{P}^B = \mu$.

Thus $B \in \mathcal{E}$ is of type $\langle \mu, \lambda\rangle$.

By $XSR_{\mathcal{E}}(\lambda)$ there is a structure A in $\mathcal{E}$ of type $\langle \nu, \lambda\rangle$, with $\nu < \mu$, and an elementary embedding of A into B.

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Proof continued

Since $A \in \mathcal{E}$, let X be an elementary submodel of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and let $\tau : X \to V_\lambda$ be a bijection such that $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$, and

$$x \in y \iff \tau(x)\ \dot{E}^A\ \tau(y)$$

holds for all $x, y \in X$. Now letting

$$i \ := \ \pi^{-1} \circ j \circ \tau : X \to V_\zeta.$$

we have that i is an exact embedding at λ.

Lemma

If λ is an exacting cardinal, then for every $n \geqslant 2$ there is a cardinal $\mu < \lambda$ such that for every Σ_n-definable, with λ as a parameter, class $\mathcal{C}$ of $\mathcal{L}$-structures, the principle $XSR_{\mathcal{C}}(\lambda)$ holds, witnessed by μ.

Proof

Suppose λ is exacting and let $j : X \to V_\zeta$ be an elementary embedding witnessing it, with ζ being the least element of $C^{(n)}$ greater than λ.

Let κ be the critical point of j, and let $\mu = j(\kappa)$.

Let $\mathcal{C}$ be a Σ_n-definable, with λ as a parameter, class of $\mathcal{L}$-structures and suppose there exists $B \in \mathcal{C}$ of type $\langle \mu, \lambda \rangle$.

Then this is true in V_ζ, and by elementarity there must exist $A \in \mathcal{C}$ of type $\langle \kappa, \lambda \rangle$ in X, and therefore also in V.

Now note that the restriction embedding $j \restriction A : A \to j(A)$ is elementary, with $j(A)$ being in $\mathcal{C}$ and of type $\langle \mu, \lambda \rangle$, so it witnesses $XSR_{\mathcal{C}}(\lambda)$.

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Exacting cardinals as *SR* principles

Theorem

A cardinal λ is exacting if and only if the principle $XSR_{\mathcal{C}}(\lambda)$ holds for all definable, with parameter λ, classes $\mathcal{C}$ (equivalently, for a particular Δ_3 class of $\mathcal{L}$-structures).

The consistency strength of exacting and ultraexacting cardinals

The strength of exacting and ultraexacting cardinals

Theorem

The consistency strength of the existence of an exacting cardinal is strictly between $\mathrm{I3}$ *and* $\mathrm{I1}$.

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The existence of an ultraexacting cardinal is equiconsistent with the existence of an $\mathrm{I0}$ *embedding.*

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The strange behavior of exacting and ultraexacting cardinals

The existence of exact or ultraexact cardinals in the presence of standard large cardinals, e.g., measurable or extendible, blows up their consistency strength. For example:

Theorem

The existence of an ultraexacting cardinal below a measurable cardinal implies the consistency of ZFC with a proper class of I0 embeddings.

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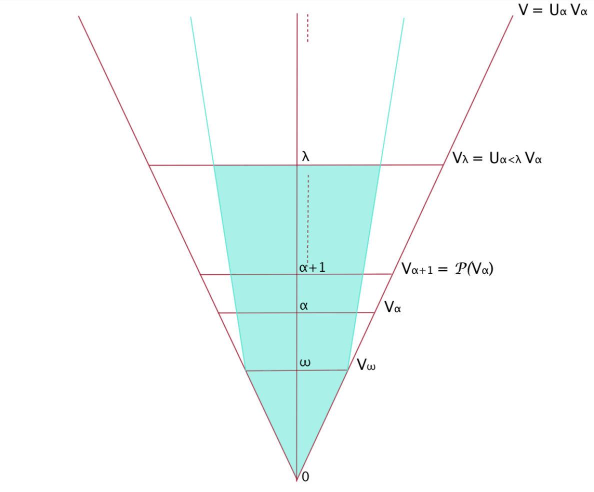
Some consequences of exacting and ultraexacting cardinals

The class Hereditarily Ordinal Definable sets (HOD)

A set is HOD if it is OD, its elements are OD, the elements of its erlements are OD, and so on.



Set Theory at a Crossroads: A Foundational Dichotomy

The HOD-Dichotomy Theorem (Woodin 2010): In the presence of large cardinals (extendible), the set-theoretic universe V is either

- very **close** to HOD or
- very **far** from HOD.

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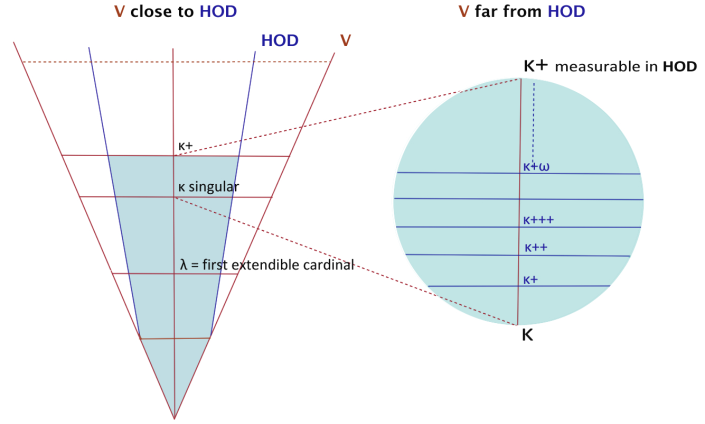

Woodin's HOD Dichotomy (weak version)

Theorem (The HOD Dichotomy. Woodin 2010)

Suppose there exists an extendible cardinal κ. Then exactly one of the following holds:

1. *Every singular cardinal $\lambda > \kappa$ is singular in HOD, and HOD computes λ^+ correctly (i.e., V is* **close** *to HOD).*
2. *Every regular cardinal $\lambda \geqslant \kappa$ is measurable in HOD (i.e., V is* **far** *from HOD).*

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ω-strong measurability

Definition (Woodin)

A cardinal κ is ω-strongly measurable in HOD if there exists a cardinal $\delta < \kappa$ with $(2^\delta)^{\mathrm{HOD}} < \kappa$ such that there is no partition $\langle S_\alpha : \alpha < \delta \rangle$ in HOD of $S^\kappa_\omega := \{\alpha < \kappa : \mathrm{cof}(\alpha) = \omega\}$ into stationary sets.

Lemma (Woodin)

If κ is ω-strongly measurable in HOD, then κ is a measurable cardinal in HOD.

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A strong consequence of ultraexacting cardinals

Theorem

If λ is ultraexacting, then λ^+ is ω-strongly measurable in HOD.

The HOD Conjecture

Woodin's HOD Conjecture: The theory

$$\text{ZFC} + \text{“There exists an extendible cardinal”}$$

proves that the first alternative of the HOD Dichotomy, i.e., V is close to HOD, is the true one.

The failure of the HOD Conjecture

Theorem

Let T be the theory "ZFC + There is an exacting cardinal above an extendible cardinal". Then,

- *T proves that V is far from HOD.*
- *$PA + CON(T)$ disproves the HOD Conjecture.*
- *The theory $NBG - AC$ + "There is a Σ_3-correct Reinhardt cardinal and a supercompact cardinal greater than the supremum of the critical sequence" proves $CON(T)$.*

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- *The theory $NBG - AC$ + "There is a Σ_3-correct Reinhardt cardinal and a supercompact cardinal greater than the supremum of the critical sequence" proves $CON(T)$.*

The failure of the HOD Conjecture

Theorem

Let T be the theory "ZFC + There is an exacting cardinal above an extendible cardinal". Then,

- *T proves that V is far from HOD.*
- *$PA + CON(T)$ disproves the HOD Conjecture.*
- *The theory $NBG - AC$ + "There is a Σ_3-correct Reinhardt cardinal and a supercompact cardinal greater than the supremum of the critical sequence" proves $CON(T)$.*

Summary

- Exacting and ultraexacting cardinals are natural large cardinals.
- They are consistent with ZFC relative to the existence of an I0 embedding.
- The existence of an exacting cardinal implies that V is not equal to HOD, and so these cardinals surpass the current hierarchy of large cardinals consistent with ZFC.
- The existence of an exacting cardinal above an extendible cardinal implies that V is far from HOD.
- The consistency of ZFC with an exacting cardinal above an extendible cardinal refutes Woodin's HOD Conjecture.
- The consistency of ZFC with the existence of an exacting cardinal above an extendible cardinal follows from the consistency of ZF with certain large cardinals beyond Choice.

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