The World of Large Cardinals

III. Reflecting measures and extenders. From strong to Woodin cardinals.

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Beyond supercompactness

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Recall: κ is extendible if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$.

Proposition
Every extendible cardinal is supercompact.

Proposition
If κ is extendible, then $\kappa \in C^{(3)}$.

Corollary:

Every extendible cardinal is a supercompact limit of supercompact cardinals.

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Theorem

The following are equivalent for a cardinal κ*:*

1. κ *is the first extendible cardinal.*
2. κ *is the least cardinal such that* $SR_\kappa(\Pi_2)$ *holds.*

For the higher levels ($n > 3$) we need the notion of $C^{(n)}$-extendible cardinal.

Definition

κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$, i.e., $V_{j(\kappa)} \preceq_{\Sigma_n} V$.

Note: κ is extendible if and only if it is $C^{(1)}$-extendible.

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Theorem

The following are equivalent for a cardinal κ, and $n \geqslant 1$:

1. κ *is the first $C^{(n)}$-extendible cardinal.*
2. κ *is the least cardinal such that $SR_\kappa(\Pi_{n+1})$ holds.*

The proof is left as an **Exercise**.

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The following are equivalent for a cardinal κ, and $n \geqslant 1$:

1. κ *is the first $C^{(n)}$-extendible cardinal.*
2. κ *is the least cardinal such that $SR_\kappa(\Pi_{n+1})$ holds.*

The proof is left as an **Exercise**.

Corollary:

The following schemata are equivalent:

1. *SR (i.e., $SR(\mathcal{C})$ holds for all $\mathcal{C}$).*
2. *There exists a $C^{(n)}$-extendible cardinal, for every n.*
3. *There exist a proper class of $C^{(n)}$-extendible cardinals, for every n.*
4. *Vopěnka's Principle (VP): for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ with an elementary embedding $j : A \to B$.*

Reflecting Measures

The reflecting club classes

Recall that, for each $n > 0$, $C^{(n)}$ is the Π_n-definable club proper class of cardinals κ that are Σ_n-correct, i.e., V_κ is a Σ_n-elementary substructure of V.

Reflecting mesures

Definition

Let κ be an infinite cardinal, and let $\lambda \geqslant \kappa$. An *n-reflecting measure on $\mathcal{P}_\kappa\lambda$* is a κ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$ which contains the set $T^n_{\kappa,\lambda}$ of all $\sigma \in \mathcal{P}_\kappa\lambda$ such that $\mathrm{ot}(\sigma) \in C^{(n)}$.

Notice that $T^{n+1}_{\kappa,\lambda} \subseteq T^n_{\kappa,\lambda}$, and therefore every $(n+1)$-reflecting measure on $\mathcal{P}_\kappa\lambda$ is also n-reflecting.

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Reflecting mesures

Recall that an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa S$ is **fine** if for every $\mu \in S$ the set $\{\sigma \in \mathcal{P}_\kappa S : \mu \in \sigma\}$ belongs to $\mathcal{U}$.

And $\mathcal{U}$ is **normal** if it is closed under diagonal intersections, i.e., if $\mathcal{X} = \langle X_i : i \in S \rangle$ is a sequence of elements of $\mathcal{U}$, then

$$\Delta\mathcal{X} = \{\sigma : \sigma \in \bigcap_{i \in \sigma} X_i\} \in \mathcal{U}.$$

Equivalently, every function $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ that is regressive (i.e., $f(\sigma) \in \sigma$) on a set in $\mathcal{U}$, is constant on a set in $\mathcal{U}$. (**Exercise.**)

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Reflecting mesures

Lemma

If there exists a fine and normal 1*-reflecting measure on* $\mathcal{P}_\kappa\lambda$*, then* $\lambda \in C^{(1)}$.

Supercompact reflecting mesures

Recall: A cardinal κ is supercompact if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Theorem

A cardinal $\kappa \leqslant \lambda$ is λ-supercompact with $\lambda \in C^{(1)}$ iff there exists a fine and normal 1-reflecting measure on $\mathcal{P}_\kappa\lambda$.

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Reflecting mesures

Corollary

A cardinal κ *is supercompact iff there exists a fine and normal 1-reflecting measure on* $\mathcal{P}_\kappa\lambda$ *for every* λ *in* $C^{(1)}$ *greater than or equal to* κ*.*

2-reflecting measures and extendible cardinals

Recall: κ is extendible if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$.

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A cardinal κ is extendible iff there exists a fine and normal 2-reflecting measure on $\mathcal{P}_\kappa\lambda$ for every (equivalently, for a proper class of) λ in $C^{(2)}$ greater than or equal to κ.

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$(n+1)$-reflecting measures and $C^{(n)}$-extendible cardinals

Recall: κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$.

A similar proof yields the following:

Theorem

A cardinal κ is $C^{(n)}$-extendible iff there exists a fine and normal $(n+1)$-reflecting measure on $\mathcal{P}_\kappa\lambda$, for all (equivalently, a proper class of) cardinals $\lambda \in C^{(n+1)}$ greater than or equal to κ.

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Reflecting extenders

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A characterization in terms of reflecting extenders can also be given for large cardinals between the first strong cardinal and the principle "$\mathtt{Ord}$ is Woodin" (i.e., the existence of a Π_n-strong cardinal, for every n).

Strong cardinals

Recall: A cardinal κ is λ-strong iff there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.

κ is strong if it is λ-strong for all (equivalently, for a proper class of) λ.

Exercise: Show that κ is λ-strong iff there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$.

Show that if κ is strong, then $\kappa \in C^{(2)}$.

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Reflecting extenders

Given a cardinal κ and an ordinal λ greater than or equal to κ, a (κ, λ)-n-reflecting extender is a set $\mathcal{E} := \{E_a : a \in [\lambda]^{<\omega}\}$ such that

1. 1.1 Each E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$.
   1.2 If $a \subseteq C^{(n)}$, then E_a is n-reflecting, i.e., $[C^{(n)} \cap \kappa]^{|a|} \in E_a$.
   1.3 E_a is not κ^+-complete for some a.
   1.4 For each $\xi \in \kappa$, there is some $a \in [\lambda]^{<\omega}$ with

$$\{s \in [\kappa]^{|a|} : \xi \in s\} \in E_a.$$

2. *Coherence:* If $a \subseteq b$ are in $[\lambda]^{<\omega}$ and such that $b = \{\alpha_1, \ldots, \alpha_n\}$ and $a = \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$, and $\pi_{ba} : [\kappa]^{|b|} \to [\kappa]^{|a|}$ is the projection map given by $\pi_{ba}(\{\xi_1, \ldots, \xi_n\}) = \{\xi_{i_1}, \ldots, \xi_{i_n}\}$, then

$$X \in E_a \quad \text{if and only if} \quad \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b\,.$$

3. *Normality:* Whenever $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$ are such that $\{s \in [\kappa]^{|a|} : f(s) \in \mathtt{max}(s)\} \in E_a$, there is $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b\,.$$

4. *Well-foundedness:* Whenever $a_m \in [\lambda]^{<\omega}$ and $X_m \in E_{a_m}$ for $m \in \omega$, there is a function $d : \bigcup_m a_m \to \kappa$ such that $d``a_m \in X_m$ for every m.

Given by a (κ, λ)-extender $\mathcal{E} := \{E_a : a \in [\lambda]^{<\omega}\}$, for each $a \in [\lambda]^{<\omega}$, let

$$j_a : V \to M_a \cong \mathrm{Ult}(V, E_a)$$

with M_a transitive, be the ultrapower embedding. (As usual, we denote the elements of M_a by their corresponding elements in $\mathrm{Ult}(V, E_a)$.)

For each $a \subseteq b$ in $[\lambda]^{<\omega}$, let π_{ba} be the projection map given as in (2) above, and let $i_{ab} : M_a \to M_b$ be the map given by

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}$$

for all $f : [\kappa]^{|a|} \to V$. By coherence, the maps i_{ab} are well-defined and commute with the ultrapower embeddings j_a.

Thus we can form the direct limit $M_{\mathcal{E}}$ of the directed system

$$\langle\langle M_a : a \in [\lambda]^{<\omega}\rangle, \langle i_{ab} : a \subseteq b\rangle\rangle$$

and let $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ be the corresponding limit elementary embedding, given by

$$j_{\mathcal{E}}(x) = [a, [c^a_x]_{E_a}]$$

for some (any) $a \in [\lambda]^{<\omega}$, and where $c^a_x : [\kappa]^{|a|} \to \{x\}$.

For each $a \in [\lambda]^{<\omega}$, let $k_{a\mathcal{E}} : M_a \to M_{\mathcal{E}}$ be given by

$$k_{a\mathcal{E}}([f]_{E_a}) = [a, [f]_{E_a}]\,.$$

It is easily checked that $j_{\mathcal{E}} = k_{a\mathcal{E}} \circ j_a$ and $k_{b\mathcal{E}} \circ i_{ab} = k_{a\mathcal{E}}$, for all $a \subseteq b$ in $[\lambda]^{<\omega}$.

Reflecting extenders

Definition: (B.-Wilson 2022)
A cardinal κ is λ-Π_n-strong if for every Π_n-definable (without parameters) class A there is an elementary embedding $j : V \to M$, with M transitive, crit$(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

A cardinal κ is Π_n-strong if it is λ-Π_n-strong for every ordinal λ.

Every strong cardinal is Π_1-strong, and every Π_n-strong cardinal belongs to $C^{(n+1)}$.

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Reflecting extenders

Theorem (B.-Goldberg 2024)

If $n \geqslant 1$ *and* $\lambda \in C^{(1)}$*, then a cardinal* $\kappa < \lambda$ *is* λ*-*Π_n*-strong if and only if there exists a* $(\kappa, \lambda + 1)$*-*n*-reflecting extender.*

Product Structural Reflection

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$PSR(\mathcal{C})$: There exists a cardinal κ such that for every A in $\mathcal{C}$ there exists a set S of structures of the same type as $\mathcal{C}$ (although not necessarily in $\mathcal{C}$) with $A \in S$ and an elementary embedding $j : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$.

Theorem:

$PSR(\Sigma_1)$ is witnessed by every $\kappa \in C^{(1)}$. **(Exercise.)**

Proposition

If κ witnesses $PSR(\Pi_n)$, then $\kappa \in C^{(n+1)}$. **(Exercise.)**

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Theorem: (B.-Wilson 2022)

The following are equivalent:

1. $PSR(\Pi_1)$
2. *There exists a strong cardinal.*

Π_n-strong cardinals

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The following are equivalent for $n > 0$:

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Π_n-strong cardinals

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The following are equivalent for $n > 0$:

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“Ord is Woodin”

Definition:

“Ord is Woodin” if for every definable $A \subseteq V$ there exists some κ which is A-strong, i.e., for every γ there is an elementary embedding $j : V \to M$ with crit$(j) = \kappa$, $\gamma < j(\kappa)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Theorem: (B.-Wilson 2022)

The following schemata are equivalent:

1. *PSR, i.e., PSR(Π_n) for all n.*
2. *There exists a Π_n-strong cardinal, for every n.*
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4. *"Ord is Woodin".*

PSR and Woodin cardinals

Recall: A cardinal δ is Woodin if it is inaccessible and for every $A \subseteq V_\delta$ there exists some $\kappa < \delta$ which is $<\delta$-A-strong, i.e., for every $\gamma < \delta$ there is an elementary embedding $j : V \to M$ with $\operatorname{crit}(j) = \kappa$, $\gamma < j(\kappa)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Theorem: (B.-Lücke 2023)

The following are equivalent for every uncountable cardinal δ:

1. *δ is a Woodin cardinal.*
2. *For every set $\mathcal{C} \subseteq V_\delta$ of structures of the same type, there exists a cardinal $\kappa < \delta$ such $PSR(\mathcal{C})$ holds, witnessed by κ.*

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