AN INTRODUCTION TO LARGE CARDINALS

SOME EXERCISES FOR THE ADVANCED COURSE AT THE FUDAN LOGIC SUMMER SCHOOL ENTITLED

THE WORLD OF LARGE CARDINALS

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JOAN BAGARIA

Day 1

(1) **The Complete Reflection Principle (CR) (Levy 1960):** For every formula $\varphi(x_1, \ldots, x_n)$ of the language of set theory, and every set a, there exists a transitive set A closed under subsets (i.e, all elements of the elements of A and all subsets of elements of A belong to A) which contains a and such that for every $a_1, \ldots, a_n \in A$,

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad A \models \varphi(a_1, \ldots, a_n).$$

Prove that

$$\text{Extensionality} + \text{Separation} + \text{Foundation} + \text{CR}$$

is equivalent to Zermelo-Fraenkel (ZF) set theory.

(2) Show that the Reflection Principle of Levy also holds with proper classes as additional predicates: for every (definable, with set parameters) proper class A, and every n, there is an ordinal α such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A).$$

(3) Show that κ is inaccessible iff for each subset A of V_κ, there is some $\alpha < \kappa$ such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_{\Sigma_1} (V_\kappa, \in, A).$$

(4) (Reinhardt) Prove that one cannot have Reflection at any cardinal κ, i.e., at $\langle V_\kappa, \in, A\rangle$, for Π^1_1 formulas with arbitrary third-order parameters A.

Hint: Suppose κ is such a cardinal and consider $A = \{V_\alpha : \alpha < \kappa\}$ seen as a third-order predicate, i.e., as a subset of $\mathcal{P}(V_\kappa)$. Consider the sentence saying that every element of A is a set.

(5) Show that $C^{(1)}$ is the club proper class of uncountable cardinals κ such that $V_\kappa = H_\kappa$. Hence $C^{(1)}$ is Π_1-definable.

More generally, show that $C^{(n)}$ is Π_n-definable.

Hint: Use the fact that the Σ_n-satisfaction relation (for $n \geq 1$) is Σ_n-definable, and also that the function $\alpha \longmapsto V_\alpha$ is Π_1-definable.

(6) Show that if $j : V \to M$ is an elementary embedding that is non-trivial, i.e., not the identity, then its critical point, namely the first ordinal moved by j, is an inaccessible cardinal.

Day 2

(1) Let $M \subseteq N$, where M and N are transitive sets or proper classes. Show that Σ_0 formulas, i.e., with only bounded quantifiers, are absolute for M, N. Namely, for every Σ_0 formula $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in M$.

$$M \models \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad N \models \varphi(a_1, \ldots, a_n).$$

Show that Σ_1 formulas are upwards absolute and that Π_1 formulas are downwards absolute for M, N.

(2) Show that he following are equivalent:
   (a) $\lambda \in C^{(1)}$
   (b) λ is an uncountable cardinal and $V_\lambda = H_\lambda$
   (c) λ is uncountable and $|V_\lambda| = \lambda$
   (d) λ is a fixed point of the $\beth$ function.

(3) Fill in the details in the background notes (4.6 to 4.10) showing that a cardinal κ is supercompact iff for every $\lambda > \kappa$ there exists a κ-complete, normal, fine ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda)$.

(4) Show that every supercompact cardinal is strong.

(5) Show that every extendible cardinal is a supercompact limit of supercompact cardinals.

Day 3

(1) Show that the following are equivalent for a cardinal κ, and $n \geq 1$:
   (a) κ is the first $C^{(n)}$-extendible cardinal.
   (b) κ is the least cardinal such that $SR_\kappa(\Pi_{n+1})$ holds.

   This is a straightforward generalization of the proof for the $n = 2$ case, done in class.

(2) Show that an ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa\lambda$ is normal iff every function $f : \mathcal{P}_\kappa\lambda \to \lambda$ that is regressive (i.e., $f(\sigma) \in \sigma$) on a set in $\mathcal{U}$, is constant on a set in $\mathcal{U}$.

(3) Show that a cardinal κ is $C^{(n)}$-extendible iff there exists a fine and normal $(n+1)$-reflecting measure on $\mathcal{P}_\kappa\lambda$, for all cardinals $\lambda \in C^{(n+1)}$ greater than or equal to κ.

   This is an easy generalization of the proof for the $n = 1$ case, done in class.

(4) Show that $PSR(\Sigma_1)$ is witnessed by every $\kappa \in C^{(1)}$.

(5) Show that if κ witnesses $PSR(\Pi_n)$, then $\kappa \in C^{(n+1)}$.

ICREA Research Professor. Department of mathematics and Computer Science, University of Barcelona.

Email address: bagaria@.ub.edu