The World of Large Cardinals

II. Supercompact cardinals. From supercompactness to Vopěnka's Principle.

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Supercompact cardinals

Supercompact and extendible cardinals as SR principles

Recall: A cardinal κ is supercompact if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Equivalently, if for every $\lambda > \kappa$ there exists a κ-complete, normal, fine ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda)$.

(Recall that $\mathcal{U}$ is *normal* if whenever $\langle X_\alpha : \alpha < \lambda \rangle$ is a sequence of sets from $\mathcal{U}$, then its diagonal intersection $\Delta_{\alpha<\lambda} X_\alpha := \{x : x \in \bigcap_{\alpha \in x} X_\alpha\}$ belongs to $\mathcal{U}$.

And $\mathcal{U}$ is *fine* if for every $\alpha < \lambda$, $\{X \in \mathcal{P}_\kappa(\lambda) : \alpha \in X\} \in \mathcal{U}$.)

Fact
If κ is supercompact, then $V_\kappa \preceq_{\Sigma_2} V$.

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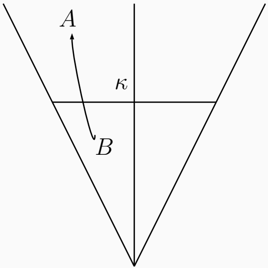
Structural Reflection

A class of relational structures $\mathcal{C}$ (of the same kind) is given by some formula $\varphi(x)$, so that

$$\mathcal{C} = \{A = \langle X, \in, \langle R_i \rangle_{i \in I} \rangle : \varphi(A)\}.$$

Structural Reflection:

$SR(\mathcal{C})$: There exists a cardinal κ that reflects $\mathcal{C}$, i.e., for all $A \in \mathcal{C}$ there exists $B \in \mathcal{C} \cap V_\kappa$ and an elementary embedding $j : B \to A$.



The *SR* principle is properly formulated in the first-order language of set theory as a schema. Namely, for each natural number n let

$SR(\Sigma_n)$: (Σ_n-*Structural Reflection*) There exists a cardinal κ that reflects every Σ_n-definable, with parameters in V_κ, class $\mathcal{C}$ of relational structures of the same type.

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Supercompactness

Theorem:
$SR(\Sigma_1)$ holds, witnessed by every κ such that $V_\kappa \preceq_{\Sigma_1} V$.

Fact: $SR(\Pi_1)$ is equivalent to $SR(\Sigma_2)$. More generally, $SR(\Pi_n)$ is equivalent to $SR(\Sigma_{n+1})$, for all $n > 0$.

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Supercompactness

Theorem: (Magidor 1971, B. 2012)

The following are equivalent:

1. $SR(\Pi_1)$
2. *There exists a supercompact cardinal.*

Notation: $SR_\kappa(\Gamma)$ means that κ witnesses $SR(\Gamma)$.

Note: If $\kappa \leqslant \lambda$ and $SR_\kappa(\Gamma)$, then also $SR_\lambda(\Gamma)$.

Corollary:

The following are equivalent for a cardinal κ:

1. κ *is the first supercompact cardinal.*
2. κ *is the least cardinal such that $SR_\kappa(\Pi_1)$ holds.*

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Beyond supercompactness

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Recall: κ is extendible if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$.

Proposition
Every extendible cardinal is supercompact.

Proposition
If κ is extendible, then $\kappa \in C^{(3)}$.

Corollary:

Every extendible cardinal is a supercompact limit of supercompact cardinals.

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Theorem

The following are equivalent for a cardinal κ:

1. κ *is the first extendible cardinal.*
2. κ *is the least cardinal such that* $SR_\kappa(\Pi_2)$ *holds.*

For the higher levels ($n > 3$) we need the notion of $C^{(n)}$-extendible cardinal.

Definition

κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$, i.e., $V_{j(\kappa)} \preceq_{\Sigma_n} V$.

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Theorem

The following are equivalent for a cardinal κ, and $n \geqslant 1$:

1. κ *is the first $C^{(n)}$-extendible cardinal.*
2. κ *is the least cardinal such that $SR_\kappa(\Pi_{n+1})$ holds.*

The proof is left as an **Exercise**.

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The following are equivalent for a cardinal κ, and $n \geqslant 1$:

1. κ *is the first $C^{(n)}$-extendible cardinal.*
2. κ *is the least cardinal such that $SR_\kappa(\Pi_{n+1})$ holds.*

The proof is left as an **Exercise**.

Corollary:

The following schemata are equivalent:

1. *SR (i.e., $SR(\mathcal{C})$ holds for all $\mathcal{C}$).*
2. *There exists a $C^{(n)}$-extendible cardinal, for every n.*
3. *There exist a proper class of $C^{(n)}$-extendible cardinals, for every n.*
4. *Vopěnka's Principle (VP): for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ with an elementary embedding $j : A \to B$.*