Weak alternation hierarchy in the modal μ-calculus

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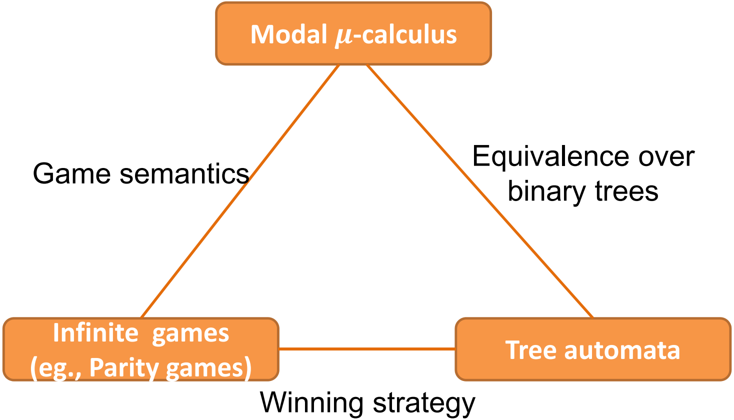
Interaction of logic, games, and automata



We will introduce the weak fragment of modal μ-calculus.

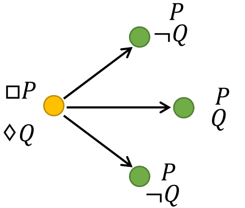
Modal μ-calculus

Modal μ-calculus is an extension of proposition logic by adding

- modalities
  At a state in a transition system (directed graph):

  $\Box P$: P holds in all successors.

  $\Diamond P$: P hold in some successor.



- fixpoint operators (second order operators),
  μ (least fixpoint), and ν (greatest fixpoint).

Example

- $\mu X.p \vee \Diamond X$ expresses that there is a path where p eventually eventually.
- $\nu Y.\mu X.(p \wedge \Diamond Y) \vee \Diamond X$ expresses that p holds infinitely many times.

A warm-up example

Example.

- Suppose $K_i\varphi$ means "the agent i knows that φ holds" , $i = 1, 2, \cdots n$
- Let E be the "everyone knows" modality:

$$E\varphi := K_1\varphi \wedge \cdots \wedge K_n\varphi$$

- Then common knowledge $C\varphi$ can be given as an infinite conjunction:

$$C\varphi \leftrightarrow \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge E^4\varphi \wedge \cdots \wedge E^n\phi \wedge \cdots$$

With greatest fixed-point operator, common knowledge has an elegant finite characterization:

$$C\varphi := \nu X.\varphi \wedge EX$$

Common knowledge $C\varphi := \nu X.\varphi \wedge EX$

- νX denotes the greatest fixed-point of the equation $X = \varphi \wedge E(X)$.

Layer 0:	φ	φ is true
Layer 1:	$E\varphi$	Everyone knows φ
Layer 2:	$EE\varphi$	Everyone knows that everyone knows φ
Layer 3:	$EEE\varphi$	Everyone knows that everyone knows that everyone knows φ
$\vdots$	$\vdots$	$\vdots$

Intuitively, X updates "the things that everyone knows":

$$X = \{\varphi, E\varphi, EE\varphi, EEE\varphi \cdots\}.$$

- The greatest fixed-point of $X = \varphi \wedge E(X)$ captures largest possible set that meets "things that everyone knows".

Basics of μ-calculus: syntax

The formulas of μ-calculus are generated by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

where P denotes an atomic proposition. Let $\top := P \vee \neg P$ and $\bot := P \wedge \neg P$.

The negation is allowed to use only if a negated formula can be transformed to a regular formula by the following rules:

$$\begin{aligned} \neg(\neg P) &= P, & \neg(\neg X) &= X, \\ \neg(\psi \vee \varphi) &= \neg\psi \wedge \neg\varphi, & \neg(\psi \wedge \varphi) &= \neg\psi \vee \neg\varphi, \\ \neg\Box\varphi &= \Diamond\neg\varphi, & \neg\Diamond\varphi &= \Box\neg\varphi, \\ \neg\mu X.\varphi(X) &= \nu X.\neg\varphi(\neg X), & \neg\nu X.\varphi(X) &= \mu X.\neg\varphi(\neg X). \end{aligned}$$

Notice that for a formula $\eta X.\varphi(X)$ ($\eta = \mu$ or ν), X appears only positively in $\varphi(X)$, namely within an even number of the scopes of negations.

Semantics

A Kripke model, (a.k.a. transition system), is a triple $M = (W, R, V)$, where (W, R) is a directed graph and V is a function from atomic propositions to the subsets of W. By $w \in V(P)$, we mean that P holds in a state or world $w \in W$.
Given a set $A \subseteq W$, the *augmented model* $M[X := A]$ is obtained by $V(X) := A$.

For a μ-formula φ, we define the valuation $\|\varphi\|^M$ on a Kripke model M inductively:

- $\|P\|^M := V(P)$; $\|X\|^{M[X:=A]} := A$; $\|\neg\varphi\|^M := W \setminus \|\varphi\|^M$;
- $\|\varphi \wedge \psi\|^M := \|\varphi\|^M \cap \|\psi\|^M$; $\|\varphi \vee \psi\|^M := \|\varphi\|^M \cup \|\psi\|^M$;
- $\|\Box\varphi\|^M := \{w \in W \mid \forall v.wRv \to v \in \|\varphi\|^M\}$;
  $\|\Diamond\varphi\|^M := \{w \in W \mid \exists v.wRv \wedge v \in \|\varphi\|^M\}$;
- $\|\mu X.\varphi\|^M$ is the least fixpoint of Γ_φ; and $\|\nu X.\varphi\|^M$ is the greatest fixpoint of Γ_φ,

where $\Gamma_\varphi : \mathcal{P}(W) \to \mathcal{P}(W)$ maps $A \subseteq W$ to $\|\varphi(X)\|^{M[X:=A]}$, abbrev. by $\|\varphi(A)\|^M$.
We also write $\Gamma_\varphi(X) = \|\varphi(X)\|^M$. As X occurs positively in $\varphi(X)$, the operator Γ_φ is monotone and its least and greatest fixed-points are well-defined.

Semantics via approximations

We can also generate the least fixpoints by approximating from the below and the greatest fixpoints from the above.

Recall that $\varphi(X)$ defines an operator

$$\Gamma_\varphi^M : \mathcal{P}(W) \to \mathcal{P}(W)$$
$$S' \mapsto [\![\varphi]\!]^{M[X:=S']}$$

We can define inductively,

- $X^0 := \emptyset$
- $X^{\alpha+1} := \varphi^M(X^\alpha)$
- $X^\lambda := \bigcup_{\alpha<\lambda} \varphi^M(X^\alpha)$, where λ ranges over limit ordinals.

There is an inductive sequence $X^0 \subseteq \cdots \subseteq X^\alpha \subseteq X^{\alpha+1} \subseteq \cdots$, which finally reaches a fixpoint $X^\beta = X^{\beta+1} := X^\infty$. We have

$$[\![\mu X.\varphi]\!] := X^\infty$$

Example

The formula $\mu X.p \vee \Diamond X$ expresses that there exists a path which leads to states where p holds. This is called liveness / reachability property. The approximation process is as follows:

$$\begin{aligned}
\mu^0 &= \emptyset \\
\mu^1 &= [\![p \vee \Diamond X]\!]^{M[X:=\mu^0]} = [\![p \vee \Diamond \emptyset]\!] = [\![p]\!] = V(p) \\
\mu^2 &= [\![p \vee \Diamond X]\!]^{M[X:=\mu^1]} = [\![p]\!] \cup [\![\Diamond p]\!] = \mu^1 \cup \{v : \exists w, (v, w) \in E \wedge w \in V(p)\} \\
\mu^3 &= [\![p \vee \Diamond X]\!]^{M[X:=\mu^2]} = [\![p]\!] \cup [\![\Diamond p]\!] \cup [\![\Diamond \Diamond p]\!] \\
&= \mu^2 \cup \{v : \exists w, u, (v, w) \in E \wedge (w, u) \in E \wedge u \in V(p)\} \\
&\vdots
\end{aligned}$$

Intuitively, μ^1 is the set of vertices where p holds, $\mu^2 = \mu^1 \cup [\![\Diamond p]\!]$ consists of vertices v such that either p holds at v or there is a successor of v such that p holds and so on.

- This process produces an inductive sequence $\mu^0 \subseteq \mu^1 \subseteq \mu^2 \subseteq \mu^3 \subseteq \cdots \subseteq \mu^n \subseteq \ldots$.
- Such a sequence reaches a fixpoint $\mu^\omega = \mu^{\omega+1} = \bigcup_{n<\omega} \mu^n$, which means that there exists n such that p holds in the n-th stage.

$\nu Z.\mu X.(p \wedge \Diamond Z) \vee \Diamond X$ expresses that p holds infinitely many times

- $\nu^0 = W \qquad \mu^{0,0} = \emptyset$

$$\mu^{0,1} = [\![(p \wedge \Diamond Z) \vee \Diamond X]\!]^{M[X:=\mu^{0,0}][Z:=W]} = [\![p \wedge \Diamond W]\!] \cup [\![\Diamond \emptyset]\!] = [\![p \wedge \Diamond W]\!]$$

$$\mu^{0,2} = [\![(p \wedge \Diamond Z) \vee \Diamond X]\!]^{M[X:=\mu^{0,1}][Z:=W]} = [\![p \wedge \Diamond W]\!] \cup [\![\Diamond \mu^{0,1}]\!]$$

$$\mu^{0,3} = [\![p \wedge \Diamond W]\!] \cup [\![\Diamond \mu^{0,2}]\!]$$

$$\vdots$$

- $\nu^1 = \mu^{0,\infty}$ eventually p $\qquad \mu^{1,0} = \emptyset$

$$\mu^{1,1} = [\![(p \wedge \Diamond Z) \vee \Diamond X]\!]^{M[X:=\mu^{1,0}][Z:=\nu^1]} = [\![(p \wedge \Diamond \nu^1) \vee \Diamond \emptyset]\!] = [\![p \wedge \Diamond \nu^1]\!]$$

$$\mu^{1,2} = [\![(p \wedge \Diamond \nu^1) \vee \Diamond \mu^{1,1}]\!] = [\![(p \wedge \Diamond \nu^1) \vee (\Diamond p \wedge \Diamond\Diamond \nu^1)]\!]$$

$$\mu^{1,3} = [\![(p \wedge \Diamond \nu^1) \vee \Diamond_a \mu^{1,2}]\!]$$

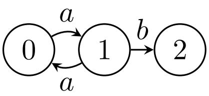
$$\vdots$$

- $\nu^1 = \mu^{1,\infty}$ eventually p followed by (eventually p)

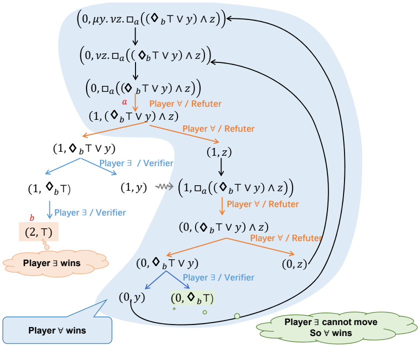
$$\vdots$$

$^{\infty}$ infinitely many

Semantics in terms of games



- Given a sentence of modal μ-calculus φ and a transition system $M = (W, R, V)$, we define the *evaluation game* $\mathcal{E}(M, s, \varphi)$ with players $\exists$ and $\forall$ moving a token along positions of the form (ψ, s), where ψ is a subformula of φ and $s \in W$.
- Player $\exists$'s purpose is to show φ is satisfied at s, while player $\forall$'s goal is opposite.



Rules of evaluation game for modal μ-calculus

Positions for player $\exists$	Admissible moves for player $\exists$
$(\psi_1 \vee \psi_2, s)$	$\{(\psi_1, s), (\psi_2, s)\}$
$(\Diamond\psi, s)$	$\{(\psi, t) \mid (s, t) \in R\}$
$(\bot, s)$	$\emptyset$
(P, s) and $s \notin V(P)$	$\emptyset$
$(\neg P, s)$ and $s \in V(P)$	$\emptyset$
$(\mu X.\psi_X, s)$	$\{(\psi_X, s)\}$
(X, s) for some subformula $\mu X.\psi_X$	$\{(\psi_X, s)\}$
Positions for player $\forall$	**Admissible moves for player $\forall$**
$(\psi_1 \wedge \psi_2, s)$	$\{(\psi_1, s), (\psi_2, s)\}$
$(\Box\psi, s)$	$\{(\psi, t) \mid (s, t) \in R\}$
$(\top, s)$	$\emptyset$
(P, s) and $s \in V(P)$	$\emptyset$
$(\neg P, s)$ and $s \notin V(P)$	$\emptyset$
$(\nu X.\psi_X, s)$	$\{(\psi_X, s)\}$
(X, s) for some subformula $\nu X.\psi_X$	$\{(\psi_X, s)\}$

In an evaluation game $M = (W, R, V)$ with an initial position $(\varphi, s_{\mathsf{in}})$, the two players can produce a sequence of positions obeying the above rules as follows,

$$\rho = (\varphi_0, s_0)(\varphi_1, s_1)(\varphi_2, s_2)\dots \text{ with } (\varphi_0, s_0) = (\varphi, s_{\mathsf{in}})$$

which is called a *play* in the evaluation game $M = (W, R, V)$.

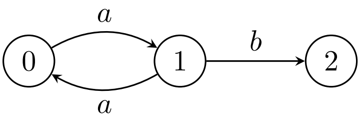
Table: Winning conditions

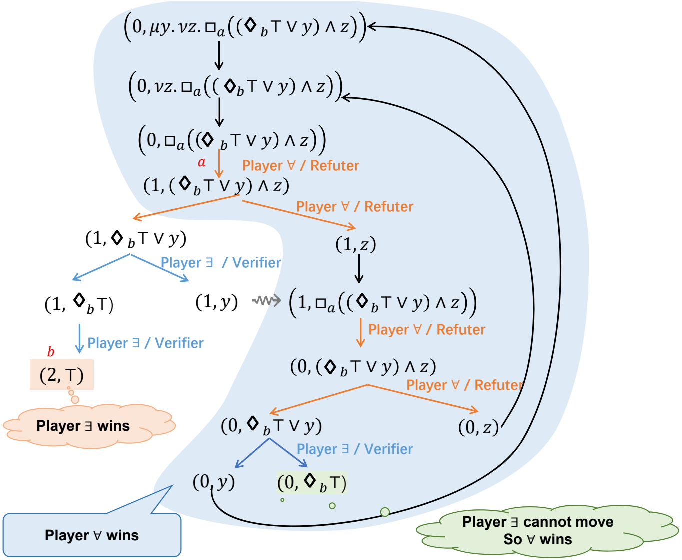
	player $\exists$ wins	player $\forall$ wins
if ρ is finite	player $\forall$ has no admissible move	player $\exists$ has no admissible move
if ρ is infinite	the outermost subformula visited infinite many times is of the form $\nu x.\varphi$	the outermost subformula visited infinite many times is of the form $\mu x.\varphi$

Example

Consider M as follows, where the only atomic proposition is p, and $V(p) = W$ (i.e., p is always true).

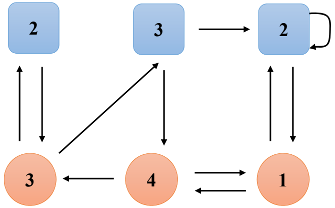
$$\mathcal{E}(M, 0, \mu y.\nu z.\Box_a((\Diamond_b \top \vee y) \wedge z)).$$





Parity games

- A *parity game* $\mathcal{G} = (V_\exists, V_\forall, E, \Omega)$ with index n is played on a colored directed graph, where each node is colored by the priority function $\Omega : V_\exists \cup V_\forall \to \{0, \ldots, n\}$.
- Parity condition: Player $\exists$ ($\forall$) wins an infinite play if the largest priority occurring infinitely often in the play is even (odd).



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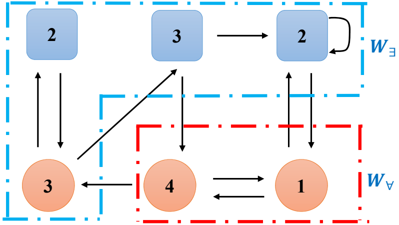
- Parity condition: Player $\exists$ ($\forall$) wins an infinite *play* (produced by their choices) if the largest priority occurring infinitely often in the play is even (odd).



- Winning region: the set of vertices from which that player has a winning strategy.
- Parity games are positionally determined (i.e., from any vertex, either player has a memoryless winning strategy).

Evaluation game and parity game

Theorem

The following are equivalent.

- *Player $\exists$ has a winning strategy in the evaluation game $\mathcal{E}(M, s, \varphi)$.*
- $M, s \models \varphi$.

To show the above theorem, the following facts are usefull.

(1) If $M, s \models \varphi$ then $\exists$ has a **memoryless** winning strategy in $\mathcal{E}(M, s, \varphi)$.

(2) If $M, s \not\models \varphi$ then $\forall$ has a **memoryless** winning strategy in $\mathcal{E}(M, s, \varphi)$.

Theorem (Calude CS, Jain S, Khoussainov B, Li W, Stephan F., 2017)

The parity game can be solved in quasipolynomial time.

Consider the following formulas in a Kripke model M at the root r:

- "always p holds"

$$\nu X.p \wedge \Box X$$

- "eventually p holds"

$$\mu X.p \vee \Diamond X$$

- "p holds infinitely many times"

$$\nu Y.\mu X.(p \wedge \Diamond Y) \vee \Diamond X$$

Question

Does the expressive power become stronger by increasing the number of the fixpoints?

To measure the complexity of such formulas,

- alternation hierarchy, classifying by by the numbers of μ and ν operators that appear alternatively.
- variable hierarchy, classifying the numbers of distinct bind variables.

Alternation hierarchy

Definition

The alternation hierarchy of modal μ-calculus is defined as follows.

- Σ^μ_0, Π^μ_0 : the class of formulas with no fixpoint operators
- Σ^μ_{n+1} : containing $\Sigma^\mu_n \cup \Pi^\mu_n$ and closed under the following operations
  - (i) if $\varphi_1, \varphi_2 \in \Sigma^\mu_{n+1}$, then $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\Box_R \varphi_1$, $\Diamond_R \varphi_1 \in \Sigma^\mu_{n+1}$,
  - (ii) if $\varphi \in \Sigma^\mu_{n+1}$, then $\mu Z.\varphi \in \Sigma^\mu_{n+1}$, and
  - (iii) if $\varphi(X)$, $\psi \in \Sigma^\mu_{n+1}$ and ψ a closed formula (namely, no free variables), then $\varphi(X \backslash \psi) \in \Sigma^\mu_{n+1}$.
- dually for Π^μ_{n+1}
- $\Delta^\mu_n := \Sigma^\mu_n \bigcap \Pi^\mu_n$

Example. $\nu Y.\Diamond Y \wedge \mu Z.p \vee \Diamond Z$ is in Δ^μ_2.
$\mu X.\nu Y.\Diamond Y \wedge \mu Z.\Diamond(X \vee Z)$ is in Σ^μ_3, but not Π^μ_3, since there are no closed subformulas.

Question

Does the alternation hierarchy for modal μ-calculus collapse?

No

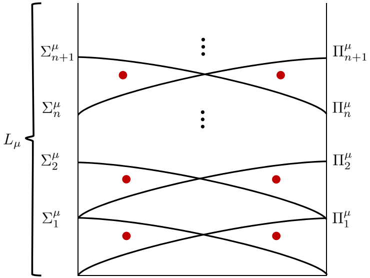
(1) Bradfield's proof using the strictness results arithmetic μ-calculus

(2) Lenzi's Σ_n^μ and Π_n^μ formula on n-ary trees (1998).

(3) Arnold's automata-theoretic method to show the strictness over binary trees (1999).

Subsequently, Walukiewicz pointed out the strict formulas in fact express the winning positions of parity games.

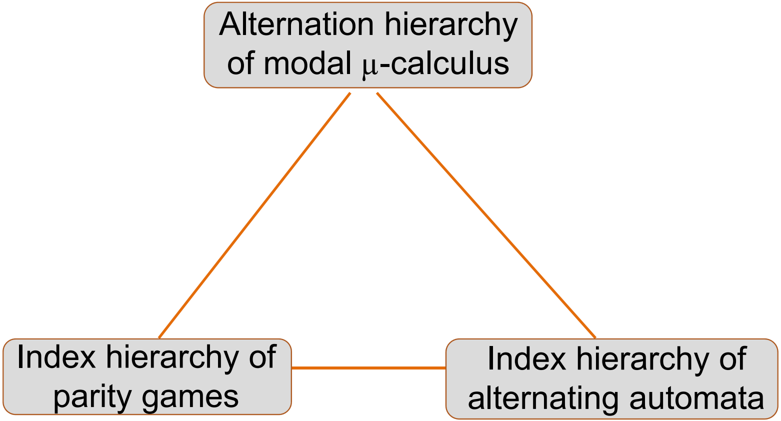
The alternation hierarchy of modal μ-calculus is strict



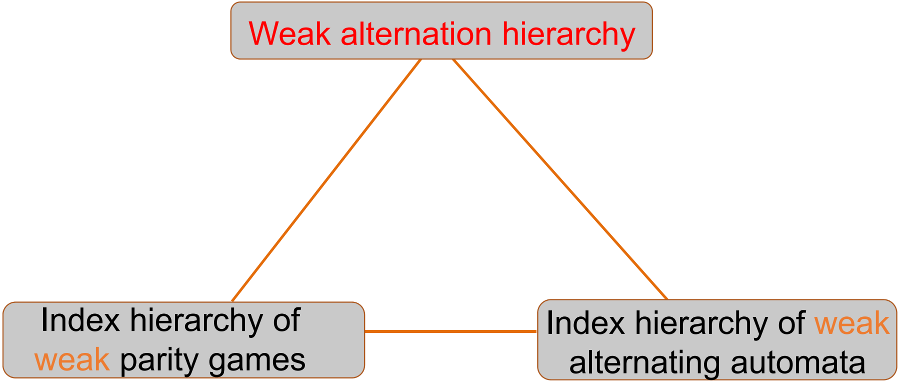
Witness of strictness:

$$\varphi_n = \mu\nu X_n. \cdots \nu X_0. \left(\Big(\bigvee_{0 \leq i \leq n} p \wedge p_i \wedge \Diamond X_i \Big) \vee \Big(\bigvee_{0 \leq i \leq n} \neg p \wedge p_i \wedge \Box X_i \Big) \right)$$

where p denotes the position of player $\exists$, p_i the color of i and $\eta = \nu$ if n is even and $\eta = \mu$ if n is odd.


Alternation hierarchy of modal μ-calculus
Index hierarchy of parity games
Index hierarchy of alternating automata

Focus of this study



Definition (Weak alternation hierarchy of L_μ)

The weak alternation hierarchy of modal μ-calculus is defined as follows.

- $\Sigma_0^{W\mu} = \Sigma_0^\mu$, $\Pi_0^{W\mu} = \Pi_0^\mu$: the class of formulas with no fixpoint operators
- $\Sigma_{n+1}^{\mathrm{W}\mu}$: is the least class of formulas containing $\Sigma_n^{\mathrm{W}\mu} \cup \Pi_n^{\mathrm{W}\mu}$ and closed under the operations $\vee, \wedge, \Box, \Diamond$ and the *substitution*: for a $\varphi(X) \in \Sigma_1^\mu$ and a closed $\psi \in \Sigma_{n+1}^{\mathrm{W}\mu}$, $\varphi(X \backslash \psi) \in \Sigma_{n+1}^{\mathrm{W}\mu}$.
- $\Pi_{n+1}^{\mathrm{W}\mu}$: is the least class of formulas containing $\Sigma_n^{\mathrm{W}\mu} \cup \Pi_n^{\mathrm{W}\mu}$ and closed under the operations $\vee, \wedge, \Box, \Diamond$ and the *substitution*: for a $\varphi(X) \in \Pi_1^\mu$ and a closed $\psi \in \Sigma_{n+1}^{\mathrm{W}\mu}$, $\varphi(X \backslash \psi) \in \Pi_{n+1}^{\mathrm{W}\mu}$.

For $n > 1$, $\Sigma_n^{\mathrm{W}\mu}$ / $\Pi_n^{\mathrm{W}\mu}$ is not closed under μX / νX.
Example. $\nu X.\Box \nu Z.((\mu Y.\Diamond Y) \wedge \Box X) \vee Z$ is in $\Pi_2^{\mathrm{W}\mu}$.

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Example. $\nu X.\Box \nu Z.((\mu Y.\Diamond Y) \wedge \Box X) \vee Z$ is in $\Pi_2^{\mathrm{W}\mu}$.

Theorem (Pacheco-L.-Tanaka)

The weak alternation hierarchy is strict.

Strictness of weak alternation hierarchy witness by weak parity games

- A parity game $\mathcal{G} = (V_\exists, V_\forall, E, \Omega)$ is said to be weak if the coloring function Ω has the following additional property:

$$\text{for all } v, v' \in V_\exists \cup V_\forall, \text{ if } (v, v') \in E, \text{ then } \Omega(v) \geq \Omega(v').$$

- If p denotes a position of player $\exists$'s turn, and p'_i a position with priority i, then

$$\mathcal{W}_0 = \nu X.(p \wedge p'_0 \wedge \Diamond X) \vee (\neg p \wedge p'_0 \wedge \Box X),$$

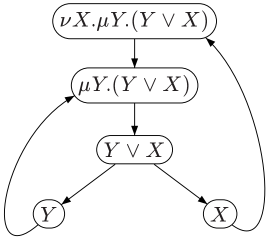
$$\mathcal{W}_{n+1} = \eta X.(p \wedge p'_{n+1} \wedge \Diamond X) \vee (\neg p \wedge p'_{n+1} \wedge \Box X) \vee \mathcal{W}_n \qquad \text{for } n \geq 0.$$

where η is μ if n is even, otherwise ν. Notice that $\mathcal{W}_{2n}$ is a $\Pi^{W\mu}_{2n+1}$-formula, and $\mathcal{W}_{2n+1}$ is a $\Sigma^{W\mu}_{2n+2}$-formula.

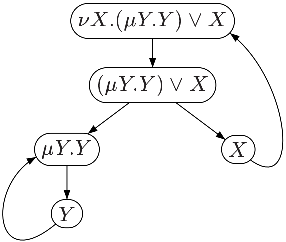
- $\mathcal{W}_n$ indeed describes the winning positions for $\exists$ in a weak parity game with colors up to n.

How far can the weak alteration hierarchy reach?

Observations on syntax tree



(a) alternation



(b) alternation-free

The weak alternation hierarchy captures the alternation-free fragment (i.e., no nested fixed-point operators).

Theorem (Pacheco-L.-Tanaka)

The weak AH syntactically exhausts Δ_2^μ, i.e., every formula in Δ_2^μ belongs to some level $\Sigma_n^{W\mu}$ or $\Pi_n^{W\mu}$ of the weak hierarchy and vice versa.

Proof. To show weak AH $\subseteq \Delta_2^\mu$ By induction on n:

- **Base Case ($n = 0$)**: $\Sigma_0^{W\mu}$ and $\Pi_0^{W\mu}$ contain only fixpoint-free formulas, which are in $\Sigma_1^\mu \cap \Pi_1^\mu \subseteq \Delta_2^\mu$.
- **Inductive Step**: Assume $\Sigma_n^{W\mu}, \Pi_n^{W\mu} \subseteq \Delta_2^\mu$. For $\Sigma_{n+1}^{W\mu}$:
  - Formulas are built from $\Sigma_n^{W\mu} \cup \Pi_n^{W\mu}$ (already in Δ_2^μ by IH).
  - Substitution of $\psi \in \Sigma_{n+1}^{W\mu}$ into $\varphi(X) \in \Sigma_1^\mu$ preserves Δ_2^μ.

To show $\Delta_2^\mu \subseteq$ weak AH Every Δ_2^μ formula ξ can be constructed via:

- Decomposing ξ into Σ_1^μ or Π_1^μ subformulas.
- Using the weak hierarchy's substitution closure to inductively build ξ in some $\Sigma_n^{W\mu}$ or $\Pi_n^{W\mu}$.

Theorem (Pacheco-L.-Tanaka)

On infinite binary trees, there exist Δ_2^{μ}-definable properties that cannot be expressed by any finite level $\Sigma_n^{\mathrm{W}\mu}$ or $\Pi_n^{\mathrm{W}\mu}$ of the weak AH, but require the transfinite extension $\Sigma_\omega^{\mathrm{W}\mu}$.

Setup: Weak parity games and their formulas

Let $\{\mathcal{W}_n\}_{n\in\mathbb{N}}$ be a family of **weak parity games**, where:

- Each $\mathcal{W}_n$ has priorities $\{0, 1, \ldots, n\}$.
- The winning condition: parity condition + weak

By the strictness of the weak AH:

- The winning region of $\mathcal{W}_n$ is definable by a $\Sigma_{n+1}^{\mathrm{W}\mu}$ formula, but **not** by any $\Sigma_n^{\mathrm{W}\mu}$ or $\Pi_n^{\mathrm{W}\mu}$ formula.

Translation from weak to non-weak parity games

For each weak parity game $\mathcal{W}_n$, we construct a corresponding **non-weak parity game** $\mathcal{W}'_n$ with only two priorities $\{\mathbf{0}, \mathbf{1}\}$, where

- priority $\mathbf{0}$ encodes even priorities in $\mathcal{W}_n$, and $\mathbf{1}$ encodes odd priorities in $\mathcal{W}_n$,
- the winning condition remains parity (smallest priority is $\mathbf{0}$).

The key is:

- The winning regions of $\mathcal{W}'_n$ can be expressed as:

$$\mu X_0.\nu X_1.(p \wedge p'_0 \wedge \Diamond X_0) \vee (p \wedge p'_1 \wedge \Diamond X_1) \vee (\neg p \wedge p'_0 \wedge \Box X_0) \vee (\neg p \wedge p'_1 \wedge \Box X_1),$$

- Since each node has at a unique color, that is $\mathrm{V}(p'_0) \cap \mathrm{V}(p'_1) = \emptyset$, by Bekič Principle, we have

$$\nu X_1.\mu X_0.(p \wedge p'_0 \wedge \Diamond X_0) \vee (p \wedge p'_1 \wedge \Diamond X_1) \vee (\neg p \wedge p'_0 \wedge \Box X_0) \vee (\neg p \wedge p'_1 \wedge \Box X_1).$$

- Thus the winning regions of $\mathcal{W}'_n$ can be captured by a Δ_2^μ formula.

Constructing Δ_2^μ Property

- Define a Δ_2^μ property φ that **describes the winning regions of all** $\mathcal{W}'_n$:
  - ψ_n holds at a node if there exists some n s.t. the node is in the winning region of $\mathcal{W}'_n$.
  - Since each $\mathcal{W}'_n$ is Δ_2^μ-definable, and Δ_2^μ is closed under countable disjunction (for properties on trees), φ as a disjunction of all such ψ_n is also Δ_2^μ.
- φ escapes all finite levels of the weak AH
- φ belongs to $\Sigma_\omega^{\mathrm{W}\mu}$
  φ can be constructed as a **limit**:
  - For each n, the winning region of $\mathcal{W}_n$ is $\Sigma_{n+1}^{\mathrm{W}\mu}$-definable.
  - The union $\bigcup_{n\in\mathbb{N}} \Sigma_n^{\mathrm{W}\mu}$ gives $\Sigma_\omega^{\mathrm{W}\mu}$.

□

Relation to the variable hierarchy

For any n, $L_\mu[n]$ denotes the set of modal μ formulas that have at most n distinct bound variables, and likewise for $\Sigma_i^\mu[n]$, $\Pi_i^\mu[n]$ for all level i and the weak AH.

Example

The following formula φ_1 is purely a one-variable formula $(\Pi_2^\mu[1])$. For readability, it may be rewritten as φ_2, a one-variable formula in a broad sense.

- $\varphi_1 = \nu X.\Box(\mu X.\Diamond X) \vee X$.
- $\varphi_2 = \nu X.\Box(\mu Y.\Diamond Y) \vee X$.

And, the following formula φ_3 is a weak modal μ-formula (in fact $\Pi_2^{\mathrm{W}\mu}$), but not one-variable.

- $\varphi_3 = \nu X.\Box\nu Z.((\mu Y.\Diamond Y) \wedge \Box X) \vee Z$.

Applying to variable hierarchy $L_\mu[n]$

Theorem (Pacheco-L.-Tanaka)

The AH of $L_\mu[1]$ (the one-variable fragment of modal μ-calculus) is strict, which is in fact witness by the weak parity games.

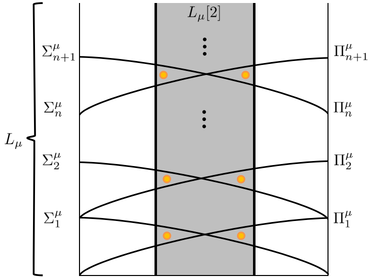
Let p denote a position of player $\exists$'s turn, and p'_i a position with priority i.

$$\mathcal{W}_0 = \nu X.(p \wedge p'_0 \wedge \Diamond X) \vee (\neg p \wedge p'_0 \wedge \Box X),$$

$$\mathcal{W}_{n+1} = \eta X.(p \wedge p'_{n+1} \wedge \Diamond X) \vee (\neg p \wedge p'_{n+1} \wedge \Box X) \vee \mathcal{W}_n \qquad \text{for } n \geq 0.$$

where η is μ if n is even, otherwise ν. Notice that $\mathcal{W}_{2n}$ is a $\Pi^\mu_{2n+1}[1]$-formula, and $\mathcal{W}_{2n+1}$ is a $\Sigma^\mu_{2n+2}[1]$-formula.

Recall that



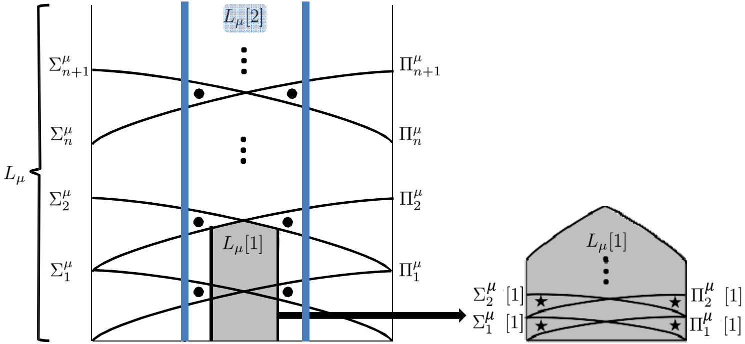
Theorem (Berwanger, 2003)

The AH of $L_\mu[2]$ is strict and not contained in any finite level of the full logic.

Theorem (Berwanger, Grädel and Lenzi, 2007)

For any n, there exists formula $\phi \in L_\mu[n]$ which is not equivalent to any $L_\mu[n-1]$ formula.

One-variable AH in the modal μ-calculus



Future work

- extending the notion of *weak* to study Δ_n^μ $(n > 2)$, the ambiguous class of L_μ.
- applications in studying the collapsing phenomenon when we restrict the Kripke models to some special class.

Class of transition systems	Alternation hierarchy of modal μ-calculus	References
$\mathbb{T}^{\mathsf{rp}}$	strict	Brad96,Brad98a
$\mathbb{T}^{n\text{-}tree}$	strict	Lenzi96
$\mathbb{T}^{2\text{-}tree}$	strict	Arnold99,Brad99a
$\mathbb{T}^{\mathsf{R}}$	strict	AF09
$\mathbb{T}^{\mathsf{RS}}$	strict	DAL12
$\mathbb{T}^{\mathsf{fda}}$	collapse to AFMC	Mateescu
$\mathbb{T}^{t}$	collapse to AFMC	AF09,DAL10,DO09
$\mathbb{T}^{t'}$	collapse to AFMC	GKM14
$\mathbb{T}^{\mathsf{tud}}$	collapse to ML	AF09,DO09
$\mathbb{T}^{\mathbf{REG}_\omega}$	collapse to AFMC	Roope
$\mathbb{T}^{\mathbf{VPL}_\omega}$	collapse to AFMC	GKM14

AFMC: alternation free fragment of L_μ(no nested μ and ν); ML: modal logic.

$\mathbb{T}^{rp}$: the class of recursive presentive transition systems

$\mathbb{T}^{n\text{-}tree}$: the class of n-ary trees

$\mathbb{T}^{2\text{-}tree}$: the class of binary trees

$\mathbb{T}^{R}$: the class of reflexive transition sytsems

$\mathbb{T}^{RS}$: the class of reflexive and symmetric transition systems

$\mathbb{T}^{fda}$: the class of finite directed acyclic transition sytsems

$\mathbb{T}^{t}$: the class of transitive transition sytsems

$\mathbb{T}^{t'}$: $\mathbb{T}^{t}$ with feedback vertex sets of a bounded size

$\mathbb{T}^{tud}$: the class of transitive and undirected graphs

$\mathbb{T}^{\mathbf{REG}_\omega}$: the class of ω-regular languages, and

$\mathbb{T}^{\mathbf{VPL}_\omega}$: the class of visibly pushdown ω-languages.

Reference

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Thank you for your attention!

Example

The negation of the formula $\nu X.p \land \Box X$ expressing "always p holds" is

$$\begin{aligned} &\neg(\nu X.p \land \Box X) \\ =&\mu X.\neg(p \land \Box \neg X) \\ =&\mu X.\neg p \lor \Diamond X \end{aligned}$$

which means "eventually $\neg p$ holds"

Note that $\mu X.\Diamond X$ is false. The approximation process is as follows:

$$\begin{aligned}\mu^0 &= \emptyset \\ \mu^1 &= [\![\Diamond X]\!]^{M[X:=\mu^0]} = \{v \in S : \; \exists w, (v,w) \in E \wedge w \in [\![X]\!]^{M[X:=\emptyset]}\} \\ &= \{v \in S : \; \exists w, (v,w) \in E \wedge w \in \emptyset\} = \emptyset\end{aligned}$$

The approximation process of $\nu X.\Diamond X$ is as follows:

$$\begin{aligned}\nu^0 &= S \\ \nu^1 &= [\![\Diamond X]\!]^{M[X:=\nu^0]} = \{v \in S : \; \exists w, (v,w) \in E \wedge w \in [\![X]\!]^{M[X:=S]}\} \\ &= \{v \in S : \; \exists w, (v,w) \in E \wedge w \in S\} = S\end{aligned}$$

- For the common syntax trees of formulas with distinct fixpoint variables, every fixpoint variable has a unique binding definition, that is, any leaf of an occurrence of a fixpoint variable Z links to its unique binding definition $\mu Z.\psi$ or $\nu Z.\psi$.
- But when the formulas can be renamed by a single variable, we need brackets to restrict the operator precedence. A leaf of an occurrence of the fixpoint variable links to the nearest fixpoint formula in the form of $\mu Z.(\ldots Z \ldots)$ or $\nu Z.(\ldots Z \ldots)$

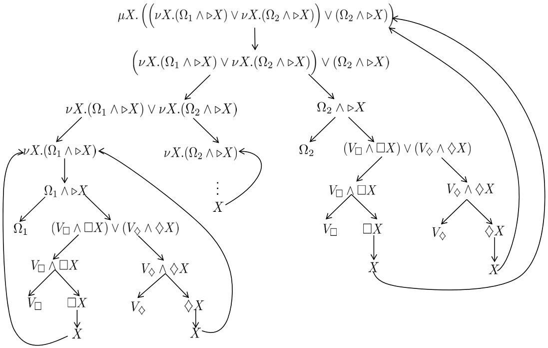


Figure: The syntax tree of φ_2

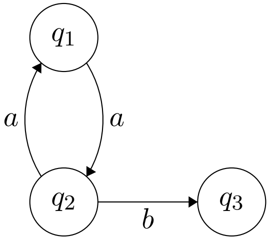
Parity games

- We can think the evaluation game of a (weak) modal μ-formula as a (weak) parity game.
- Given a pointed transition systems $(\mathbb{S}, s_0)$ and a (weak) modal μ-formula φ, we can define a (weak) parity game $\mathcal{G}$ on a tree, which is equivalent to the evaluation game $\mathcal{E}$ of $(\mathbb{S}, s_0) \models \varphi$.
- The arena of $\mathcal{G}$ is defined to be a tree constructed as follows:
  1. each node ρ is a partial play (*i.e.*, a finite sequence of admissible moves) of the evaluation game $\mathcal{E}$; the ownership of each node is inherited from the evaluation game,
  2. the relation of the arena is inherited from the admissible moves in the evaluation game $\mathcal{E}$.

  The coloring function Ω of game $\mathcal{G}$ for a (weak) modal μ-formula φ is defined by cases mainly on the last element of a partial play ρ in $\mathcal{G}$.

Example

Let $\mathcal{K} = (S, (E_\ell)_{\ell \in \{a,b\}}, V)$ be a Kripke structure as follows, with $V(p) = \{q_3\}$ and a interpretation function $\mathcal{V}$.



Example (Continued)

(1) We will first give the semantics of $\varphi_1 = \nu X.\Box_a(\Diamond_b p \vee X)$.

$$\nu^0 = S$$

$$\nu^1 = [\![\Box_a(\underbrace{\Diamond_b p}_{\{q_2\}} \vee X)]\!]_{\mathcal{V}[X\backslash\nu^0]} = \Box_a(\{q_2\} \cup S) = \{q_1, q_2, q_3\}$$

$$\underbrace{= \nu^0}_{\text{fixpoint}}$$

Example (Continued)

(2) Next we give the semantics of $\varphi_2 = \mu X.\Box_a(\Diamond_b p \lor X)$.

$$\mu^0 = \emptyset$$

$$\mu^1 = [\![\Box_a(\underbrace{\Diamond_b p}_{\{q_2\}} \lor X)]\!]_{\mathcal{V}[X \backslash \mu^0]} = \Box_a(\{q_2\} \cup \emptyset) = \{q_1, q_3\}$$

$$\mu^2 = [\![\Box_a(\Diamond_b p \lor X)]\!]_{\mathcal{V}[X \backslash \mu^1]} = \Box_a(\{q_2\} \cup \{q_1, q_3\}) = \{q_1, q_2, q_3\}$$

$$\mu^3 = [\![\Box_a(\Diamond_b p \lor X)]\!]_{\mathcal{V}[X \backslash \mu^2]} = \Box_a(\{q_2\} \cup \{q_1, q_2, q_3\}) = \{q_1, q_2, q_3\} = \mu^2$$

Example (Continued)

(3) On the other hand, the semantics of $\varphi_2 = \nu Z.\mu X.\Box_a\Big((\Diamond_b p \wedge Z) \vee X\Big)$ with respect to $\mathcal{K}$ can be computed as follows.

$\bullet \nu^0 = S = \{q_1, q_2, q_3\}$

$$\mu^{0,0} = \emptyset$$

$$\mu^{0,1} = [\![\Box_a\Big((\Diamond_b p \wedge Z) \vee X\Big)]\!]_{\mathcal{V}[X\backslash\mu^{0,0}]} = \Box_a\Big((\{q_2\} \wedge \{q_1, q_2, q_3\}) \vee \emptyset\Big) = \{q_1, q_3\}$$

$$\mu^{0,2} = [\![\Box_a\Big((\Diamond_b p \cap Z) \cup X\Big)]\!]_{\mathcal{V}[X\backslash\mu^{0,1}]} = \Box_a\Big((\{q_2\} \cap \{q_1, q_2, q_3\}) \cup \{q_1, q_3\}\Big)$$

$$= \{q_1, q_2, q_3\}$$

$$\mu^{0,3} = [\![\Box_a\Big((\Diamond_b p \wedge Z) \vee X\Big)]\!]_{\mathcal{V}[X\backslash\mu^{0,2}]} = \{q_1, q_2, q_3\} = \mu^{0,2}$$

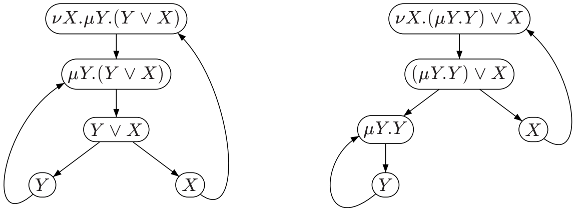
$\bullet \nu^1 = \mu^2 = \{q_1, q_2, q_3\} = \nu^0$

Example (Continued)

- φ_1, φ_2 and φ_3 are semantically equivalent over the Kripke structure $\mathcal{K}$, in the sense that φ_1, φ_2 and φ_3 define the same set of vertices over $\mathcal{K}$.
- The equivalence of φ_1 and φ_2 shows that the semantics of the least and greatest operator makes no difference when the transition system contain no infinite paths.
- The equivalence of φ_3 and φ_2 shows that a syntactically complex formula may be as expressive as some simple formula over a certain transition system.

Another view of alternation free: syntax tree

An L_μ-formula is called alternation-free if no ν-variable occurs free in the scope of a μ-operator, and *vice versa*.



(a) alternation (b) alternation-free

Fig. 2. Alternation $(\nu X.(\mu Y.Y \vee X))$ vs. alternation-free $(\nu X.(\mu Y.Y) \vee X)$

In term of syntax tree, φ is alternation free iff its syntax tree contains no cycle of a μ-variable and a ν-variable. Figure (2a) has a cycle of both X and Y. Figure (2b) has two maximal strongly connected component, one on X and the other on Y.

Another view of alternation free: syntax tree

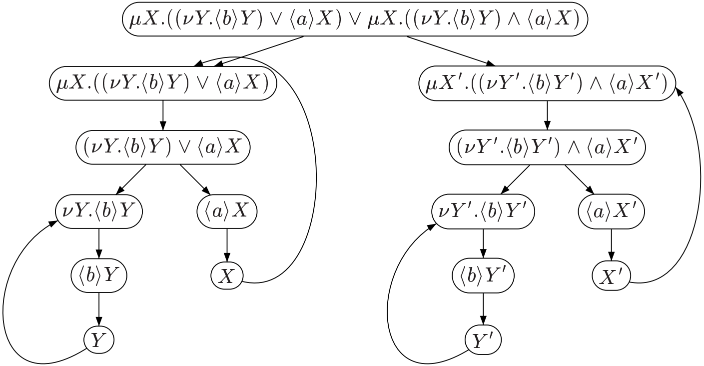


Fig. 1. The graph for $\mu X.((\nu Y.\langle b\rangle Y) \vee \langle a\rangle X) \vee \mu X'.((\nu Y'.\langle b\rangle Y') \wedge \langle a\rangle X')$.

Source: Local parallel model checking for the alternation free μ-caclsulus, technical report, 2002...

Given n, Berwanger (2003) introduced the following formulas for all $i = 1, \ldots, n$.

- for all odd $i \leq n$,

$$\varphi_i^n(X) := \mu Z. \left((\Omega_i \wedge \triangleright Z) \vee \left(\bigvee_{j=1}^{i-1} \Omega_j \wedge X \right) \vee \left(\bigvee_{j=i+1}^{n} \Omega_j \wedge \varphi_{i+1}^n(Z) \right) \right),$$

- for all even $i \leq n$,

$$\varphi_i^n(Z) := \nu X. \left((\Omega_i \wedge \triangleright X) \vee \left(\bigvee_{j=1}^{i-1} \Omega_j \wedge Z \right) \vee \left(\bigvee_{j=i+1}^{n} \Omega_j \wedge \varphi_{i+1}^n(X) \right) \right).$$

where

$$\triangleright X := (V_\Diamond \wedge \Diamond X) \vee (V_\Box \wedge \Box X).$$

$$\begin{cases} \varphi^n_n(X) = \mu Z. \left((\Omega_n \wedge \triangleright Z) \vee \left(\bigvee_{j=1}^{n-1} \Omega_j \wedge X \right) \right) \in \Sigma^{S\mu}_1[1] \\ \vdots \\ \varphi^n_{i+1}(Z) = \nu X. \left((\Omega_{i+1} \wedge \triangleright X) \vee \underbrace{\left(\bigvee_{j=1}^{i} \Omega_j \wedge Z \right)}_{Z \text{ is a free varibale in } \varphi^n_{i+1}} \vee \left(\bigvee_{j=i+2}^{n} \Omega_j \wedge \varphi^n_{i+2}(X) \right) \right) \in \Pi^{S\mu}_{n-i}[2] \\ \varphi^n_i(X) = \mu Z. \left((\Omega_i \wedge \triangleright Z) \vee \left(\bigvee_{j=1}^{i-1} \Omega_j \wedge X \right) \vee \underbrace{\left(\bigvee_{j=i+1}^{n} \Omega_j \wedge \varphi^n_{i+1}(Z) \right)}_{Z \text{ is a bounded variable in } \varphi^n_i} \right) \in \Sigma^{S\mu}_{n-i+1}[2] \\ \vdots \\ \widehat{W}^n_{[2]} = \varphi^n_1 = \underbrace{\mu Z. \left((\Omega_i \wedge \triangleright Z) \vee \left(\bigvee_{j=i+1}^{n} \Omega_j \wedge \varphi_{i+1}(Z) \right) \right)}_{\text{No free variable in } \varphi^n_1} \in \Sigma^{S\mu}_n[2] \end{cases}$$

$L_\mu[2] = L_\mu$?

Formulas in $L_\mu[2]$ can express properties in arbitrary level of alternation hierarchy of L_μ. Then it is natural to ask whether $L_\mu[2] = L_\mu$ or not.

- It is negatively answered by showing the strictness of variable hierarchy.

Theorem (Berwanger, Grädel and Lenzi, 2007)

For any n, there exists formula $\phi \in L_\mu[n]$ which is not equivalent to any formula in $L_\mu[n-1]$.

Question

How is the one-variable fragment of L_μ, namely $L_\mu[1]$?

$L_\mu[1]$ consists of formulas each of which only contains one fixpoint variable.
We can define the simple alternation hierarchy of $L_\mu[1]$ by modifying the definition of simple alternation hierarchy for L_μ, via level-by-level restricting the formulas with only one fixpoint variable, for instance, $\Sigma_n^{S\mu}[1] = \Sigma_n^{S\mu} \bigcap L_\mu[1]$.
We first note that one-variable fragment of modal μ-calculus is contained in the whole weak alternation hierarchy. By definition, it is obvious that the relation

$$\underbrace{\bigcup_{n<\omega} \Sigma_n^{S\mu}[1]}_{\text{Simple altern. hierar. of } L_\mu[1]} \qquad \subseteq \Delta_2^{N\mu}$$

We will show that $L_\mu[1]$ is enough to express the winning region of weak parity games. A *weak game* can be given as a rooted structure $\mathcal{G}, v_0$ with $\mathcal{G} = (V, V_\Diamond, V_\Box, E, \Omega, n)$. Player I wins with a play x if the priority sequence of x is nonincreasing.
Given n, we consider the following formulas for $i = 1, \dots, n$,

$$\begin{cases} \varphi_i := \nu X.\Big(\varphi_{i-1} \vee (\Omega_i \wedge \triangleright X)\Big), & \text{if } i \text{ is odd} \\ \varphi_i := \mu X.\Big(\big(\varphi_{i-1} \vee \nu X.(\Omega_i \wedge \triangleright X)\big) \vee (\Omega_i \wedge \triangleright X)\Big), & \text{if } i \text{ is even} \end{cases} \qquad (\clubsuit)$$

where

$$\triangleright X := (V_\Diamond \wedge \Diamond X) \vee (V_\Box \wedge \Box X).$$

The formula φ_n describes that player $\Diamond$ has a winning strategy in a weak parity game with priority n.

Example

For n=2, $\varphi_1 = \nu X.(\Omega_1 \wedge \triangleright X)$,
$\varphi_2 = \mu X.\Big(\big(\nu X.(\Omega_1 \wedge \triangleright X) \vee \nu X.(\Omega_2 \wedge \triangleright X)\big) \vee (\Omega_2 \wedge \triangleright X)\Big)$ note that $\varphi_2 \in \Sigma_2^{S\mu}[1]$.

$$\mu X.p \vee (q \wedge \Diamond X),$$

means that there is a path in which p eventually holds and q holds before p holds. Similarly

$$\varphi_2 = \mu X.(\underbrace{\Big(\nu X.(\Omega_1 \wedge \triangleright X) \vee \nu X.(\Omega_2 \wedge \triangleright X)\Big)}_{\text{Property } \varrho} \vee (\Omega_2 \wedge \triangleright X)),$$

means that there is a path where property ϱ eventually holds and Ω_2 is true before ϱ holds.

Theorem

The simple alternation hierarchy of $L_\mu[1]$ is strict over finitely branching transition systems. Moreover, the simple alternation hierarchy of $L_\mu[1]$ exhausts the weak alternation hierarchy.

I: Simple alternation hierarchy

Counting simply syntactic alternation of μ and ν results in the following definition. The superscript S means simple or syntactic.

Definition

- $\Sigma_0^{S\mu}, \Pi_0^{S\mu}$: the class of formulas with no fixpoint operators
- $\Sigma_{n+1}^{S\mu}$: containing $\Sigma_n^{S\mu} \cup \Pi_n^{S\mu}$ and closed under the following operations
  - (i) if $\varphi_1, \varphi_2 \in \Sigma_{n+1}^{S\mu}$, then $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\Box\varphi_1$, $\Diamond\varphi_1 \in \Sigma_{n+1}^{S\mu}$,
  - (ii) if $\varphi \in \Sigma_{n+1}^{S\mu}$, then $\mu X.\varphi \in \Sigma_{n+1}^{S\mu}$
- dually for $\Pi_{n+1}^{S\mu}$

A formula is strict $\Sigma_{n+1}^{S\mu}$ if it is not $\Sigma_n^{S\mu} \cup \Pi_n^{S\mu}$.

Example: $\mu X.(p \vee \mu Y.(X \vee \Diamond Y)) \in \Sigma_1^{S\mu}$.

- Notice that simple alternation does not capture the complexity of feedbacks between fixpoints.
- For instance, it does not distinguish the following two formulas:
  - $\Phi_1 = \nu Y.\mu X.(p \wedge \Diamond Y) \wedge \Diamond X$
  - $\Phi_2 = \nu Y.\Diamond Y \wedge (\mu Z.p \vee \Diamond Z)$
- Both Φ_1 and Φ_2 are strict $\Pi_2^{S\mu}$.
- But the former is more complex: inner fixpoint depends on the outer one. Observe that in Φ_2, the subformula $\mu Z.p \vee \Diamond_b Z$ is a closed formula (namely, no free variable).

II: Emerson-Lei alternation hierarchy

Definition

The Emerson-Lei alternation hierarchy of modal μ-calculus is defined as follows.

- $\Sigma_0^{EL\mu}, \Pi_0^{EL\mu}$: the class of formulas with no fixpoint operators
- $\Sigma_{n+1}^{EL\mu}$: containing $\Sigma_n^{EL\mu} \cup \Pi_n^{EL\mu}$ and closed under the following operations
  - (i) if $\varphi_1, \varphi_2 \in \Sigma_{n+1}^{EL\mu}$, then $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\Box_R \varphi_1$, $\Diamond_R \varphi_1 \in \Sigma_{n+1}^{EL\mu}$,
  - (ii) if $\varphi \in \Sigma_{n+1}^{EL\mu}$, then $\mu Z.\varphi \in \Sigma_{n+1}^{EL\mu}$, and
  - (iii) if $\varphi(X), \psi \in \Sigma_{n+1}^{EL\mu}$ and ψ a closed formula (namely, no free variables), then $\varphi(X \backslash \psi) \in \Sigma_{n+1}^{EL\mu}$.
- dually for $\Pi_{n+1}^{EL\mu}$

Example. $\nu Y.\Diamond Y \wedge \mu Z.p \vee \Diamond Z$ is Delta_2^{μ}

III: Niwiński alternation hierarchy

Definition

- $\Sigma_0^{N\mu}, \Pi_0^{N\mu}$: the class of formulas with no fixpoint operators
- $\Sigma_{n+1}^{N\mu}$: containing $\Sigma_n^{N\mu} \cup \Pi_n^{N\mu}$ and closed under the following operations
  - (i) if $\varphi_1, \varphi_2 \in \Sigma_{n+1}^{N\mu}$, then $\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2, \Box\varphi_1, \Diamond\varphi_1 \in \Sigma_{n+1}^{N\mu}$,
  - (ii) if $\varphi \in \Sigma_{n+1}^{N\mu}$, then $\mu Z.\varphi \in \Sigma_{n+1}^{N\mu}$, and
  - (iii) if $\varphi(X), \psi \in \Sigma_{n+1}^{N\mu}$ and no free variable of ψ is captured by φ, then $\varphi(\psi) \in \Sigma_{n+1}^{N\mu}$.
- dually for $\Pi_{n+1}^{N\mu}$

The Niwiński alternation depth of a formula ϕ is the least n such that $\phi \in \Sigma_n^{N\mu} \bigcap \Pi_n^{N\mu}$.

Fact: $\Sigma_n^{S\mu} \subsetneq \Sigma_n^{EL\mu} \subsetneq \Sigma_n^{N\mu}$ for $n \geq 2$,
$\Sigma_1^{S\mu} = \Sigma_1^{EL\mu} = \Sigma_1^{N\mu}$.

Example

$$\Phi_1 = \nu Y.\mu X.(p \wedge \Diamond Y) \wedge \Diamond X$$

$$\Phi_2 = \nu Y.\Diamond Y \wedge \mu Z.p \vee \Diamond Z$$

- Φ_1 and Φ_2 are in $\Pi_2^{N\mu}$.
- Φ_2 is also in $\Sigma_2^{N\mu}$. Thus Φ_2 is in $\Delta_2^{N\mu}$ and $\Delta_2^{EL\mu}$.

Example

$\Phi_3 = \mu X.\nu Y.\Diamond Y \wedge \mu Z.\Diamond(X \vee Z)$

1. Φ_3 is in $\Sigma_3^{S\mu}$, but not $\Pi_3^{S\mu}$.
2. Φ_3 is in $\Sigma_3^{EL\mu}$, but not $\Pi_3^{EL\mu}$, since there are no closed subformulas.
3. But for Niwiński alternation hierarchy, Φ_3 is in $\Sigma_2^{N\mu}$. Because

$$\underbrace{\mu Z.\Diamond(X \vee Z)}_{\in \Sigma_1^{N\mu} \subseteq \Sigma_2^{N\mu}} \qquad \underbrace{\nu Y.\Diamond Y \wedge W}_{\in \Pi_1^{N\mu} \subseteq \Sigma_2^{N\mu}}$$

$$\searrow \qquad \swarrow$$

$$\underbrace{\nu Y.\Diamond Y \wedge \mu Z.\Diamond(X \vee Z)}_{\in \Sigma_2^{N\mu}}$$

$$\downarrow$$

$$\underbrace{\mu X.\nu Y.\Diamond Y \wedge \mu Z.\Diamond(X \vee Z)}_{\in \Sigma_2^{N\mu}}$$