The World of Large Cardinals

I. Large cardinals as principles of Structural Reflection: An overview.

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Large-cardinal hypotheses (a.k.a. large-cardinal axioms) assert the existence of infinite cardinal numbers, collectively known as large cardinals, whose associated properties make them very large and imply they cannot be proved to exist in the set theory axiom system of Zermelo-Fraenkel with the axiom of Choice (ZFC).

Some early examples of large cardinals (in chronological order)

- κ is weakly inaccessible if it is regular, uncountable, and a limit cardinal. (Hausdorff 1908)
- κ is inaccessible if it is regular, uncountable, and a strong limit, i.e., $2^\lambda < \kappa$ for all $\lambda < \kappa$. (Sierpiński-Tarski, Zermelo 1930)
- κ is measurable if it is uncountable and there is a κ-additive non-trivial two-valued measure on $\mathcal{P}(\kappa)$. (Ulam 1930)
- Vopěnka's Principle (VP) asserts that here is no rigid proper class of graphs. (Vopěnka 1940)
- κ is weakly compact if it is uncountable and $\kappa \to (\kappa)^2_2$. (Erdös-Tarski 1961)

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Empirical Fact: *Every natural statement φ of the language of set theory (hence every natural mathematical statement) that is independent of ZFC is either equiconsistent with ZFC or equiconsistent with ZFC plus a large cardinal axiom.*

The consistency strength of a statement φ can be measured by large cardinal axioms if there are large cardinal axioms A_1 and A_2 such that

$$CON(ZFC + \varphi) \Rightarrow CON(ZFC + A_1)$$

and

$$CON(ZFC + A_2) \Rightarrow CON(ZFC + \varphi)$$

We refer to A_1 as a lower bound for the consistency of φ and A_2 as an upper bound. When the lower and upper bound coincide, we obtain an exact measure of the consistency strength of φ.

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Some equiconsistencies

The following pairs of statements are equiconsistent:

1. - Every co-analytic uncountable set of real numbers contains a perfect set.
   - There exists an inaccessible cardinal.
2. - There is a countably-additive measure that extends the Lebesgue measure and measures all sets of reals.
   - There exists a measurable cardinal.
3. - Every full subcategory of a locally presentable category $\mathcal{K}$ closed under colimits is coreflective in $\mathcal{K}$.
   - Vopěnka's Principle. (In fact, they are equivalent!)
4. - There exist no ω_2-Aronszajn trees.
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The hierarchy of large cardinals (by consistency strength)

Berkeley, Reinhardt

$\neg$AC

I0, I1, I3

huge

Vopěnka

extendible

supercompact

Woodin

strong

measurable

Jónsson

$V \neq L$

subtle

strongly unfoldable

weakly compact

Mahlo

inaccessible

Some (related) questions

- What is a large cardinal?

- Why do large cardinals fall into a well-ordered hierarchy, with respect to (consistency) strength?

- Is there a **single principle** that yields in a *uniform* way the whole hierarchy of large cardinals?

- **Are large-cardinal axioms true?**

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The intrinsic justification of large cardinals

- Large Cardinal Axioms have been immensely successful in terms of mathematical consequences.
- They provide a method to calibrate the strength of statements that are independent from ZFC.
- The study of large cardinals seems to be inexhaustible, as new hypotheses keep cropping up in set-theoretic work.
- But their status as new axioms of set theory is not completely settled. Many large cardinal notions don't have a straightforward intuitive appeal. Moreover, they don't seem to follow from the *concept of set*, and thus do not appear to be *intrinsically* justified.

Intrinsicness Issue:

Are large cardinals justified in light of the **concept of set**?

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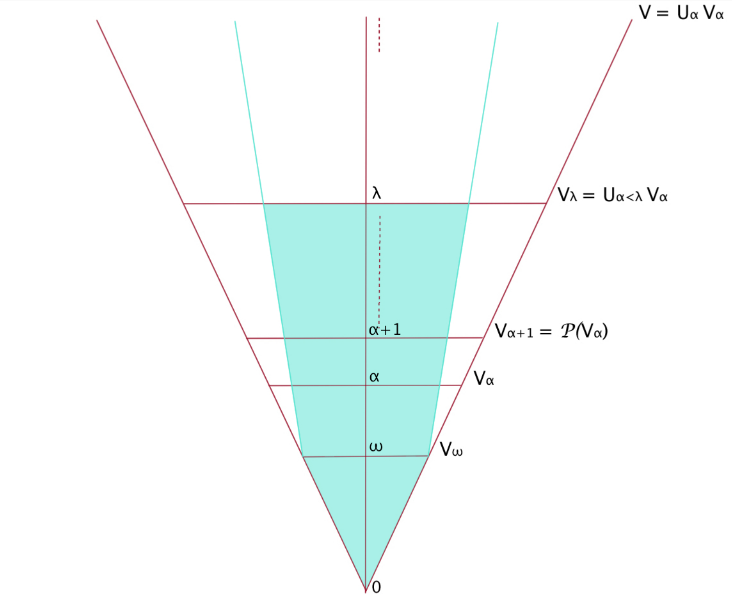
The universe V of all pure sets



Infinity as Self-Similarity

Self-similarity

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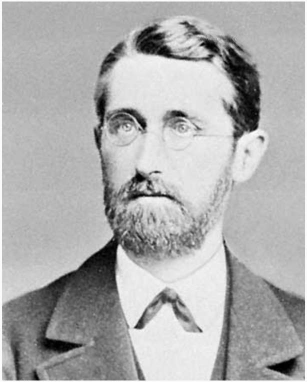
Dedekind's definition of infinite set



Richard Dedekind
(1831 - 1916)

A set S is said to be **infinite** *when it is similar [i.e., bijectable] to a proper subset of itself, otherwise it is said to be finite.*

Was sind und was sollen die Zahlen? (1888).

Infinity as self-similarity

Thus, according to Dedekind, what makes a set infinite is not its size, but its being self-similar. Size is, one might say, a side-effect.

Question: *Can all large cardinal axioms be formulated as strong forms of self-similarity?*

(Spoiler) Recent work on **Structural Reflection (SR)** reveals that, indeed, all known large cardinal axioms, correspond to different and ever stronger self-similarity properties of the set-theoretic universe.

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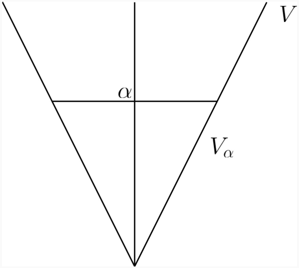
Some notions of self-similarity
I. Reflection

Reflection

The Reflection Principle: If V has some property P, then there exists an ordinal α such that V_α has P.

Theorem (Reflection Theorem (Levy 1960, Montague 1961))
For every n, there exists a closed and unbounded proper class $C^{(n)}$ of ordinals such that $V_\alpha \preceq_{\Sigma_n} V$, for every $\alpha \in C^{(n)}$.



Proof: See the Background notes, Theorem 2.1.

Complete Reflection

The Complete Reflection Principle (CR) (Levy 1960): For every formula $\varphi(x_1, \ldots, x_n)$ of the language of set theory, and every set a, there exists a transitive set A closed under subsets (i.e, all elements of the elements of A and all subsets of elements of A belong to A) which contains a and such that for every $a_1, \ldots, a_n \in A$,

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad A \models \varphi(a_1, \ldots, a_n).$$

Theorem (Levy 1960)

The theory

$$\textit{Extensionality} + \textit{Separation} + \textit{Foundation} + \textit{CR}$$

is equivalent to Zermelo-Fraenkel (ZF) set theory.

Proof: Exercise.

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Levy's Strong Reflection Principles

In the series of papers (1960-1962) Levy considers stronger principles of reflection and shows them equivalent to the existence of inaccessible, Mahlo, and Hyper-Mahlo cardinals. The unifying idea behind such principles is stated at the beginning of Levy 1961:

> *If we start with the idea of the impossibility of distinguishing, by specific means, the universe from partial universes we shall be led to [some] axiom schemata [which] will be called principles of reflection since they state the existence of standard models (by models we shall mean [...] models whose universes are sets) which reflect in some sense the state of the universe.*

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This idea is also implicit in earlier work of Ackermann (1956), but it is Levy who demonstrates how reflection can be used to find new natural theories strengthening ZFC.

In his review of Levy's 1962 article,[1] Solomon Feferman writes:

> *The author's earlier work demonstrated very well that the diversity of known set-theories could be viewed with more uniformity in the light of various reflection principles, and that these also provided a natural way to "manufacture" new theories.*

[1] Azriel Lévy. On the principles of reflection in axiomatic set theory. In *Logic, Methodology and Philosophy of Science (Proc. 1960 Internat. Congr.)*, pages 87–93. Stanford University Press, Stanford, Calif., 1962.

Large cardinals from second-order reflection

The Reflection Principle also holds with proper classes as additional predicates (**Exercise**): for every (definable, with set parameters) proper class A, and every n, there is an ordinal α such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A).$$

Thus, if we interpret second-order quantifiers over V as ranging only over definable classes, then for each n, the following second-order sentence, call it φ_n, is true in V

$$\forall A \exists \alpha (V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A).$$

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Applying second-order reflection, φ_n must reflect to some V_κ. The second-order universal quantifier in φ_n is now interpreted in V_κ, hence ranging over all subsets of V_κ, which are now available in $V_{\kappa+1}$. So we have that for each subset A of V_κ, there is some $\alpha < \kappa$ such that

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If this is so, and even just for $n = 1$, then κ must be an inaccessible cardinal, and this property actually characterises inaccessible cardinals (Levy). (**Exercise**)

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Similar arguments yield weakly-compact cardinals, etc.

Why is second-order reflection problematic?

- Second-order quantification over V is ill-defined.
- Second-order reflection cannot yield large cardinals up and beyond the first ω-Erdös cardinal (Koellner 2009).

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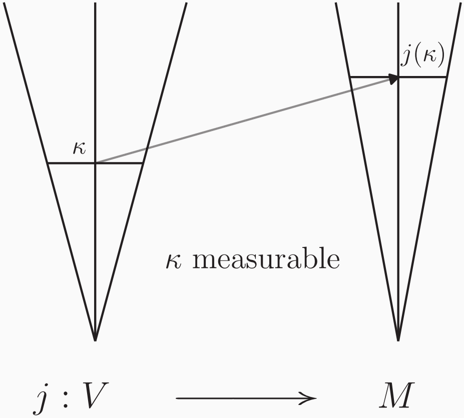
Some notions of self-similarity
2. Ressemblance

Larger cardinals as critical points of elementary embeddings

- κ is measurable iff it is the critical point of an elementary embedding $j : V \to M$, with M transitive.
  (Scott 1961)



Larger cardinals from stronger embeddings

κ is strong if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $V_\lambda \subseteq M$. (Dodd-Jensen 1982)

κ is supercompact if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $j(\kappa) > \lambda$ and M is closed under λ-sequences. (Solovay-Reinhardt, late 1960's)

Note that

$$\text{measurable} \Rightarrow \text{strong} \Rightarrow \text{supercompact}.$$

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Kunen's Theorem

Theorem: (Kunen 1971)

There is no non-trivial elementary embedding $j : V \to V$. In fact, there is no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, for any λ.

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Larger cardinals from set elementary embeddings

κ is extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$. (Reinhardt 1974)

Vopěnka's Principle holds iff for every proper class of structures $\mathcal{C}$ of the same type, there exist distinct M and N in $\mathcal{C}$ and an elementary embedding $j : N \to M$.

Larger cardinals from set elementary embeddings

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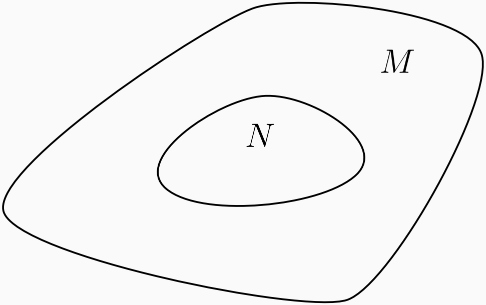
Some notions of self-similarity
3. Local Self-Similarity

The Downwards Löwenheim-Skolem-Tarski Theorem:

For every infinite structure M in a first-order language $\mathcal{L}$, there exists an elementary substructure N of M of size $\leq |\mathcal{L}|$.



Note: N need not be of the same kind as M. E.g., if M is transitive, then N need not be so. (**Exercise:** Give an example.)

Löwenheim-Skolem-Tarski Reflection

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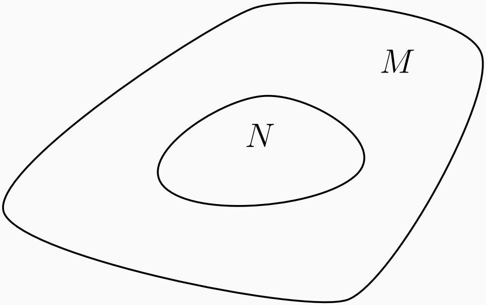

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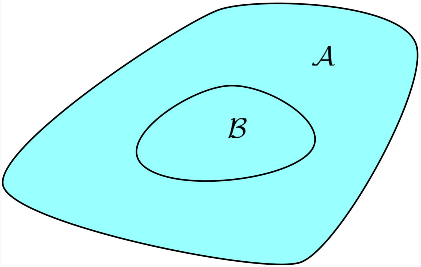
Supercompactness as local second-order self-similarity

Theorem: (Magidor 1971)

The first supercompact cardinal is the first cardinal κ *that reflects every second order formula* φ*, i.e., given a second order formula* φ*, for every structure* $\mathcal{A}$ *that satisfies* φ *there exists a substructure* $\mathcal{B} \subseteq \mathcal{A}$ *of cardinality less than* κ *that also satisfies* φ*.*



Main question

Universality Issue:

Is there a (natural) set-theoretic principle able to yield all known large cardinal notions?

Question: *Can all large cardinal axioms be shown equivalent to some (general, uniform, strong) notion of self-similarity?*

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Gödel's insight (after Ackermann, and Cantor)

Kurt Gödel
(1906 - 1978)

The universe of sets cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation in it [...].

Quoted in Wang 1996

Structural Reflection

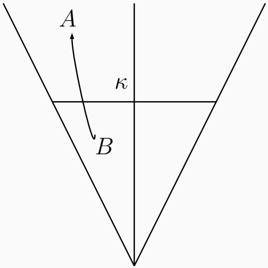
Structural Reflection

A class of relational structures $\mathcal{C}$ (of the same kind) is given by some formula $\varphi(x)$, so that

$$\mathcal{C} = \{A = \langle X, \in, \langle R_i \rangle_{i \in I} \rangle : \varphi(A)\}.$$

Structural Reflection:

$SR(\mathcal{C})$: There exists a cardinal κ that reflects $\mathcal{C}$, i.e., for all $A \in \mathcal{C}$ there exists $B \in \mathcal{C} \cap V_\kappa$ and an elementary embedding $j : B \to A$.



Supercompactness

Theorem:
$SR(\Sigma_1)$ holds, witnessed by every κ such that $V_\kappa \preceq_{\mathbf{\Sigma_1}} V$.

Supercompactness

Recall: A cardinal κ is supercompact if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Theorem: (Magidor 1971, B. 2012)

The following are equivalent:

1. $SR(\Pi_1)$
2. *There exists a supercompact cardinal.*

Fact: $SR(\Pi_1)$ is equivalent to $SR(\Sigma_2)$. More generally, $SR(\Pi_n)$ is equivalent to $SR(\Sigma_{n+1})$, for all $n > 0$.

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Recall: κ is extendible if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$.

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For the higher levels ($n > 2$) we need the notion of $C^{(n)}$-extendible cardinal.

Recall: $C^{(n)}$ is the class of cardinals κ such that $V_\kappa \preceq_{\Sigma_n} V$.

Definition: (B. 2012)

κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with crit$(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$.

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Theorem: (B. 2012)

The following are equivalent for $n \geq 1$:

1. $SR(\Pi_{n+1})$
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Corollary:

The following schemata are equivalent:

1. *SR (i.e., $SR(\mathcal{C})$ holds for all $\mathcal{C}$).*
2. *There exists a $C^{(n)}$-extendible cardinal, for every n.*
3. *There exist a proper class of $C^{(n)}$-extendible cardinals, for every n.*
4. *Vopěnka's Principle (VP): for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ with an elementary embedding $j : A \to B$.*

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SR from Supercompact to Vopěnka's Principle

SR	Strength
Σ_1	ZFC
Π_1	Supercompact
Π_2	Extendible
Π_3	$C^{(2)}$-Extendible
$\vdots$	$\vdots$
Π_n	$C^{(n-1)}$-Extendible
$\vdots$	$\vdots$
Π_n, all n	VP

SR and Vopěnka Cardinals

Recall: δ is a Vopěnka cardinal if it is inaccessible and for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exist $A \neq B \in \mathcal{C}$ with an elementary embedding $j : A \to B$.

Theorem: (B.-Lücke 2023)

The following are equivalent for every uncountable cardinal δ:

1. *δ is a Vopěnka cardinal.*
2. *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exists a cardinal $\kappa < \delta$ such that $SR(\mathcal{C})$ holds, witnessed by κ.*

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Below supercompactness

Product Structural Reflection

Product Structural Reflection

PSR($\mathcal{C}$): There exists a cardinal κ such that for every A in $\mathcal{C}$ there exists a set S of structures of the same type as $\mathcal{C}$ (although not necessarily in $\mathcal{C}$) with $A \in S$ and an elementary embedding $j : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$.

Theorem:

$PSR(\Sigma_1)$ is witnessed by every $\kappa \in C^{(1)}$.

Theorem: (B.-Wilson 2022)

The following are equivalent:

1. *$PSR(\Pi_1)$*
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Π_n-strong cardinals

Definition: (B.-Wilson 2022)
A cardinal κ is λ-Π_n-strong if for every Π_n-definable (without parameters) class A there is an elementary embedding $j : V \to M$, with M transitive, $crit(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

A cardinal κ is Π_n-strong if it is λ-Π_n-strong for every ordinal λ.

Theorem: (B.-Wilson 2022)
The following are equivalent for $n > 0$:

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“Ord is Woodin"

Definition:

“Ord is Woodin" if for every definable $A \subseteq V$ there exists some κ which is A-strong, i.e., for every γ there is an elementary embedding $j : V \to M$ with crit$(j) = \kappa$, $\gamma < j(\kappa)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Theorem: (B.-Wilson 2022)

The following schemata are equivalent:

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SR from Strong to Ord is Woodin

PSR	Strength
Σ_1	ZFC
Π_1	Strong
Π_2	Π_2-Strong
Π_3	Π_3-Strong
$\vdots$	$\vdots$
Π_n	Π_n-Strong
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PSR and Woodin cardinals

Recall: A cardinal δ is Woodin if it is inaccessible and for every $A \subseteq V_\delta$ there exists some $\kappa < \delta$ which is $<\delta$-A-strong, i.e., for every $\gamma < \delta$ there is an elementary embedding $j : V \to M$ with $\operatorname{crit}(j) = \kappa$, $\gamma < j(\kappa)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Theorem: (B.-Lücke 2023)

The following are equivalent for every uncountable cardinal δ:

1. *δ is a Woodin cardinal.*
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Below measurability

Weak Structural Reflection

Weak Structural Reflection

WSR($\mathcal{C}$): There exists a cardinal κ such that $\mathcal{C} \cap V_\kappa \neq \varnothing$, and for every A in $\mathcal{C}$ *of cardinality* κ, there exists some B in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of B into A.

Definition: (Villaveces 1998)

An inaccessible cardinal κ is strongly unfoldable if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \to N$ with $crit(j) = \kappa$ and $j(\kappa) \geq \lambda$.

First totally indescribable < First strongly unfoldable < First strong

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Theorem: (B-Lücke 2023)

The following are equivalent for every cardinal κ:

1. κ *is either strongly unfoldable or a limit of supercompact cardinals.*
2. κ *witnesses* $\mathit{WSR}(\Pi_1)$.

Since supercompact cardinals are strongly unfoldable, we get

Corollary:

The following are equivalent for every cardinal κ:

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Theorem: (B-Lücke 2023)

The following are equivalent for every cardinal κ and $n > 1$:

1. *κ is either $C^{(n)}$-strongly unfoldable*[2] *or a limit of $C^{(n-1)}$-extendible cardinals.*
2. *κ witnesses $WSR(\Pi_n)$.*

Since $C^{(n)}$-extendible cardinals are $C^{n+1)}$-strongly unfoldable, we get the following:

Corollary:

The following are equivalent for every cardinal κ and every $n > 0$:

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Subtle cardinals

If δ is an ordinal, a sequence $\langle E_\alpha : \alpha < \delta \rangle$ is called an δ-list if $E_\alpha \subseteq \alpha$ holds for every $\alpha < \delta$.

Definition: (Jensen-Kunen 1969)

An infinite cardinal δ is subtle if for every δ-list $\langle E_\gamma : \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with $E_\beta = E_\gamma \cap \beta$.

First totally indescribable $<$ First subtle $<$ First strongly unfoldable

If δ is subtle, then in V_δ there are stationary-many strongly unfoldable cardinals.

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If δ is subtle, then in V_δ there are stationary-many strongly unfoldable cardinals.

Theorem: (B.-Lücke 2023)

The following are equivalent for every uncountable cardinal δ:

1. *δ is either subtle or a limit of subtle cardinals.*
2. *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exists a cardinal $\kappa < \delta$ that witnesses WSR($\mathcal{C}$).*

Corollary:

The following are equivalent for every uncountable cardinal δ:

1. *δ is the least subtle cardinal.*
2. *δ is the least cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exists a cardinal $\kappa < \delta$ that witnesses WSR($\mathcal{C}$).*

Theorem: (B.-Lücke 2023)

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WSR from Strong to Ord is subtle

WSR	Strength
Σ_1	ZFC
Π_1	Strongly unfoldable
Π_2	$C^{(2)}$-Strongly unfoldable
Π_3	$C^{(3)}$-Strongly unfoldable
$\vdots$	$\vdots$
Π_n	$C^{(n)}$-Strongly unfoldable
$\vdots$	$\vdots$
Π_n, all n	Ord is subtle

Patterns of Structural Reflection

$$\frac{\textit{Vopěnka}}{\textit{supercompact}} = \frac{\textit{Woodin}}{\textit{strong}} = \frac{\textit{subtle}}{\textit{strongly unfoldable}}$$

Beyond Vopěnka's Principle

Exact Structural Reflection

Given infinite cardinals $\kappa < \lambda$ and a class $\mathcal{C}$ of structures of the same type, let

Exact Structural Reflection

$ESR_{\mathcal{C}}(\kappa, \lambda)$: For every $A \in \mathcal{C}$ of rank λ, there exists some $B \in \mathcal{C}$ of rank κ and an elementary embedding $j : B \to A$.

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The strength of $ESR(\kappa, \lambda)$

The strength of $ESR(\kappa, \lambda)$, goes beyond VP, for it implies the existence of almost huge cardinals.

Recall: A cardinal κ is **almost huge** if there exists a transitive class M and a non-trivial elementary embedding $j : V \to M$ with $\mathrm{crit} j = \kappa$ and ${}^{<j(\kappa)}M \subseteq M$.

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The strength of $ESR(\kappa, \lambda)$

Recall: κ is **huge** if there exists a transitive class M and a non-trivial elementary embedding $j : V \to M$ with $\text{crit} j = \kappa$ and ${}^{j(\kappa)}M \subseteq M$.

- If κ is **huge**, then $ESR_{\Pi_1}(\mu, \nu)$ holds for some $\mu \leq \kappa$ and $\nu > \mu$.
- $ESR_{\Pi_1}(\kappa, \lambda)$ implies the existence of many almost-huge cardinals, so it is stronger than VP.

Recall: κ is 2-**huge** if there exists a transitive class M and a non-trivial elementary embedding $j : V \to M$ with $\text{crit} j = \kappa$ and ${}^{j^2(\kappa)}M \subseteq M$.

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The strength of $ESR(\kappa, \lambda)$

$ESR(\kappa, \lambda)$	Strength
Π_1	Between almost huge and huge
Π_n	Below 2-huge, all n
Σ_{n+1}	Below 2-huge, all n

Beyond huge

The following strengthenings of *SR* were introduced and studied by **Nai-Chung Hou** in his master's thesis (UB, 2024):

Level-by-level Structural Reflection

LSR($\mathcal{C}$): There exists a cardinal κ that reflects $\mathcal{C}$ *level-by-level*, i.e., for every ordinal β there is some ordinal $\alpha < \kappa$ such that for every $B \in \mathcal{C}$ of rank β, there exists some $A \in \mathcal{C}$ of rank α and an elementary embedding $j : A \to B$.

Capturing Structural Reflection

CSR($\mathcal{C}$): There exists a cardinal κ such that for every $B \in \mathcal{C}$ there exists some $A \in \mathcal{C} \cap V_\kappa$ that *captures* B, i.e., for every $b \in B$ there is an elementary embedding $j : A \to B$ with b in its range.

Beyond huge

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The strength of *LSR* and *CSR*

Definition (Hou)

κ is *m*-supercompact if for every $\lambda > \kappa$ there exists $\bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_{\lambda}$ with $j^m(\mathrm{crit}(j)) = \kappa$.

Theorem (Hou)

The following are equivalent:

1. *LSR*(Π_1)
2. *CSR*(Π_1)
3. *There exists a 2-supercompact cardinal.*

The strength of *LSR* and *CSR*

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2. $CSR(\Pi_1)$
3. *There exists a 2-supercompact cardinal.*

The strength of *LSR* and *CSR*

Definition (B-Poveda, Hou)

A cardinal κ is $C^{(n)}$-2-*fold extendible* if for every $\eta > \kappa$ there is an elementary embedding $j : V_{j(\eta)} \to V_\delta$, for some δ, with $\mathrm{crit}(j) = \kappa$, $\eta < j(\kappa)$, and $j(\kappa) \in C^{(n)}$.

Theorem (Hou)

The following are equivalent for every $n > 0$:

1. $LSR(\Pi_{n+1})$
2. $CSR(\Pi_{n+1})$
3. *There exists a $C^{(n)}$-2-fold extendible cardinal.*

If κ is almost 2-huge, then V_κ is a model of ZFC plus there is a proper class of $C^{(n)}$-2-fold extendible cardinals, for every n.

The strength of *LSR* and *CSR*

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LSR, *CSR*	Strength
Π_1	2-supercompact
Π_2	$C^{(1)}$-2-fold extendible
Π_3	$C^{(2)}$-2-fold extendible
$\vdots$	$\vdots$
Π_n	$C^{(n-1)}$-2-fold extendible
$\vdots$	$\vdots$
Π_n, all n	Below 2-huge

Structural Reflection at the edge

The strongest large cardinals consistent with AC

- An I3 embedding is a non-trivial elementary embedding $j : V_\lambda \to V_\lambda$, with λ a limit ordinal.

- An I1 embedding is a non-trivial elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$, with λ a limit ordinal.

  (**Kunen's inconsistency:** There is no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, with λ a limit ordinal.)

- An I0 embedding (Woodin 1984) is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, with λ a limit ordinal, and critical point below λ.

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Definition (Aguilera-B.-Lücke)

A cardinal λ is exacting (ultraexacting) iff there is an elementary embedding $j : X \to V_\zeta$, where ζ is some cardinal in $C^{(2)}$ above λ (equivalently, the least such) and X is an elementary substructure of V_ζ that contains $V_\lambda \cup \{\lambda\}$, such that $j(\lambda) = \lambda$, $j \restriction \lambda \neq \mathrm{Id}_\lambda$ (and $j \restriction V_\lambda \in X$).

Let $\mathcal{L}$ be a first-order language with a unary predicate symbol $\dot{P}$.

We say that an $\mathcal{L}$-structure $A = \langle A, \dot{P}^A, \ldots \rangle$ has type $\langle \mu, \lambda \rangle$ is A has rank λ and $\dot{P}^A$ has rank μ.

Exacting Structural Reflection for a class $\mathcal{C}$ of $\mathcal{L}$-structures
$XSR_{\mathcal{C}}(\lambda)$: There is a cardinal $\mu < \lambda$ such that for every $A \in \mathcal{C}$ of type $\langle \mu, \lambda \rangle$ there exists $B \in \mathcal{C}$ of type $\langle \nu, \lambda \rangle$, for some $\nu < \mu$, and an elementary embedding $j : B \to A$.

Note: If $\mathcal{C} = \{\langle V_\xi, \in, \alpha \rangle : \alpha < \xi\}$, then $XSR_{\mathcal{C}}(\lambda)$ is equivalent to the existence of an I3 embedding $j : V_\lambda \to V_\lambda$.

Let $\mathcal{L}$ be a first-order language with a unary predicate symbol $\dot{P}$.

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Exacting cardinals as *SR* principles

Theorem (A-B-L)

A cardinal λ is exacting if and only if the principle $XSR_{\mathcal{C}}(\lambda)$ holds for all definable, with parameter λ, classes $\mathcal{C}$ (equivalently, for a particular Δ_3 class of $\mathcal{L}$-structures).

I3 embeddings with parameters in OD predicates

Recall: OD is the class of all Ordinal Definable sets. OD is a Σ_2 definable class, with a Σ_2-definable well-ordering, $<_{OD}$.

Theorem (Aguilera-B.-Goldberg-Lücke)

A cardinal λ is exacting if and only if for every nonempty OD subset A of $V_{\lambda+1}$, there exist $x, y \in A$ and an elementary embedding $j : (V_\lambda, x) \to (V_\lambda, y)$.

I1 embeddings with *OD* predicates

Theorem (A-B-G-L)

A cardinal λ is ultraexacting if and only if for every OD subset A of $V_{\lambda+1}$, there exists an elementary embedding $j : (V_{\lambda+1}, A) \to (V_{\lambda+1}, A)$.

The strength of exacting and ultraexacting cardinals

Theorem (A-B-G-L)

The consistency strength of the existence of an exacting cardinal is just slightly above the existence of an $\mathrm{I3}$ *embedding, modulo ZFC.*

Theorem (A-B-G-L)

The existence of an ultraexacting cardinal is equiconsistent with the existence of an $\mathrm{I0}$ *embedding, modulo ZFC.*

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