On a problem of Fritz, Netzer, & Thom (joint w/ J. Sinclair)

Question (FNT) Is the standard presentation of $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ computable?

C^*-algebra: Banach $*$-algebra, $\|x^*x\| = \|x\|^2$ (C^*-equality)

example $B(H)$, $\langle T^*x, y \rangle = \langle x, Ty \rangle$ $\forall x, y \in H$

<u>example</u> C^* algebras coming from groups

G ctbl discrete group

$\mathbb{C}G$ = set of finite formal sums $\sum_{g\in G} c_g u_g$ $\quad c_g \in \mathbb{C}$

$*$-algebra

$C^*(G)$ = universal group C^*-algebra associated to G

universal property: whenever $\pi: G \to B(H)$ is a unitary representation it extends uniquely to a $*$-hom. $\pi: C^*(G) \to B(H)$.

$C^*(\mathbb{F}_2 \times \mathbb{F}_2)$

Presentations of C*-algebras

If A is a C^*-alg, a presentation of A is a countable sequence $\underbrace{(a_n)}_{\text{special points}}{}_{n\in\mathbb{N}}$ from A s.t. the $\underbrace{\mathbb{Q}(i)\text{-}*\text{-subalgebra}}_{\text{generated points}}$ is dense in A.

$A^{\dagger}$, $A^{\#}$ presentations

Standard presentation of $C^*(G)$

$\mathbb{F}_2 = \langle u, v \mid \quad \rangle$

<u>example</u> $C^*(\mathbb{F}_2)$ standard presentation $(u, v, u, v, u, v, \ldots)$

<u>example</u> $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ $(u_1, v_1, u_2, v_2, u_1, v_1, u_2, v_2, \ldots)$

<u>example</u> $C^*(G)$ view some enumeration of G as the special points

If A^+ is a presentation of A, we call it <u>computable</u> if there is an algorithm so that, upon input a generated point x of A^+ and $\varepsilon \in \mathbb{Q}^{>0}$, returns $r \in \mathbb{R}^{>0}$ so that $\big|\, \|x\| - r \,\big| < \varepsilon$

$\|[u_1, u_2]\|$

Q (FNT) Is the st pres of $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ computable?

Thm (G., Sinclair) No!

Context Connes Embedding Problem $\rightsquigarrow$ von Neumann algebra theory

Kirchberg CEP $\Leftrightarrow$ $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ is residually finite-dimensional (RFD)

Def A C^*-alg A is RFD if: $\forall x \in A \setminus \{0\}$, there is $*$-hom $\pi : A \to M_n(\mathbb{C})$ for which $\pi(x) \neq 0$.

Fact (Choi) $C^*(\mathbb{F}_2)$ is RFD.

Fact (Bekka) $C^*(\underbrace{SL_3(\mathbb{Z})}_{\text{res. finite}})$ not RFD.

Thm (FNT) If G is a fin. pres. RFD group, then the stand pres of $C^*(G)$ is computable.

In particular, $C^*(\mathbb{F}_2)$ is computable.

Thm (Fox) If A^+ is any c.e. presentation of A, with A RFD, then A^+ is computable

$C^*(\mathbb{F}_2 \times \mathbb{F}_2)$

Cor If $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ is not computable, then CEP fails

2020 $MIP^* = RE \implies$ CEP fails

Our result uses $MIP^{co} = coRE$.

Some applications

① $C^*(\mathbb{F}_2 \times \mathbb{F}_2)_\varepsilon$ = Universal C^*-algebra generated by u_1, u_2, v_1, v_2,
subject to $\|[u_i, v_j]\| \leq \varepsilon$. $\underline{\varepsilon = 0}$ $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$

<u>Enders-Shulman</u> RFD if $\varepsilon > 0$ $\therefore$ (by Fox) computable pres.

But $\varepsilon < \varepsilon' \Rightarrow C^*(\mathbb{F}_2 \times \mathbb{F}_2)_{\varepsilon'} \longrightarrow C^*(\mathbb{F}_2 \times \mathbb{F}_2)_\varepsilon$

Direct limit $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$

<u>Cor</u> $p(x_1, x_2, y_1, y_2)$ $*$-poly

$\alpha_{m,p} = \|p(\vec{u}, \vec{v})\|_{C^*(\mathbb{F}_2 \times \mathbb{F}_2)_{1/m}}$

$\alpha_p = \|\cdots\|_{C^*(\mathbb{F}_2 \times \mathbb{F}_2)}$

(So $\alpha_{m,p} \xrightarrow{m \to \infty} \alpha_p$) There is no algorithm s.t. upon input p, $\varepsilon > 0$, returns m s.t. $|\alpha_{m,p} - \alpha_p| < \varepsilon$.

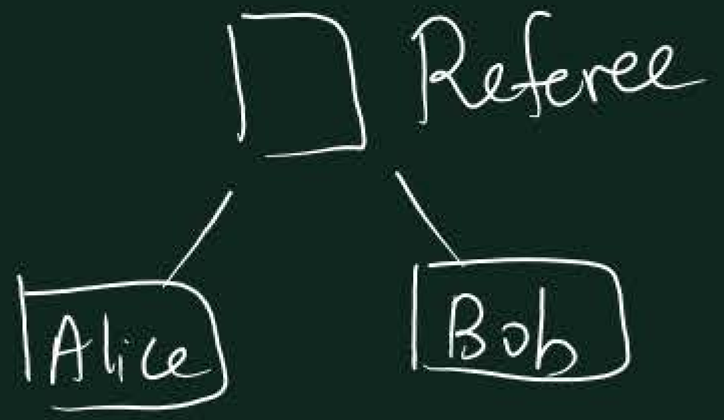
Application ② Question (Ozawa) $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ have lifting property (LP)?

This means: given $C^*(\mathbb{F}_\infty) \twoheadrightarrow C^*(\mathbb{F}_2 \times \mathbb{F}_2)$,

does the identity map on $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ admit a ucp lift $C^*(\mathbb{F}_2 \times \mathbb{F}_2) \longrightarrow C^*(\mathbb{F}_\infty)$?

Cor There can be not computable such ucp lift.

The proof



Nonlocal game $\mathcal{G} = (\pi, D)$

k questions

n answers

π prob dist on $[k]^2 = [k] \times [k]$

$D : [k]^2 \times [n]^2 \longrightarrow \{0, 1\}$

win

lose

"Strategy"

$p(a,b|x,y) \in [0,1]^{k^2 n^2}$

$$\mathrm{val}(\mathcal{G}, p) = \sum_{(x,y)} \pi(x,y) \sum_{(a,b)} p(a,b|x,y) \cdot D(x,y,a,b)$$

expect value

Quantum commuting strategies $C_{qc}(k,n)$

$p \in C_{qc}(k,n)$ iff there is a Hilb space H, $\xi \in H$ unit vector ("state"), <u>POVMs</u> A^x, B^y on H for $x, y \in [k]$ $[A^x_a, B^y_b] = 0$

$$A^x = (\underbrace{A^x_1, \ldots, A^x_n}_{\text{positive operators}}), \quad \sum A^x_a = 1.$$

$$val^{co}(G) = \sup_{p \in C_{qc}(k,n)} val(G,p)$$

$$p(a,b|x,y) = \langle A^x_a B^y_b \xi, \xi \rangle$$

<u>Fact</u> Right-c.e.

<u>Thm</u> (Lin) $MIP^{co} = coRE \Rightarrow$ $val^{co}(G)$ <u>not</u> computable (unif in G)

$C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$

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Fact (Fritz; Junge, et. al.)

$p \in C_{qc}(k,n) \iff$ there is a state ϕ on $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ and POVMs A^x, B^y in $C^*(\mathbb{F}_2)$ s.t.

$$p(a,b|x,y) = \phi(A^x_a \otimes B^y_b)$$

there A s.t. A has a c.e. pres. but no comp. pres?

Iden: $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ comp $\Rightarrow val^{co}(\mathcal{G})$ comp

$C^*(SL_3(\mathbb{Z}) \times SL_3(\mathbb{Z}))$ not comp

Q (FNT) $C^*(SL_3(\mathbb{Z}))$ computable?

$\mathbb{F}_2 \leq SL_3(\mathbb{Z})$

Do all C^*-alg have WEP?

1. WEP?

$\iff C^*(\mathbb{F}_2)$ have WEP?

A WEP $\quad C^*(\mathbb{F}_\infty) \otimes A$ is unique. (Kirchberg)

A LLP $\quad B(H) \otimes A$ is unique.