SOME APPLICATIONS OF MODEL THEORY TO ALGEBRAIC COMBINATORICS - EXERCISES

Contents

Throughout, we list some definitions from [1] for convenience.

1. Overview of the Elekes-Szabó problem

Definition 1.1. V ***admits no power-saving*** if there are no $\epsilon, c > 0$ such that $|V \cap \prod_{i=1}^n X_i| \leq c(N^{\dim V - \epsilon})$ for all $X_i \subseteq \mathbb{C}$ of size $\leq N$.

Definition 1.2. A ***generically finite algebraic correspondence*** between irreducible algebraic varieties V and V' is a closed irreducible subvariety of the product $\Gamma \subseteq V \times V'$ such that the projections $\pi_V(\Gamma) \subseteq V$ and $\pi_{V'}(\Gamma) \subseteq V'$ are Zariski dense, and $\dim(\Gamma) = \dim(V) = \dim(V')$.

Definition 1.3. Suppose $W_1, \ldots, W_n$ and $W'_1, \ldots, W'_n$ are irreducible algebraic varieties, and $V \subseteq \prod_{i=1}^n W_i$ and $V' \subseteq \prod_{i=1}^n W'_i$ are irreducible subvarieties. Then we say V and V' are in ***co-ordinatewise correspondence*** if there is a generically finite algebraic correspondence $\Gamma \subseteq V \times V'$ and a permutation $\sigma \in \mathrm{Sym}(n)$ such that for each i, the closure of the projection $(\pi_i \times \pi'_{\sigma i})(\Gamma) \subseteq W_i \times W'_{\sigma i}$ is a generically finite algebraic correspondence (between the closure of $\pi_i(V)$ and the closure of $\pi'_{\sigma i}(V')$).

Definition 1.4. An irreducible algebraic set $V \subseteq \mathbb{C}^n$ is ***special*** if it is in co-ordinatewise correspondence with a product $\prod_i H_i \leq \prod_i G_i^{n_i}$ of connected subgroups H_i of powers $G_i^{n_i}$ of 1-dimensional complex algebraic groups, where $\sum_i n_i = n$.

1.1. Exercises for Monday 17 August.

(1) Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic set. Prove that there is a constant $c > 0$ such that for any $N \in \mathbb{N}$ and $X_1, \ldots, X_n \subseteq \mathbb{C}$ of size at most N we have $|V \cap \prod_{i=1}^n X_i| \leq cN^{\dim V}$

Hint: Use the fact that there is an algebraic $W \subseteq V$ with $\dim W < \dim V$ and a co-ordinate projection $V \setminus W \to \mathbb{C}^{\dim V}$ with fibres of bounded finite size.

(2) Let $V = \{(x, y, x + y, xy) : x, y \in \mathbb{C}\} \subseteq \mathbb{C}^4$. It can be shown V is not special, hence by [1] it admits power-saving. Deduce from this the sum-product phenomenon:

Fact(Erdős–Szemerédi, 1983) There are $\delta, c > 0$ such that for every finite set $A \subseteq \mathbb{Z}$, one has

$$\max\{|A + A|,\ |A \cdot A|\} \geq c|A|^{1+\delta}.$$

(3) Prove that if $V \subseteq \mathbb{C}^3$ is an irreducible surface satisfying one of the two conditions below, then V is special:

(i) $V \subseteq \mathbb{C}^3$ is in co-ordinatewise correspondence with the graph $\Gamma = \{(g, h, g + h) : g, h \in G\} \subseteq G^3$ of the group operation of a 1-dimensional connected complex algebraic group G,

(ii) $\dim(\pi_{ij}(V)) = 1$ for some $i \neq j \in \{1, 2, 3\}$.

(4) Prove the converse to the above: if $V \subseteq \mathbb{C}^3$ is an irreducible surface that is special, then it satisfies (i) or (ii).

Hint: use the following fact:

Fact 1.5. *Let G be a one-dimensional connected complex algebraic group, and let $H \leq G^3$ be a connected algebraic subgroup of codimension one. Then there exist endomorphisms*

$$\alpha_1, \alpha_2, \alpha_3 \in \mathrm{End}(G),$$

not all zero, such that H is the connected component of

$$\left\{(x_1, x_2, x_3) \in G^3 : \alpha_1(x_1) + \alpha_2(x_2) + \alpha_3(x_3) = 0\right\}.$$

(5) Prove directly that if $V \subseteq \mathbb{C}^3$ is an irreducible surface satisfying (i) or (ii) then it admits no power-saving.

2. The coarse pseudo-finite dimension

We work with the ultraproduct $K = \prod_{s\to\mathcal{U}} K_s$ where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

We let ${}^*\mathbb{R} := \mathbb{R}^{\mathcal{U}} = \prod_{s\to\mathcal{U}} \mathbb{R}$.

Using existence of ultralimits (see Exercise 1 below) we let $\mathrm{st} : {}^*\mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ be the ***standard part map***, namely $\mathrm{st}(\xi)$ is ∞ (resp. $-\infty$) if $\xi = (\xi_s)_{s\geq 0} \in {}^*\mathbb{R}$ is larger (resp. smaller) than any standard real, and otherwise it is the ultralimit along $\mathcal{U}$ of the sequence $(\xi_s)_s$.

We say that a subset $X \subseteq K^n$ is ***internal*** if $X = \prod_{s\to\mathcal{U}} X_s$ for some subsets $X_s \subseteq K_s^n$. Note that every definable subset of K is internal.

We fix a non-standard real $\xi \in {}^*\mathbb{R}$ with $\xi > \mathbb{R}$, which we call the ***scaling constant***.

Definition 2.1. Given an internal set $X = \prod_{s\to\mathcal{U}} X_s \subseteq K^n$, define the non-standard cardinality of X by $|X| := \prod_{s\to\mathcal{U}} |X_s| \in {}^*\mathbb{R} \cup \{\infty\}$ and its ***coarse pseudo-finite dimension*** $\boldsymbol{\delta}(X)$ by

$$\boldsymbol{\delta}(X) := \mathrm{st}\left(\frac{\log |X|}{\log \xi}\right) \in \mathbb{R}_{\geq 0} \cup \{-\infty, \infty\}$$

By a ***$\bigwedge$-definable set*** (type definable set) over C we mean a subset X of K^n, for some n, which is the intersection of countably many C-definable sets. We say a set is ***$\bigwedge$-internal*** if it is the intersection of a countable collection of internal sets; so $\bigwedge$-definable sets are $\bigwedge$-internal.

For a $\bigwedge$-internal set $X \subseteq K^n$, we define

$$\boldsymbol{\delta}(X) := \inf\{\boldsymbol{\delta}(Y) : Y \supseteq X,\ Y \text{ internal}\}.$$

For any countable set $C \subseteq K$ and any $\overline{a} \in K^n$ the set $\mathrm{tp}(\overline{a}/C)(K)$ is $\bigwedge$-definable, so this allows to define $\boldsymbol{\delta}(\overline{a}/C)$, the ***coarse dimension of the tuple*** $\overline{a}$ over C, as $\boldsymbol{\delta}(\mathrm{tp}(\overline{a}/C)(K))$.

2.1. **Exercises for Tuesday 18 August.**

(1) Prove that every bounded sequence r_i of real numbers has a limit r with respect to $\mathcal{U}$, meaning: for every $\epsilon > 0$ we have $\{i \in \mathbb{N} : |r_i - r| < \epsilon\} \in \mathcal{U}$.
(2) Prove that $K = \prod_{s \to \mathcal{U}} K_s$ has the following $\aleph_1$-compactness property: given a chain $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ of internal subsets of K^n, if $\bigcap_{i \in \mathbb{N}} X_i = \emptyset$ then $X_i = \emptyset$ for some i.
(3) Given internal sets $A, B \subseteq K^n$, prove that
   - If A is non-empty then $\boldsymbol{\delta}(A) \geq 0$
   - If $A \subseteq B$ then $\boldsymbol{\delta}(A) \leq \boldsymbol{\delta}(B)$
   - $\boldsymbol{\delta}(A \cup B) = \max(\boldsymbol{\delta}(A), \boldsymbol{\delta}(B))$
(4) Prove in detail that if $\{a_1, \dots, a_n\} = \{b_1, \dots, b_k\}$ where a_i and b_j are elements of K (we allow repetitions in both sets), then $\boldsymbol{\delta}(a_1, \dots, a_n) = \boldsymbol{\delta}(b_1, \dots, b_k)$.
(5) Recall we say $\boldsymbol{\delta}$ is ***continuous*** if given $n, m \geq 1$, $\alpha \in \mathbb{R}$, $\epsilon > 0$ and a $\emptyset$-definable set $Y \subseteq K^n \times K^m$ there is a $\emptyset$-definable set $W \subseteq K^m$ such that

$$\{\overline{b} \in K^m : \boldsymbol{\delta}(Y_{\overline{b}}) \geq \alpha + \epsilon\} \subseteq W \subseteq \{\overline{b} \in K^m : \boldsymbol{\delta}(Y_{\overline{b}}) \geq \alpha\},$$

where $Y_{\overline{b}}$ is the fibre $\{\overline{x} \in K^n : (\overline{x}, \overline{b}) \in Y\}$.

Prove that if $\boldsymbol{\delta}$ is continuous, then for every Y and α as above, the sets

$$\{\overline{b} \in K^m : \boldsymbol{\delta}(Y_{\overline{b}}) \geq \alpha\}$$

and

$$\{\overline{b} \in K^m : \boldsymbol{\delta}(Y_{\overline{b}}) \leq \alpha\}$$

are $\bigwedge$-definable over $\emptyset$.

2.2. **Exercises for Wednesday 19 August.**

(1) Deduce from the $\aleph_1$-compactness property that if a $\bigwedge$-internal set is contained in the union of countably many internal sets, then it is contained in the union of finitely many of them.
(2) Prove that $\boldsymbol{\delta}(A \times B) = \boldsymbol{\delta}(A) + \boldsymbol{\delta}(B)$ for any non-empty internal sets A and B.
(3) Prove that $\boldsymbol{\delta}(A \times B) = \boldsymbol{\delta}(A) + \boldsymbol{\delta}(B)$ for any non-empty $\bigwedge$-internal sets A and B. Hint: Given an internal $Z \supseteq A \times B$, use $\aleph_1$-compactness to find internal X and Y with $A \times B \subseteq X \times Y \subseteq Z$.
(4) Prove that $\boldsymbol{\delta}(\overline{a}\overline{b}) \leq \boldsymbol{\delta}(\overline{a}) + \boldsymbol{\delta}(\overline{b})$. Note this holds without assuming that $\boldsymbol{\delta}$ is continuous.
(5) Assume δ is continuous. We know in that case $\boldsymbol{\delta}(\overline{a}\overline{b}/C) = \boldsymbol{\delta}(\overline{a}/\overline{b}C) + \boldsymbol{\delta}(\overline{b}/C)$. Deduce the following property, which may be seen as symmetry of '$\boldsymbol{\delta}$-independence': $\boldsymbol{\delta}(\overline{a}/C) = \boldsymbol{\delta}(\overline{a}/\overline{b}C)$ if and only if $\boldsymbol{\delta}(\overline{b}/C) = \boldsymbol{\delta}(\overline{b}/\overline{a}C)$.
(6) Recall the locus $\mathrm{loc}(\overline{a}/C_0)$ of a tuple $\overline{a}$ over a subfield C_0 is the smallest algebraic set defined over C_0 containing $\overline{a}$, and that if C_0 is algebraically closed then $\mathrm{loc}(\overline{a}/C_0)$ is irreducible. Suppose C_0 is algebraically closed, $V \subseteq W_1 \times W_2 \times W_3$ and $V' \subseteq W_1' \times W_2' \times W_3'$ are irreducible and defined over C_0, $\overline{a} = (a_1, a_2, a_3)$ is generic in V over C_0 (meaning $\mathrm{loc}(\overline{a}/C_0) = V$) and $\overline{b} = (b_1, b_2, b_3)$ is generic in V' over C_0, and a_i is interalgebraic with b_i over C_0 for $i = 1, 2, 3$, then $\mathrm{loc}(\overline{a}, \overline{b}/C_0)$ is a co-ordinatewise correspondence between V and V'.

2.3. **Exercises for Thursday 20 August.** We are working in the (algebraically closed) field $K = \mathbb{C}^{\mathcal{U}}$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

(1) Suppose X, Y, Z are non-empty internal sets with $Z \subseteq X \times Y$. Without assuming continuity or additivity of $\boldsymbol{\delta}$, prove that if for every $\overline{b} \in Y$ we have $\boldsymbol{\delta}(Z_{\overline{b}}) \geq \alpha$, then $\boldsymbol{\delta}(Z) \geq \alpha + \boldsymbol{\delta}(Y)$.
(2) Prove the same for $\bigwedge$-internal X, Y, Z.
(3) Consider the finite field $\mathbb{F}_p$ and let P be the set of all points in $\mathbb{F}_p^2$ and L the set of all (affine) lines in $\mathbb{F}_p^2$. Let $I = \{(q, \ell) : q \in P, \ell \in L, q \in \ell\}$ be the set of incidences between the points in P and the lines in L. Observe that the classical Szemeredi-Trotter bound $|I| \leq |P|^{2/3}|L|^{2/3} + |P| + |L|$ does **not** hold for these P and L (for sufficiently large p).
(4) By an irreducible component of an algebraic set V we mean a maximal irreducible algebraic subset of V. Observe that if $W \subseteq V$ is an irreducible algebraic set of the same dimension as V, then W is an irreducible component of V.
(5) Prove that if $\overline{a}$ is a finite tuple of elements of K and k is a subfield of K, then $\mathrm{loc}(\overline{a}/k^{\mathrm{alg}})$ is an irreducible component of $\mathrm{loc}(\overline{a}/k)$, where k^{alg} denotes the algebraic closure of k (in K).

3. Szemeredi-Trotter, wgp, and the Orchard Problem

3.1. Szemeredi-Trotter-type bounds.

Fact 3.1 ([2, Theorem 9])**.** *If $V \subseteq \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ is a complex algebraic subvariety there is $\epsilon_0 = \epsilon_0(n_2) > 0$ such that the following holds. Let $B \in \mathbb{N}$. Let $X_1 \subseteq \mathbb{C}^{n_1}$ and $X_2 \subseteq \mathbb{C}^{n_2}$ be finite subsets. Write $V(y) := \{x \in \mathbb{C}^{n_1} : (x, y) \in V\}$ for the fibre above $y \in \mathbb{C}^{n_2}$. Assume that for any two distinct $y, y' \in X_2$ the intersection $V(y) \cap V(y')$ contains at most B points from X_1. Then the number I of incidences $(x, y) \in X_1 \times X_2$ with $x \in V(y)$ satisfies:*

$$I \leq O_{B,V,n_1}(|X_1|^{\frac{1}{2}(1+\epsilon_0)}|X_2|^{1-\epsilon_0} + |X_1| + |X_2| \log |X_1|).$$

Moreover, the dependence in B of the big-O is sublinear, that is $O_{B,V,n_1} \leq B \cdot O_{V,n_1}$.

Corollary 3.2. *Let $X_1 \subseteq K^{n_1}$ and $X_2 \subseteq K^{n_2}$, suppose each X_i is $\bigwedge$-internal, and let $X = (X_1 \times X_2) \cap V$ where $V \subseteq K^{n_1+n_2}$ is a K-Zariski closed subset. Assume that $\boldsymbol{\delta}(X_1), \boldsymbol{\delta}(X_2)$ are both finite. Set*

$$\beta := \sup_{a,b \in X_2;\ a \neq b} \boldsymbol{\delta}(X(a) \cap X(b)),$$

where $X(y) := \{x \in X_1 : (x, y) \in X\}$. Then for some $\epsilon_0 > 0$ depending only on n_2,

$$\boldsymbol{\delta}(X) \leq \beta + \max\left(\frac{1}{2}\boldsymbol{\delta}(X_1) + \boldsymbol{\delta}(X_2) - \epsilon_0(\boldsymbol{\delta}(X_2) - \frac{1}{2}\boldsymbol{\delta}(X_1)), \boldsymbol{\delta}(X_1), \boldsymbol{\delta}(X_2)\right).$$

3.2. **The Orchard Problem and wgp.** In this subsection, assume that $\boldsymbol{\delta}$ is continuous.

Given a set of points P, by a triple line we mean a line passing through at least three of them.

For a natural number n, let $t(n)$ denote the maximal number such that there exists a set of n points on the plane with $t(n)$ many triple lines.

Write $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\boldsymbol{\delta}}_C \overline{b}$ for $\boldsymbol{\delta}(\overline{a}/C) = \boldsymbol{\delta}(\overline{a}/C\overline{b})$.

Definition 3.3. For a countable set $C \subseteq K$ and a tuple $a \in K$, the complete type $\mathrm{tp}(a/C)$ is *in weak general position* (or is *wgp*) if $\boldsymbol{\delta}(a/C) < \infty$ and for any tuple $b \in K$ such that $\mathrm{trd}^0(a/Cb) < \mathrm{trd}^0(a/C)$, we have $\boldsymbol{\delta}(a/Cb) < \boldsymbol{\delta}(a/C)$.

Equivalently: for any b, if $a \mathop{\smile\!\!\!\!\!\!\mid}^{\boldsymbol{\delta}}_C b$ then $a \mathop{\smile\!\!\!\!\!\!\mid}^{0}_C b$.

Definition 3.4. A $\bigwedge$-internal set A is *wgp in* a constructible set $X \supseteq A$ if $\boldsymbol{\delta}(A) < \infty$ and $\boldsymbol{\delta}(A \cap Y) < \boldsymbol{\delta}(A)$ for any constructible $Y \subseteq X$ over K with $\dim(Y) < \dim(X)$.

Fact 3.5. *Let V be a variety and let $a \in V(K)$. Then $\operatorname{tp}(a/C)$ is wgp if and only if the $\bigwedge$-internal set $\operatorname{tp}(a/C)(K)$ is wgp in $\operatorname{locus}(a/C)$.*

Fact 3.6. *Assume that $\boldsymbol{\delta}(a/C) < \infty$. Then $\operatorname{tp}(a/C)$ is wgp if and only if $a \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} d$ for all tuples $d \in K$ with $\boldsymbol{\delta}(d/C) = 0$.*

3.3. **Exercises for Friday 21 August.**

(1) Work out the details of how Corollary 3.2 follows from Fact 3.1.
(2) Check carefully that $t(6)$ is indeed equal to 4.
(3) Consider a set P of n points on the plane such that no line passes through more than three of them. Prove that $\frac{n(n-1)}{2} = N_2 + 3N_3$ where N_2 is the number of lines passing through exactly two points of P, and N_3 the number of triple lines.
(4) For every natural number n divisible by 6, find n points in $\{(x, y) \in \mathbb{R}^2 : y(y-1)(y-2) = 0\}$ with at least $n^2/18$ triple lines.
(5) Let $a, b \in K$ be tuples, let $C \subseteq K$ be countable, and suppose $\boldsymbol{\delta}(a/C), \boldsymbol{\delta}(b/C) < \infty$. Prove that:
    (i) If $\operatorname{tp}(a/C)$ is wgp and $e \in \operatorname{acl}(aC)$, then $\operatorname{tp}(e/C)$ is wgp.
    (ii) If $\operatorname{tp}(b/C)$ and $\operatorname{tp}(a/Cb)$ are wgp, then so is $\operatorname{tp}(ab/C)$.
    (iii) If $a \mathop{\smile\!\!\!\!\!\!|}^{\boldsymbol{\delta}}_{C} b$ and $\operatorname{tp}(a/C)$ is wgp, then $\operatorname{tp}(a/Cb)$ is wgp.
    Hint: Fact 3.6 can help.

References

[1] M. Bays and E. Breuillard. Projective geometries arising from Elekes-Szabó problems.
[2] G. Elekes and E. Szabó. How to find groups? (and how to use them in Erdös geometry?)