Minimality of difference-differential equations

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Joint work with Thomas Scanlon

When can a function satisfy both a differential equation and a difference equation?

As a rule of thumb, if a function satisfies a nontrivial difference equation, it is difficult for this function to satisfy a nontrivial differential equation.

For example, the fact that the Γ function satisfies the difference equation $\Gamma(t+1) = t\Gamma(t)$ is used in the proof that it is hypertranscendental, that is, satisfies no nontrivial algebraic differential equation over $\mathbb{C}(t)$.

Some functions break this rule.

- $y := f(t) = t$ satisfies $y' = 1$ and $y(t+1) = y(t) + 1$.
- $y := f(t) = \exp(t)$ satisfies $y' = y$ and $y(t+1) = ey(t)$.

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Hypertranscendence and linear difference equations

Adamczewski-Dreyfus-Hardouin prove (JAMS, 2021) in various precise senses that solutions to linear difference equations are automatically hypertranscendental, *i.e.*, satisfy no nontrivial algebraic differential equations.

For one precise sense, let

- $K = \mathbb{C}(x)$ be the field of rational functions in one variable over the complex numbers,
- F be the field of meromorphic functions, and
- F_0 be any subfield of F closed under the difference operator $\sigma : f(x) \mapsto f(x+1)$ with $F_0^\sigma = \{f \in F_0 : \sigma(f) = f\} = \mathbb{C}$ for which $F_0 \cap \mathbb{C}(x, \{\exp(\lambda x) : \lambda \in \mathbb{C}\}) = \mathbb{C}(x)$.

Then, if $f \in F_0$ satisfies a nontrivial linear difference equation with coefficients from K, either $f \in K$ or f is hypertranscendental.

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Then, if $f \in F_0$ satisfies a nontrivial linear difference equation with coefficients from K, either $f \in K$ or f is hypertranscendental.

Dependence on the field of solutions

The exceptional difference-differential field generated by rational functions and exponentials is necessary. Can we change K, the difference-differential field over which (linear) difference equations are defined, and F, the difference-differential field in which solutions are taken to obtain a similar hypertranscendence result?

In general, no. Algebraically, linear difference equations are compatible with arbitrarily complicated algebraic differential equations.

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Nonlinear variants

For the case already considered of difference equations over $\mathbb{C}(x)$ with solutions in the field of meromorphic functions, it remains a challenge to determine to what extent being a transcendental solution to a nonlinear difference equation forces hypertranscendence.

We consider a complementary problem: if y is a solution of (nonlinear) algebraic differential equations, under what conditions can we obtain algebraicity / difference-transcendence dichotomy for y?

We answer this question in the case that the differential equations define a *strongly minimal set* relative to the theory of differentially closed fields.

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Difference-differential fields

By a difference-differential-field (or σ-δ-field), we mean a field K of characteristic zero equipped with

- a field endomorphism $\sigma : K \to K$ (i.e. $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$) and
- a derivation $\delta : K \to K$ (i.e. $\delta(x+y) = \delta(x) + \delta(y)$) and $\delta(xy) = x\delta(y) + \delta(x)y$

for which δ and σ commute. Denote by (K, σ, δ).

Example: $(\mathbb{C}(x), \frac{d}{dx}, \sigma : f(x) \to f(x+1))$

- Field of differential constants: $C_K := \{x \in K : \delta(x) = 0\}$;
- σ-Fixed field: $K^\sigma := \{x \in K : \sigma(x) = x\}$.

If we require merely that K is a commutative ring, then we say that (K, σ, δ) is a σ-δ-ring.

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Difference-differential polynomial equations and varieties

Given a σ-δ-field K and a tuple $x = (x_1, \ldots, x_n)$ of variables, the σ-δ-ring $K\{x\}_{\sigma,\delta}$ of σ-δ-polynomials in x with coefficients from K is the free σ-δ-ring generated over K by x. Concretely,

$$K\{x\}_{\sigma,\delta} = K[\sigma^i\delta^j(x_\ell) : i, j \in \mathbb{N}; 1 \leq \ell \leq n].$$

- Let (L, σ, δ) be a σ-δ-field extending K and $a = (a_1, \ldots, a_n) \in L^n$. There exists a unique σ-δ-homomorphism over K,

$$\phi_a : K\{x\}_{\sigma,\delta} \longrightarrow L \quad \text{with} \quad \phi_a(\sigma^i\delta^j(x_i)) = \sigma^i\delta^j(a_i) \text{ and } \phi_a|_K = id.$$

  For $f \in Ker(\phi_a)$, a is a solution of f and denote $f(a) = 0$.

- Given $\Sigma \subseteq K\{x\}_{\sigma,\delta}$, $X = \mathbb{V}(\Sigma)$ is the σ-δ-variety defined by Σ; and for a σ-δ-field (L, σ, δ) extending K, write

$$X(L) := \{a \in L^n : f(a) = 0 \text{ for all } f \in \Sigma\}.$$

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The theory of DCF_0 and theory of $DCFA_0$

- The theory of differential fields (of char 0) admits a model companion, the theory DCF_0 of differential closed field of char 0.

  Blum(1968): Axiomatic of differentially closed fields and elimination of quantifiers in the language $\mathscr{L} = \{+, -, \cdot, 0, 1, \delta\}$ of differential rings.

  One gets the following bijection over a differential field (K, δ):

$$\phi : S_n(K) \longrightarrow \{I \subset K\{y_1, \ldots, y_n\} \text{ prime differential ideals}\}$$

- The theory of σ-δ-fields (of char 0) also admits a model companion, $DCFA_0$ of σ-δ closed field of char 0 (Hrushovski, Bustamante2005).

  $DCFA_0$ does not enjoy quantifier elimination but a weak quantifier simplification.

Minimality difference-differential varieties

Let K be a σ-δ-field and $X = \mathbb{V}(\Sigma)$ a σ-δ-variety defined over K. We say that X is minimal relative to DCFA_0 if

- there is some extension L_0 with $X(L_0)$ infinite, but
- for any tower of extensions $K \subseteq L \subseteq M$ and any solution $x \in X(M)$, either x is algebraic in the sense that $x \in X(L^{\mathrm{alg}})$ or x satisfies no new equations in the sense that $\mathbb{I}_L(x) = \mathbb{I}_K(x) \otimes L$, where $\mathbb{I}_L(x) = \{f \in L\{y\}_{\sigma,\delta} : f(x) = 0\}$.

Minimal varieties and minimal types

There are various subtly distinct model theoretic notions of minimality for minimal varieties, minimal definable sets, minimal types.

- A complete type $\text{tp}(v/K)$ is minimal (or U-rank 1) iff $v \notin K^{alg}$ but for every forking extension of $\text{tp}(v/K)$ is algebraic, that is, has only finitely many realizations.
- Our definition of a σ-δ-variety being minimal is equivalent to asking that X defines an infinite definable set and every complete type extending X is minimal.

(Non)-examples of minimality

- If f, $g \in K[x]$ and the equations $\sigma(x) = f(x)$ and $\delta(x) = g(x)$ are consistent, then $X = \mathbb{V}(\sigma(x) - f(x), \delta(x) - g(x))$ is minimal.
- The σ-δ-variety $X = \mathbb{V}(\sigma(x) - e\delta(x), \delta^2 x - 2\delta x + x)$ defined over $K = \mathbb{C}$ is not minimal. If $L = \mathbb{C}(e^t)$ and $M = \mathbb{C}(t, e^t)$, then $a = te^t \in X(M) \setminus X(L^{\text{alg}})$, but it also belongs to the proper σ-δ-variety defined by the additional equation $\delta x - x = e^t$.

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Minimality for differential equations

In the definition of minimality we could work just with σ-equations or just with δ-equations. For example, a δ-variety X over a δ-field K is minimal if

- there is some δ-field extension L with $X(L)$ infinite, but
- for any tower of extensions of δ-fields $K \subseteq L \subseteq M$ and any solution $a \in X(M)$, either a is algebraic in the sense that $a \in X(L^{\mathrm{alg}})$ or a satisfies no new equations in the sense that there is no proper δ-variety $Y \subsetneq X$ defined over L with $a \in Y(M)$.

A strongly minimal set is a set X definable over some base K so that for some extension L, $X(L)$ is infinite, but for any extension M of K for every M-(q.f.)definable subset $Y \subseteq X$, either $Y(M)$ or $X(M) \smallsetminus Y(M)$ is finite.

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Examples of strongly minimal δ-varieties

- For any irreducible polynomial $f(x, y)$ in two variables, the δ-variety $X = \mathbb{V}(f(x, \delta x))$ is strongly minimal.
- By work of Jaoui (ANT, 2022) for $K = \mathbb{C}$ and $f(x, y)$ and $g(x, y)$ sufficiently general polynomials in two variables, the δ-variety $X = \mathbb{V}(\delta x_1 - f(x_1, x_2), \delta x_2 - g(x_1, x_2))$ is strongly minimal.
- The Schwarzian differential equation satisfied by Klein's j-function,

$$x^2(x-1728)^2(2\delta^3 x \delta x - 3(\delta^2 x)^2) + (x^2 - 1968x + 2654208)(\delta x)^4 = 0$$

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δ-varieties as a σ-δ-varieties

If X is a δ-variety, then we may consider the δ-equations defining X as σ-δ-equations so that X may also be regarded a σ-δ-variety.

If X is minimal as a δ-variety, does it remain minimal as a σ-δ-variety?

In general, no. For example, if $X = \mathbb{V}(\delta x)$, then for any σ-δ-field K, $X(K)$ is just the field of δ-constants and every difference field may be realized in this form. Thus, there are many inequivalent difference equations consistent with X.

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(Non)orthogonality

If X and Y are two strongly minimal sets defined over some K, then we say that X and Y are not almost-orthogonal over K, written $X \not\perp_K^a Y$, if there is some strongly minimal set $Z \subseteq X \times Y$ defined over K for which the projections to each of X and Y miss at most finitely many points.

We say that X and Y are nonorthogonal, written $X \not\perp Y$, if there is some extension L so that $X \not\perp_L^a Y$.

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Triviality

A strongly minimal set X defined over K is geometrically trivial if for any tower of extensions $K \subseteq L \subseteq M$ and distinct $v_1, \dots, v_n$ from $X(M)$, $L\langle v_1, \dots, v_n\rangle^{alg} = \cup_{i=1}^{n} L\langle v_i\rangle^{alg}$. That is, if $(v_1, \dots, v_n)$ satisfies some new relations over L not implied by their individual membership in X, then there is some new binary relation.

Examples:

- The differential equation $y' = y^3 - y^2$ defines a geometrically trivial strongly minimal set (Kolchin/Rosenlicht/Shelah, 1974)
- The Painleve equation $P_{II}(\alpha)$: $y'' = 2y^3 + ty + \alpha$ is geometrically trivial strongly minimal for $\alpha \notin 1/2 + \mathbb{Z}$ (Nagloo-Pillay, 2011).
- All generic differential equations are geometrically trivial strongly minimal sets (Develbiss-Freitag, 2021).

Trichotomy

Strongly minimal sets (and even just minimal types) satisfy the Zilber trichotomy relative to the theory of differentially closed fields of characteristic zero.

Theorem (Hrushovski-Sokolović, 1994)

Let X be a strongly minimal set in a differentially closed field $\mathscr{U}$. Then exactly one of the following holds:

(1) (Field-like) X is non-orthogonal to constants $C_{\mathscr{U}}$;

(2) (Group-like) X is non-orthogonal to a very special strongly minimal subgroup of an abelian variety which does not descend to $C_{\mathscr{U}}$;

(3) (trivial) X is geometrically trivial.

Remark. If $\operatorname{ord}(X) := \max\{\operatorname{tr.deg} K\langle a\rangle/K : a \in X\} = 1$, then X satisfies either for (1) or (3); if $\operatorname{ord}(X) > 1$, then X satisfies either for (2) or (3).

General principle of geometrically trivial and strongly minimality differential equations

Theorem. Let X be a strongly minimal and geometrically trivial differential variety with respect to $\mathrm{DCF}_{0,K}$ where K is an algebraically closed σ-δ field. Then X is minimal as a σ-δ-variety with respect to $\mathrm{DCFA}_{0,K}$. Precisely, for any tower of σ-δ-fields $K \subset L \subset M$ and any $a \in X(M)$, either $a \in X(L^{alg})$ or $\mathbb{I}_{\sigma,\delta}(a/L) = \mathbb{I}_{\sigma,\delta}(a/K) \otimes L$.

Totally disintegrated differential equations

Definition. Let X be a geometrically trivial and strongly minimal set defined over K relative to DCF_0. X is said to be totally disingrated if for any $y \in X$, we have

$$acl_X(K, y) = K\langle y\rangle^{alg} \cap X = \{y\}.$$

Theorem. Let X be a totally disingrated set defined by a δ-polynomial equation over an algebraic closed σ-δ-field K. If L is any σ-δ-field extension of K and $a \in X(L)$, then one of the following holds:

(1) $a \in X(K)$;

(2) there exists $n \in \mathbb{N}$, a differential bi-rational map $\phi : X \longrightarrow X^{\sigma^n}$ such that $\sigma^n(a) = \phi(a)$;

(3) a is differencely transcendental, i.e., satisfies no nontrivial σ-equation over K.

Strictly minimal sets

Definition. A strongly minimal set X defined over K relative to DCF_0 is said to be strictly minimal over K if for any definable equivalence relation E on X, all but finitely many of the E-equivalence classes have size one.

- Test for strictly minimality of rational 1-forms on $\mathbb{P}^1$ via residues (Hrushovski-Itai, 2003)

Theorem. Let K be a σ-δ-field of characteristic 0 and $f(y) \in C_K[y]$ be a monic polynomial of degree $n \geq 3$. Assume $\mathbb{V}(y' - f(y))$ is strictly minimal relative to DCF_0. If ξ is a solution of $y' - f(y) = 0$, then ξ is either algebraic over K, or differencely transcendental over K, unless[1] $f^{\sigma^m}(y) = c \cdot f(\frac{y-d}{c}) + d'$ for some $m \geq 1$, $c, d \in K$ with $c^{n-1} = 1$.

[1] Here, $f^{\sigma^m}(y)$ is the polynomial in $K[y]$ obtained from $f(y)$ by acting σ^m to the coefficients of f.

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Computational example: $y' = y^3 + a_2 y^2 + a_1 y + a_0$

Theorem. Let K be a σ-δ-field of characteristic 0 with $a \in C_K$. Let ξ be a transcendental solution of $y' = y^3 + a$. Then we have the following:

(1) If $a \notin \{\sigma^n(a) : n \in \mathbb{Z}_{\geq 1}\}$, then ξ is differencely transcendental over K.

(2) If $\sigma^n(a) = a$ for some $n \in \mathbb{Z}_{\geq 1}$, then either ξ is differencely transcendental over K or ξ satisfies $\sigma^n(y) - y = 0$.

For general monic cubic polynomial $f(y) = y^3 + a_2 y^2 + a_1 y + a_0$ irreducible over $\mathbb{Q}(a_2, a_1, a_0)$ with $a_i \in C_K$, we have:

Theorem. If $f^{\sigma^n}(y) \neq \pm f(\pm y + b) + b'$ for any $n \in \mathbb{Z}_{\geq 1}$ and $b \in K$, then any solution ξ of the differential equation $y' = f(y)$ is either algebraic or differencely transcendental over K.

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Generalizations / questions

The general principle of geometrically trivial strongly minimal sets also holds relative to $\mathrm{DCF}_{0,m}$ and $\mathrm{DCFA}_{0,m}$, the theory of Δ-closed field of char 0 with m commuting derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and the σ-Δ closed field of char 0.

Questions:

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Thanks!