The World of Large Cardinals

IV. Large cardinals at the edge: exacting and ultraexacting cardinals. Large Cardinals Beyond HOD.

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Dividing lines in the large-cardinal hierarchy

$\mathbf{ZF} + \neg\mathbf{AC}$

(Reinhardt, super-Reinhardt, rank-Berkeley, Berkeley, club-Berkeley, ...)

$\mathbf{ZFC} + \mathbf{V}{\neq}\mathbf{L}$

(Jónsson, Ramsey, measurable, strong, Woodin, strongly compact, supercompact, extendible, Vopěnka, huge, I3, I2, I1, I0)

$\mathbf{ZFC} + \mathbf{V}{=}\mathbf{L}$

(inaccessible, Mahlo, weakly compact, totally indescribable, strongly unfoldable, subtle, ...)

Kunen's Theorem

Theorem (Kunen)

There is no non-trivial elementary embedding $j : V \to V$. *In fact, there is no non-trivial elementary embedding* $j : V_{\lambda+2} \to V_{\lambda+2}$, *for any* λ.

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A new dividing line

ZF + ¬AC

ZFC + V≠HOD

(exacting, ultraexacting)

ZFC + V≠L

ZFC + V=L

Exacting and Ultraexacting cardinals

Exacting and Ultraexacting cardinals

Definition

A cardinal λ is exacting (ultraexacting) iff there is an elementary embedding $j : X \to V_\zeta$, where ζ is some cardinal in $C^{(2)}$ above λ (equivalently, the least such) and X is an elementary substructure of V_ζ that contains $V_\lambda \cup \{\lambda\}$, such that $j(\lambda) = \lambda$, $j \restriction \lambda \neq \mathrm{Id}_\lambda$ (and $j \restriction V_\lambda \in X$).

Exacting cardinals are natural large cardinals

A cardinal λ is Jónsson (B. Jónsson 1962) if every structure (in a countable first-order language) of cardinality λ has a proper elementary substructure of cardinality λ.

Theorem (ABL)

The following are equivalent for each cardinal λ such that $|V_\lambda| = \lambda$:

1. *λ is exacting.*
2. *For every class $\mathcal{C}$ of structures (in a countable first-order language) that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ that is (isomorphic to a structure) in $\mathcal{C}$.*

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I1 embeddings with OD predicates

Recall: An I0 embedding is an elementary embedding from $L(V_{\lambda+1})$ to itself, with critical point less than λ.

Recall: An I1 embedding is an elementary embedding from $V_{\lambda+1}$ to itself, with critical point less than λ.

If $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an I0 embedding, then $j \restriction V_{\lambda+1}$ is an I1 embedding.

Moreover, if A is subset of $V_{\lambda+1}$ that is definable in $L(V_{\lambda+1})$ with parameters in $V_{\mathrm{crit}(j)} \cup \{\lambda\}$, then $j \restriction V_{\lambda+1} : (V_{\lambda+1}, A) \to (V_{\lambda+1}, A)$ is an elementary embedding.

Recall: OD is the class of all Ordinal Definable sets. OD is a Σ_2 definable class, with a Σ_2-definable well-ordering, $<_{OD}$.

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The following results are from joint work with Juan P. Aguilera, Gabriel Goldberg, and Philipp Lücke

References:

- [**ABL**] Aguilera, J.P., Bagaria, J., and Lücke, P., (2024) *Large Cardinals, Structural Reflection, and the HOD Conjecture*. Submitted.
- [**ABGL**] Aguilera, J.P., Bagaria, J., Goldberg, G., and Lücke, P., (2025) *Large cardinals beyond HOD*. Submitted.

Both references are available at ArXiv.

I1 embeddings with OD predicates

Theorem

A cardinal λ is ultraexacting if and only if for every OD subset A of $V_{\lambda+1}$, there exists an elementary embedding $j : (V_{\lambda+1}, A) \to (V_{\lambda+1}, A)$.

Remark

The proof of the theorem above gives some additional information. Namely, given λ, if ζ is the least element of $C^{(2)}$ greater than λ, then the following are equivalent:

1. *λ is ultraexacting.*
2. *There is an elementary embedding from $(V_{\lambda+1}, A)$ to itself, where A is the Δ_2-definable, with parameters λ and ζ, subset of $V_{\lambda+1}$ used in the proof of the theorem. Namely, A is the set of all well-founded extensional relations $E \subseteq V_\lambda \times V_\lambda$ isomorphic to some elementary substructure $X \preceq V_\zeta$ with $V_\lambda \cup \{\lambda\} \subseteq X$.*

I3 embeddings with parameters in *OD* predicates

Theorem

A cardinal λ is exacting if and only if for every nonempty OD subset A of $V_{\lambda+1}$, there exist $x, y \in A$ and an elementary embedding $j : (V_\lambda, x) \to (V_\lambda, y)$.

Remark

Given any cardinal λ, if ζ is the least element of $C^{(2)}$ greater than λ, then the following are equivalent:

1. *λ is exacting.*
2. *There is an elementary embedding $j : (V_\lambda, x) \to (V_\lambda, y)$, where x, y belong to the Δ_2-definable, with parameters λ and ζ, subset $A \subseteq V_{\lambda+1}$ used in the proof of the theorem. Namely, there are well-founded extensional relations E, F on V_λ, each one isomorphic to some elementary substructure of V_ζ containing $V_\lambda \cup \{\lambda\}$, with an elementary embedding $j : (V_\lambda, E) \to (V_\lambda, F)$.*

Some corollaries

Corollary

A cardinal λ is ultraexacting if and only if there exists an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ such that for any Σ_2-formula $\varphi(x)$ and any $a \in V_{\lambda+1}$, $V \vDash \varphi(a)$ if and only if $V \vDash \varphi(j(a))$.

Proposition

If λ is an ultraexacting cardinal, then for every cardinal $\gamma > \lambda$ there is a γ-directed closed forcing notion which forces that λ remains ultraexacting and $V_\lambda \subseteq \mathrm{HOD}$.

Corollary

If λ is ultraexacting, then in some forcing extension λ remains ultraexacting and there is no exacting cardinal below λ.

Summary

- The following are equivalent:
  1. λ is ultraexacting.
  2. For every OD subset A of $V_{\lambda+1}$, there exists an elementary embedding $j : (V_{\lambda+1}, A) \to (V_{\lambda+1}, A)$.
  3. There exists an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ such that for any Σ_2-formula $\varphi(x)$ and any $a \in V_{\lambda+1}$, $V \vDash \varphi(a)$ if and only if $V \vDash \varphi(j(a))$.
- The following are equivalent:
  1. λ is exacting.
  2. For every nonempty OD subset A of $V_{\lambda+1}$, there exist $x, y \in A$ and an elementary embedding $j : (V_\lambda, x) \to (V_\lambda, y)$.
- It is consistent for λ to be ultraexacting and with $V_\lambda \subseteq \mathrm{HOD}$.
- It is consistent for λ to be ultraexacting and with no exacting cardinal below λ.

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Exacting and Ultraexacting cardinals as Structural Reflection principles

For a limit ordinal λ and a function $f : V_\lambda \to V_\lambda$, a square root of f is a function $r : V_\lambda \to V_\lambda$ with $r^+(r) = f$, where $r^+(r) = \bigcup\{r(r \cap V_\alpha) : \alpha < \lambda\}$.

Example: If $j : V_\lambda \to V_\lambda$ is an I3 embedding, then j is a square root of j^2.

Given a first-order language $\mathcal{L}$ containing a unary predicate symbol $\dot{P}$, we say that an $\mathcal{L}$-structure A has type $\langle \mu, \lambda \rangle$ if the universe of A has rank λ and $\dot{P}^A$ has rank μ.

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Ultraexact SR

Definition

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures, the Ultraexact Structural Reflection principle for $\mathcal{C}$ at a cardinal λ $(UXSR_{\mathcal{C}}(\lambda))$ asserts that there is a function $f : V_\lambda \to V_\lambda$ and a cardinal $\mu < \lambda$ with the property that for every structure B in $\mathcal{C}$ of type $\langle \mu, \lambda \rangle$, there exists a structure A in $\mathcal{C}$ of type $\langle \nu, \lambda \rangle$, for some $\nu < \mu$, and a square root r of f such that the restriction of r to the universe of A is an elementary embedding of A into B.

Let $\mathcal{L}$ be the first-order language that extends the language of set theory by a unary function symbol $\dot{f}$, a binary relation symbol $\dot{E}$, and a unary predicate symbol $\dot{P}$.

Let $\mathcal{U}$ to be the class of $\mathcal{L}$-structures A such that there exists a limit cardinal λ such that the following hold:

- The reduct of A to the language of set theory is equal to $\langle V_\lambda, \in\rangle$.
- If ζ is the least cardinal in $C^{(2)}$ greater than λ, then there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, \dot{f}^A\} \subseteq X$ and a bijection $\tau : X \to V_\lambda$ with $\tau(\lambda) = \langle 0, 0\rangle$, $\tau(x) = \langle 1, x\rangle$ for all $x \in V_\lambda$ and
$$x \in y \iff \tau(x) \ \dot{E}^A \ \tau(y)$$
for all $x, y \in X$.

Note that $\mathcal{U}$ is a Δ_3 class.

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Lemma

If $\lambda \in C^{(1)}$ *and* $UXSR_{\mathcal{U}}(\lambda)$ *holds, then* λ *is ultraexacting.*

Proof

Let $f : V_\lambda \to V_\lambda$ and $\mu < \lambda$ witness that $UXSR_{\mathcal{U}}(\lambda)$ holds.

Let ζ be the least cardinal in $C^{(2)}$ greater than λ, let Y be an elementary submodel of V_ζ of cardinality λ with $V_\lambda \cup \{\lambda, f\} \subseteq Y$, and let $\pi : Y \to V_\lambda$ be a bijection with $\pi(\lambda) = \langle 0, 0 \rangle$, $\pi(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$.

Then there is an $\mathcal{L}$-structure B extending $\langle V_\lambda, \in \rangle$ with $\dot{f}^B = f$, $\dot{E}^B = \{\langle \pi(x), \pi(y) \rangle : x, y \in Y,\ x \in y\}$ and $\dot{P}^B = \mu$.

It follows that B is an element of $\mathcal{U}$ of type $\langle \mu, \lambda \rangle$.

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Let ζ be the least cardinal in $C^{(2)}$ greater than λ, let Y be an elementary submodel of V_ζ of cardinality λ with $V_\lambda \cup \{\lambda, f\} \subseteq Y$, and let $\pi : Y \to V_\lambda$ be a bijection with $\pi(\lambda) = \langle 0, 0\rangle$, $\pi(x) = \langle 1, x\rangle$ for all $x \in V_\lambda$.

Then there is an $\mathcal{L}$-structure B extending $\langle V_\lambda, \in\rangle$ with $\dot{f}^B = f$, $\dot{E}^B = \{\langle \pi(x), \pi(y)\rangle : x, y \in Y,\ x \in y\}$ and $\dot{P}^B = \mu$.

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Proof continued

Hence, there is a structure A in $\mathcal{U}$ of type $\langle \nu, \lambda \rangle$, with $\nu < \mu$, and a square root $r : V_\lambda \to V_\lambda$ of f that is an elementary embedding of A into B.

Notice that r is an I3-embedding with $r(\nu) = \mu$. Also, we have that $r(\langle m, x \rangle) = \langle m, r(x) \rangle$ holds for all $x \in V_\lambda$ and $m < \omega$.

Since $A \in \mathcal{U}$, let X be an elementary submodel of V_ζ with $V_\lambda \cup \{\lambda, \dot{f}^A\} \subseteq X$, and let $\tau : X \to V_\lambda$ be a bijection such that $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$, and

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Now define

$$j := \pi^{-1} \circ r \circ \tau : X \to V_\zeta.$$

We claim that j is an ultraexact embedding at λ.

First note that

$$j(\lambda) = (\pi^{-1} \circ r \circ \tau)(\lambda) = (\pi^{-1} \circ r)(\langle 0, 0\rangle) = \pi^{-1}(\langle 0, 0\rangle) = \lambda$$

and for all $x \in V_\lambda$,

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Further, j is an elementary embedding: for every $a \in X$ and every formula $\varphi(x)$ in the language of set theory,

$$X \models \varphi(x) \quad \text{iff} \quad \langle V_\lambda, \dot{E}^A\rangle \models \varphi(\tau(a)) \quad \text{iff} \quad \langle V_\lambda, \dot{E}^B\rangle \models \varphi(r(\tau(a))) \quad \text{iff}$$

$$Y \models \varphi(\pi^{-1}(r(\tau(a)))) \quad \text{iff} \quad V_\zeta \models \varphi(j(a)).$$

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Claim: $j \restriction V_\lambda = \dot{f}^A$.

Why? For a contradiction, pick $m < \omega$ with $j \restriction V_{\lambda_m} \neq \dot{f}^A \restriction V_{\lambda_m}$, where $\langle \lambda_m : m < \omega \rangle$ is the critical sequence of r.

Elementarity then implies that

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This completes the proof of the lemma, because $j \restriction V_\lambda = \dot{f}^A \in X$, thus showing that j is an ultraexact embedding at λ.

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Lemma

If λ is an ultraexacting cardinal, then for every OD class $\mathcal{C}$ of $\mathcal{L}$-structures, the principle $UXSR_{\mathcal{C}}(\lambda)$ holds.

Proof

Let $\mathcal{C}$ be an ordinal definable class of $\mathcal{L}$-structures, and let $\mathcal{C}_{\lambda+1} = \mathcal{C} \cap V_{\lambda+1}$.

Thus, $\mathcal{C}_{\lambda+1}$ is an element of $V_{\lambda+2}$ that is ordinal-definable with λ as an additional parameter.

Let $j : (V_{\lambda+1}, \mathcal{C}_{\lambda+1}) \to (V_{\lambda+1}, \mathcal{C}_{\lambda+1})$ be an elementary embedding with critical point, κ, less than λ.

Let $\mu = j(\kappa)$, and let $f := j \restriction V_\lambda : V_\lambda \to V_\lambda$. We claim that $UXSR_{\mathcal{C}}(\lambda)$ holds, witnessed by f and μ.

So suppose $A \in \mathcal{C}_{\lambda+1}$ is a structure of type $\langle \mu, \lambda \rangle$.

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So suppose $A \in \mathcal{C}_{\lambda+1}$ is a structure of type $\langle \mu, \lambda \rangle$.

Proof continued

Then the elementarity of j implies that $j(A) \in \mathcal{C}_{\lambda+1}$, and the restriction map $j \restriction A : A \to j(A)$ is an elementary embedding.

Notice that $j \restriction A$ is the restriction to A of the function $j \restriction V_\lambda$, which is a square root of $j(f) = j^2$.

Thus, we have shown that:

"There is a structure (namely A) in $\mathcal{C}_{\lambda+1}$ of type $\langle j(\kappa), \lambda \rangle$ with an elementary embedding from that structure into $j(A)$ which is the restriction of a function that is a square root of $j(f)$"

Pulling back this statement via j^{-1}, we conclude that there is a structure B in $\mathcal{C}_{\lambda+1}$ of type $\langle \kappa, \lambda \rangle$ with an elementary embedding from B into A which is the restriction of a function that is a square root of f.

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Ultraexacting cardinals as *SR* principles

Theorem

A cardinal λ is ultraexacting if and only if the principle $UXSR_{\mathcal{C}}(\lambda)$ holds for all ordinal definable classes $\mathcal{C}$ (equivalently, for the particular Δ_3 class used in the proof of Lemma 9) of $\mathcal{L}$-structures.

Exacting *SR*

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Let $\mathcal{L}$ be a first-order language with a unary predicate symbol $\dot{P}$.

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures,

Exact Structural Reflection

The Exact Structural Reflection principle for $\mathcal{C}$ at a cardinal λ ($XSR_{\mathcal{C}}(\lambda)$) asserts that there is a cardinal $\mu < \lambda$ with the property that for some structure B in $\mathcal{C}$ of type $\langle \mu, \lambda \rangle$, if there is any, there exists a structure A in $\mathcal{C}$ of type $\langle \nu, \lambda \rangle$, for some $\nu < \mu$, and an elementary embedding of A into B.

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Let $\mathcal{L}$ be the first-order language that extends the language of set theory by a a binary relation symbol $\dot{E}$ and a unary predicate symbol $\dot{P}$. Define $\mathcal{E}$ to be the class of $\mathcal{L}$-structures A such that there exists a limit cardinal λ such that the following hold:

- The reduct of A to the language of set theory is equal to $\langle V_\lambda, \in \rangle$.
- If ζ is the least cardinal in $C^{(2)}$ greater than λ, then there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$ and a bijection $\tau : X \to V_\lambda$ with $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$ and

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It is easily seen that $\mathcal{E}$ is a Δ_3 class.

Lemma

If $\lambda \in C^{(1)}$ *and* $XSR_{\mathcal{E}}(\lambda)$ *holds, then* λ *is exacting.*

Proof

Let $\mu < \lambda$ witness $XSR_{\mathcal{E}}(\lambda)$.

Let ζ be the least cardinal in $C^{(2)}$ greater than λ, let Y be an elementary submodel of V_ζ of cardinality λ with $V_\lambda \cup \{\lambda\} \subseteq Y$, and let $\pi : Y \to V_\lambda$ be a bijection with $\pi(\lambda) = \langle 0, 0 \rangle$, $\pi(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$.

Then there is an $\mathcal{L}$-structure B extending $\langle V_\lambda, \in \rangle$ with $\dot{E}^B = \{\langle \pi(x), \pi(y) \rangle$ $x, y \in Y,\ x \in y\}$ and $\dot{P}^B = \mu$.

Thus $B \in \mathcal{E}$ is of type $\langle \mu, \lambda \rangle$.

By $XSR_{\mathcal{E}}(\lambda)$ there is a structure A in $\mathcal{E}$ of type $\langle \nu, \lambda \rangle$, with $\nu < \mu$, and an elementary embedding of A into B.

Proof

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By $XSR_{\mathcal{E}}(\lambda)$ there is a structure A in $\mathcal{E}$ of type $\langle \nu, \lambda\rangle$, with $\nu < \mu$, and an elementary embedding of A into B.

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Let $\mu < \lambda$ witness $XSR_{\mathcal{E}}(\lambda)$.

Let ζ be the least cardinal in $C^{(2)}$ greater than λ, let Y be an elementary submodel of V_ζ of cardinality λ with $V_\lambda \cup \{\lambda\} \subseteq Y$, and let $\pi : Y \to V_\lambda$ be a bijection with $\pi(\lambda) = \langle 0, 0\rangle$, $\pi(x) = \langle 1, x\rangle$ for all $x \in V_\lambda$.

Then there is an $\mathcal{L}$-structure B extending $\langle V_\lambda, \in\rangle$ with $\dot{E}^B = \{\langle \pi(x), \pi(y)\rangle$ $x, y \in Y,\ x \in y\}$ and $\dot{P}^B = \mu$.

Thus $B \in \mathcal{E}$ is of type $\langle \mu, \lambda\rangle$.

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Proof continued

Since $A \in \mathcal{E}$, let X be an elementary submodel of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and let $\tau : X \to V_\lambda$ be a bijection such that $\tau(\lambda) = \langle 0, 0 \rangle$, $\tau(x) = \langle 1, x \rangle$ for all $x \in V_\lambda$, and

$$x \in y \iff \tau(x) \; \dot{E}^A \; \tau(y)$$

holds for all $x, y \in X$. Now letting

$$i \; := \; \pi^{-1} \circ j \circ \tau : X \to V_\zeta.$$

we have that i is an exact embedding at λ.

Lemma

If λ *is an exacting cardinal, then for every* $n \geqslant 2$ *there is a cardinal* $\mu < \lambda$ *such that for every* Σ_n*-definable, with* λ *as a parameter, class* $\mathcal{C}$ *of* $\mathcal{L}$*-structures, the principle* $XSR_{\mathcal{C}}(\lambda)$ *holds, witnessed by* μ*.*

Proof

Suppose λ is exacting and let $j : X \to V_\zeta$ be an elementary embedding witnessing it, with ζ being the least element of $C^{(n)}$ greater than λ.

Let κ be the critical point of j, and let $\mu = j(\kappa)$.

Let $\mathcal{C}$ be a Σ_n-definable, with λ as a parameter, class of $\mathcal{L}$-structures and suppose there exists $B \in \mathcal{C}$ of type $\langle \mu, \lambda \rangle$.

Then this is true in V_ζ, and by elementarity there must exist $A \in \mathcal{C}$ of type $\langle \kappa, \lambda \rangle$ in X, and therefore also in V.

Now note that the restriction embedding $j \restriction A : A \to j(A)$ is elementary, with $j(A)$ being in $\mathcal{C}$ and of type $\langle \mu, \lambda \rangle$, so it witnesses $XSR_{\mathcal{C}}(\lambda)$.

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Exacting cardinals as *SR* principles

Theorem

A cardinal λ is exacting if and only if the principle $XSR_{\mathcal{C}}(\lambda)$ holds for all definable, with parameter λ, classes $\mathcal{C}$ (equivalently, for a particular Δ_3 class of $\mathcal{L}$-structures).