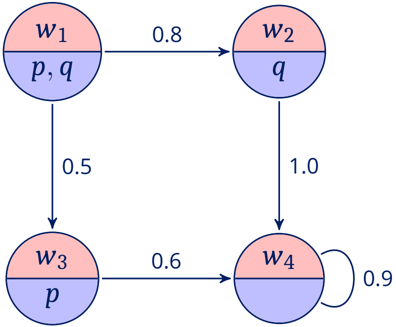
Weighted Modal Logic and Its Applications

Yì N. Wáng
Shandong University
`https://ncml.org.cn/ynw`

Shanghai, 4 August 2025

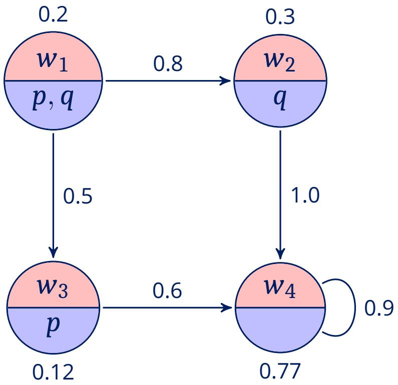
Weighted Epistemic/Doxastic Models

- Weights denote probabilities or degrees of knowledge/belief
  - Enable quantitative analysis

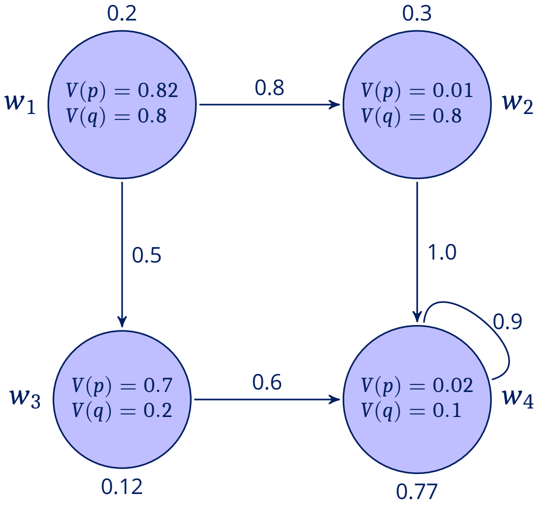


Weighted Epistemic/Doxastic Models

- Weights denote probabilities or degrees of knowledge/belief
  — Enable quantitative analysis
- Assignable to nodes or edges in the model



Weighted Epistemic/Doxastic Models



- Weights denote probabilities or degrees of knowledge/belief
  - Enable quantitative analysis
- Assignable to nodes or edges in the model
  - Sometimes can be integrated into the valuation function

Ways to Incorporate Weights into the Logics

Focus on edge weights

- **Explicitly** used in the logical language
- **Implicitly** incorporated via a capability function in the models

Epistemic Logic with Explicit Weights

Dong, H. & Li, X. & Wáng, Y.N. Weighted Modal Logic in Epistemic and Deontic Contexts. LORI 2021.

Epistemic language: $\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box^r \varphi \qquad (r \in [0,1])$

M, s: a weighted model with a designated point

Satisfaction:
$$
\begin{aligned}
M, s \models p &\iff p \in \nu(s) \\
M, s \models \neg\psi &\iff \text{not } M, s \models \psi \\
M, s \models \psi \to \chi &\iff \text{if } M, s \models \psi \text{ then } M, s \models \chi \\
M, s \models \Box^r \psi &\iff \text{for all } t \in W, \text{ if } E(s,t) \geq r \text{ then } M, t \models \psi
\end{aligned}
$$

- $E(s,t)$: Strength of indiscernibility between states s and t
- $E(s,t) \geq r$: **Agent cannot distinguish s from t with strength r or lower**
- Supports multi-agent extensions

Epistemic Logic with Implicit Weights

Liang X. & Wáng, Y.N. Epistemic Logics over Weighted Graphs. LNGAI 2022.

Epistemic language: $\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box\varphi$

M, s: a weighted model with a designated point
$C \in [0, 1]$ (global) or $C : W \to [0, 1]$ (local)

Satisfaction:

$$\begin{array}{lcl} M, s \models p & \iff & p \in \nu(s) \\ M, s \models \neg\psi & \iff & \text{not } M, s \models \psi \\ M, s \models \psi \to \chi & \iff & \text{if } M, s \models \psi \text{ then } M, s \models \chi \\ M, s \models \Box\psi & \iff & \text{for all } t \in W, \text{ if } E(s, t) \geq C \text{ then } M, t \models \psi \end{array}$$

- $E(s, t)$: Strength of indiscernibility between s and t
- $E(s, t) \geq C$: **Agent cannot discern between s and t with strength C or lower**

Conditions over Weighted Structures

- Can require a similarity metric on weights
  - Congruence implies equality
  - Symmetry
  - Triangularity (optional)
- Weighted adaptations of reflexivity, transitivity, and other relational properties

Liang & Wáng. Characterization of Similarity Metrics in Epistemic Logic. PRICAI 2024.

Different Kinds of Weights

- Numeric weights: represent degree of uncertainty
- Set-based weights: represent skill sets

Different Kinds of Weights

- Numeric weights: represent degree of uncertainty
  - Ordered linearly (e.g., real numbers in $[0, 1]$)
- Set-based weights: represent skill sets
  - $(\wp(S), \subseteq)$ forms a Boolean algebra[1]

[1] At least a distributed lattice if not all subsets of S are included.

Different Kinds of Weights

- Numeric weights: represent degree of uncertainty
  - Ordered linearly (e.g., real numbers in $[0, 1]$)
- Set-based weights: represent skill sets
  - $(\wp(S), \subseteq)$ forms a Boolean algebra[1]
- Generalized weights: structured as a lattice, a partial/pre-order, or totally incomparable (potentially less practical)

[1] At least a distributed lattice if not all subsets of S are included.

Different Kinds of Weights

- Numeric weights: represent degree of uncertainty
  - Ordered linearly (e.g., real numbers in $[0, 1]$)
- Set-based weights: represent skill sets
  - $(\wp(S), \subseteq)$ forms a Boolean algebra[1]
- Generalized weights: structured as a lattice, a partial/pre-order, or totally incomparable (potentially less practical)

- Focus today: classical and fuzzy skill sets

[1] At least a distributed lattice if not all subsets of S are included.

Why Weighted Modal Logics?

Applications: Epistemic, doxastic, temporal, deontic, preferential, probabilistic, etc.

- Expressing leveled uncertainty
- Quantitative modeling: Captures weights as probabilities, costs, rewards, time...
  - E.g., optimize systems by finding the least costly path in weighted transition systems
- Enhanced expressivity
  - Supports diverse concepts and innovative ideas

Generalization: Extends classical modal logic for broader, practical applications.

Previous Work

1. Dong & Li & Wáng. Weighted Modal Logic in Epistemic and Deontic Contexts. LORI 2021.
2. Liang & Wáng. Epistemic Logics over Weighted Graphs. LNGAI 2022.
3. Liang & Wáng. Epistemic Logic via Distance and Similairty. PRICAI 2022.
4. Liang & Wáng. Epistemic Skills: Logical Dynamics of Knowing and Forgetting. GandALF 2024.
5. Liang & Wáng. Field Knowledge as a Dual to Distributed Knowledge: A characterization by weighted modal logic. LNGAI 2024.
6. Liang & Wáng. Characterization of Similarity Metrics in Epistemic Logic. PRICAI 2024.
7. Liang & Wáng. Epistemic Skills: Reasoning about Knowledge and Oblivion. under submission.
8. Liang & Wáng. Weighted Epistemic Logic: Skill Assessment and Rough Set Applications. under submission.

- We focus on logics under various conditions, their axiomatizations and computation complexity
- Implicit weights (2-5, 7), Explicit weights (1, 6)

Multi-Agent Weighted Models over Classical Skill Sets

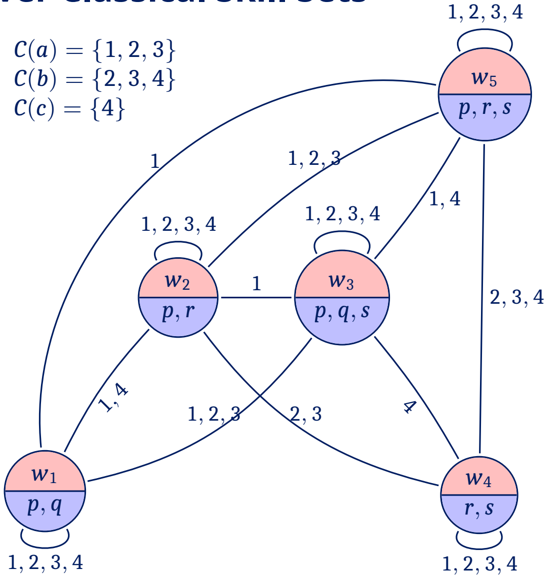
P: atoms
A: agents
S: epistemic skills

A model is a tuple (W, E, C, V):

- W: worlds / states / nodes
- $E : W \times W \to \wp(S)$: edge function
- $C : A \to \wp(S)$: capability function
- $V : W \to \wp(P)$: valuation

$M, s \models \Box_i \varphi$ iff for all $t \in W$,
if $E(s, t) \supseteq C(i)$ then $M, t \models \psi$

$C(a) = \{1, 2, 3\}$
$C(b) = \{2, 3, 4\}$
$C(c) = \{4\}$



Incorporating Group Knowledge

CK, DK, EK and FK

Notions of Group Knowledge

- Individual knowledge: $K_a\varphi$
- Mutual/Everyone's knowledge: $E_G\varphi := \bigwedge_{x\in G} K_x\varphi$
- Common knowledge: $C_G\varphi$, make sure that $\models C_G\varphi \leftrightarrow E_G(\varphi \wedge C_G\varphi)$
- Distributed knowledge: $D_G\varphi$, to be reinterpreted
- Field knowledge: $F_G\varphi$, new

Liang X. & Wáng, Y.N. Field Knowledge as a Dual to Distributed Knowledge: A Characterization by Weighted Modal Logic. LNGAI 2024.

Semantics

Model $M = (W, E, C, V)$

$$M, s \models K_a\psi \iff \text{for all } t \in W, \text{ if } C(a) \subseteq E(s,t) \text{ then } M, t \models \psi$$
$$M, s \models E_G\psi \iff \text{for all } a \in G, M, s \models K_a\psi$$
$$M, s \models C_G\psi \iff \text{for all } n \in \mathbb{N}^+, M, s \models E_G^n\psi$$

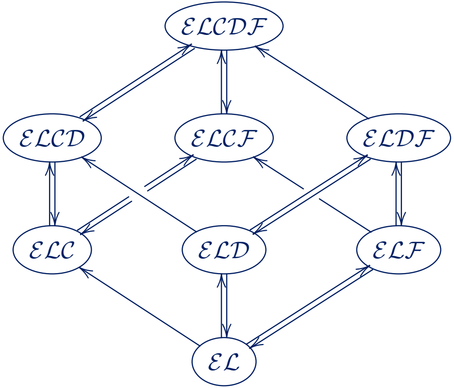
$$M, s \models D_G\psi \iff \text{for all } t \in W, \text{ if } \bigcup_{a\in G} C(a) \subseteq E(s,t) \text{ then } M, t \models \psi$$
$$M, s \models F_G\psi \iff \text{for all } t \in W, \text{ if } \bigcap_{a\in G} C(a) \subseteq E(s,t) \text{ then } M, t \models \psi$$

- Distributed knowledge: knowledge by combing the individual skills of a group
- Field knowledge: knowledge by their common skills

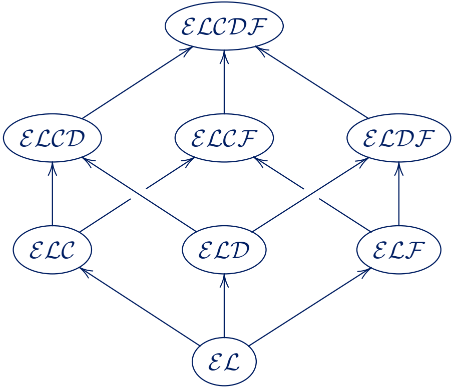
Compare with standard epistemic logic:

- $M, s \models E_G\psi \iff$ for all $t \in W$, if $(s,t) \in \bigcup_{a\in G} R_a$, then $M, t \models \psi$
- $M, s \models D_G\psi \iff$ for all $t \in W$, if $(s,t) \in \bigcap_{a\in G} R_a$, then $M, t \models \psi$

Expressivity



(a) when $|Ag| = 1$



(b) when $|Ag| \geq 2$

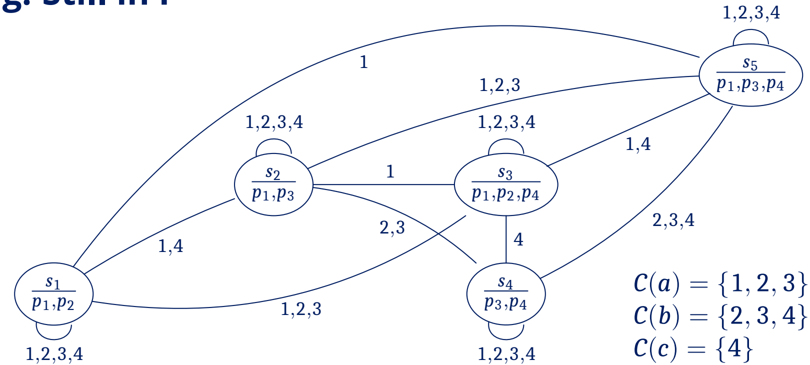
Axiomatization

- Base system: **KB**
- System **F**
  - (K$_F$) $F_G(\varphi \to \psi) \to (F_G\varphi \to F_G\psi)$
  - (F1) $F_{\{a\}}\varphi \leftrightarrow K_a\varphi$
  - (F2) $F_G\varphi \to F_H\varphi$ with $H \subseteq G$
  - (BF) $\varphi \to F_G\neg F_G\neg\varphi$
  - (NF) from φ infer $F_G\varphi$
- System **C**
  - (C1) $C_G\varphi \to \bigwedge_{a\in G} K_a(\varphi \wedge C_G\varphi)$
  - (C2) from $\varphi \to \bigwedge_{a\in G} K_a(\varphi \wedge \psi)$ infer $\varphi \to C_G\psi$
- System **D**
  - (K$_D$) $D_G(\varphi \to \psi) \to (D_G\varphi \to D_G\psi)$
  - (D1) $D_{\{a\}}\varphi \leftrightarrow K_a\varphi$
  - (D2) $D_G\varphi \to D_H\varphi$ with $G \subseteq H$
  - (BD) $\varphi \to D_G\neg D_G\neg\varphi$

Completeness proofs

- By translation of satisfiability
  - $\mathbf{KB}$
- Canonical model method
  - $\mathbf{KB}$
- Path-based canonical models (unraveling/folding)
  - $\mathbf{KB} \oplus \mathbf{D}$, $\mathbf{KB} \oplus \mathbf{F}$, $\mathbf{KB} \oplus \mathbf{D} \oplus \mathbf{F}$
- Finitary path-based canonical models
  - $\mathbf{KB} \oplus \mathbf{C}$, $\mathbf{KB} \oplus \mathbf{C} \oplus \mathbf{D}$, $\mathbf{KB} \oplus \mathbf{C} \oplus \mathbf{F}$, $\mathbf{KB} \oplus \mathbf{C} \oplus \mathbf{D} \oplus \mathbf{F}$

Model Checking: Still in P

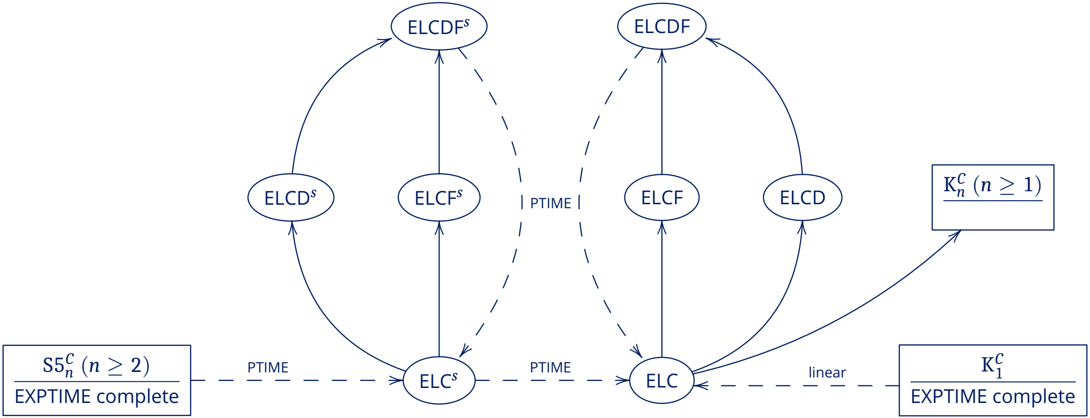


$$C(a) = \{1, 2, 3\}$$
$$C(b) = \{2, 3, 4\}$$
$$C(c) = \{4\}$$

$$s_2 \models K_a p_3$$
$$s_4 \models \neg F_{\{a,b\}} \neg p_1$$
$$s_5 \models \neg C_{\{a,c\}} p_1$$

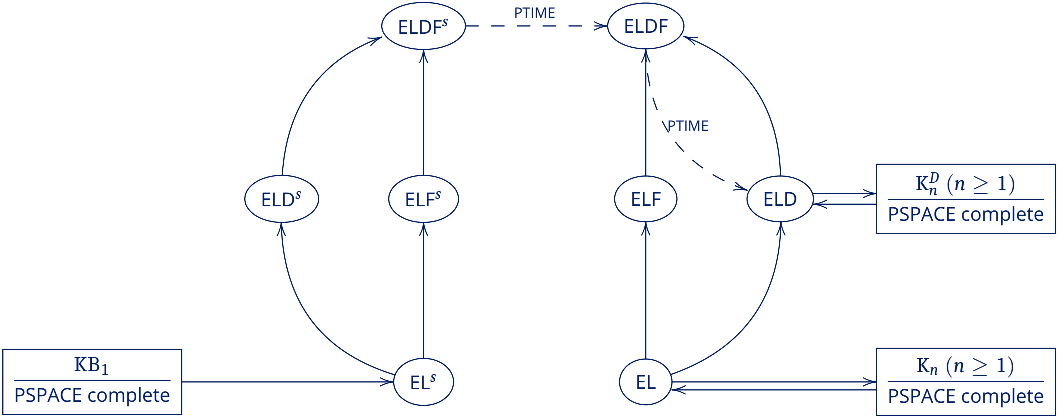
Computational complexity of SAT

Logics with CK: EXPTIME complete



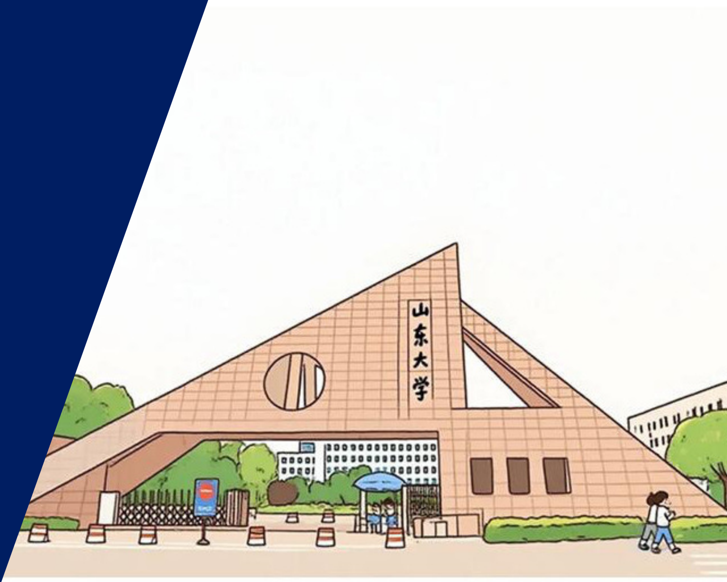
Computational complexity of SAT

Logics without CK: PSPACE complete



Dynamics

Knowing and forgetting



Upskilling, Downskilling and Reskilling

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \to \varphi) \mid K_a\varphi \mid C_G\varphi \mid D_G\varphi \mid E_G\varphi \mid F_G\varphi \mid$$
$$(+_S)_a\varphi \mid (-_S)_a\varphi \mid (=_S)_a\varphi \mid (\equiv_b)_a\varphi \mid \boxplus_a\varphi \mid \boxminus_a\varphi \mid \Box_a\varphi$$

$M, w \models (+_S)_a\psi \Leftrightarrow W, E, C^{a+S}, \beta, w \models \psi$ $\quad C^{a+S}(a)=C(a)\cup S$ and $\forall x \in A\backslash\{a\}.\ C^{a+S}(x)=C(x)$

$M, w \models (-_S)_a\psi \Leftrightarrow W, E, C^{a-S}, \beta, w \models \psi$ $\quad C^{a-S}(a)=C(a)\backslash S$ and $\forall x \in A\backslash\{a\}.\ C^{a-S}(x)=C(x)$

$M, w \models (=_S)_a\psi \Leftrightarrow W, E, C^{a=S}, \beta, w \models \psi$ $\quad C^{a=S}(a)=S$ and $\forall x \in A\backslash\{a\}.\ C^{a=S}(x)=C(x)$

$M, w \models (\equiv_b)_a\psi \Leftrightarrow W, E, C^{a\equiv b}, \beta, w \models \psi$ $\quad C^{a\equiv b}(a)=C(b)$ and $\forall x \in A\backslash\{a\}.\ C^{a\equiv b}(x)=C(x)$

$M, w \models \boxplus_a\psi \quad \Leftrightarrow$ for all finite nonempty $S \subseteq \mathsf{S}$, $M, w \models (+_S)_a\psi$

$M, w \models \boxminus_a\psi \quad \Leftrightarrow$ for all finite nonempty $S \subseteq \mathsf{S}$, $M, w \models (-_S)_a\psi$

$M, w \models \Box_a\psi \quad \Leftrightarrow$ for all finite nonempty $S \subseteq \mathsf{S}$, $M, w \models (=_S)_a\psi$

Liang X. & Wáng, Y.N. Epistemic Skills: Logical Dynamics of Knowing and Forgetting. GandALF 2024.

Slogans
Forgetting: decrease in skills, and increase in uncertainty

APAL:“Knowable as known after an announcement.”

Slogan 1. Knowable as known after upskilling.
Slogan 2. Forgettable as unknown after downskilling.

Debate: having no access is not forgetting.

Computational Complexity
The Model Checking Problem

- Logics without quantifiers: in P
- Logics with quantifiers: PSPACE complete
  - Hardness: reducing the Undirected Edge Geography (UEG) problem
- Traditional DELs with quantifiers (e.g., APAL, GAL) are of similar complexities
  - Yet less flexible and hard to model oblivion

Computational Complexity
The Model Checking Problem

- Logics without quantifiers: in P
- Logics with quantifiers: PSPACE complete
  - Hardness: reducing the Undirected Edge Geography (UEG) problem
- Traditional DELs with quantifiers (e.g., APAL, GAL) are of similar complexities
  - Yet less flexible and hard to model oblivion

Open problems for these dynamic logics:

- Complexity of the SAT problems
- Axiomatizations

Applications in Rough Sets

Understanding the Logic in Pawlak Rough Sets

Animal (U)	**Size** (R_1)	**Color** (R_2)	**Type** (R_3)	**Dangerous?** (p_0)
x_1	*small*	*black*	*bear*	✓
x_2	*medium*	*black*	*bear*	✓
x_3	*large*	*brown*	*dog*	✓
x_4	*small*	*black*	*cat*	✗
x_5	*medium*	*black*	*horse*	✗
x_6	*large*	*black*	*horse*	✓
x_7	*large*	*brown*	*horse*	✓

$\boldsymbol{Q}$-upper approx. of p: $\overline{\boldsymbol{Q}}p = \{x \in U \mid [x]_{\bigcap \boldsymbol{Q}} \cap p \neq \emptyset\}$

$\boldsymbol{Q}$-lower approx. of p: $\underline{\boldsymbol{Q}}p = \{x \in U \mid [x]_{\bigcap \boldsymbol{Q}} \subseteq p\}$

A category p is $\boldsymbol{Q}$-exact if $\underline{\boldsymbol{Q}}p = \overline{\boldsymbol{Q}}p$.

Understanding the Logic in Pawlak Rough Sets

Animal (U)	**Size** (R_1)	**Color** (R_2)	**Type** (R_3)	**Dangerous?** (p_0)
x_1	*small*	*black*	*bear*	✓
x_2	*medium*	*black*	*bear*	✓
x_3	*large*	*brown*	*dog*	✓
x_4	*small*	*black*	*cat*	✗
x_5	*medium*	*black*	*horse*	✗
x_6	*large*	*black*	*horse*	✓
x_7	*large*	*brown*	*horse*	✓

$\boldsymbol{Q}$-upper approx. of p: $\overline{\boldsymbol{Q}}p = \{x \in U \mid [x]_{\bigcap_{\boldsymbol{Q}}} \cap p \neq \emptyset\}$

$\boldsymbol{Q}$-lower approx. of p: $\underline{\boldsymbol{Q}}p = \{x \in U \mid [x]_{\bigcap_{\boldsymbol{Q}}} \subseteq p\}$

A category p is $\boldsymbol{Q}$-exact if $\underline{\boldsymbol{Q}}p = \overline{\boldsymbol{Q}}p$.

- Attributes: skills
- Approximation space: frame
- Category: atomic proposition
- $\underline{\boldsymbol{Q}}p : \Box p$ (with $C = \boldsymbol{Q}$)
- $\overline{\boldsymbol{Q}}p : \Diamond p$ (with $C = \boldsymbol{Q}$)
- $\boldsymbol{Q}$-exactness: $\Box p \leftrightarrow \Diamond p$
- Attribute selection: solvable by model checking

Our Logic allows iteration of attributes

When Weights Represented by Fuzzy Sets

Our logic allows fuzzy attribute sets

Table: Restaurant Dataset.

Restaurant	**Price Level**	**Cuisine**	**Ambiance**	**Wait Time**	p_1
Pasta Palace (x_1)	2 (medium)	1 (Italian)	4	15 mins	0.85
Taco Hut (x_2)	1 (low)	2 (Mexican)	3	10 mins	0.75
Sushi Spot (x_3)	3 (high)	3 (Japanese)	2	25 mins	0.40
Burger Bonanza (x_4)	1 (low)	4 (American)	5	5 mins	0.95
Curry Corner (x_5)	2 (medium)	5 (Indian)	3	20 mins	0.60

Table: Fuzzy approximation space $\mathbf{KB}_2 = (U, R_1, R_2, R_3, R_4)$.

R_1	x_1	x_2	x_3	x_4	x_5
x_1	1.00	0.14	0.14	0.14	1.00
x_2	0.14	1.00	0.00	1.00	0.14
x_3	0.14	0.00	1.00	0.00	0.14
x_4	0.14	1.00	0.00	1.00	0.14
x_5	1.00	0.14	0.14	0.14	1.00

R_2	x_1	x_2	x_3	x_4	x_5
x_1	1.00	0.61	0.14	0.01	0.00
x_2	0.61	1.00	0.61	0.14	0.01
x_3	0.14	0.61	1.00	0.61	0.14
x_4	0.01	0.14	0.61	1.00	0.61
x_5	0.00	0.01	0.14	0.61	1.00

R_3	x_1	x_2	x_3	x_4	x_5
x_1	1.00	0.14	0.02	0.14	0.14
x_2	0.14	1.00	0.14	0.02	1.00
x_3	0.02	0.14	1.00	0.00	0.14
x_4	0.14	0.02	0.00	1.00	0.02
x_5	0.14	1.00	0.14	0.02	1.00

R_4	x_1	x_2	x_3	x_4	x_5
x_1	1.00	0.61	0.14	0.61	0.61
x_2	0.61	1.00	0.02	0.61	0.14
x_3	0.14	0.02	1.00	0.00	0.61
x_4	0.61	0.61	0.00	1.00	0.02
x_5	0.61	0.13	0.61	0.02	1.00

Extended Logics

Definition 1 (Languages). *The languages $\mathcal{L}$ and $\mathcal{L}^+$ are generated by following grammar, where φ and ψ represent a formula in $\mathcal{L}$ and $\mathcal{L}^+$ respectively:*

$$
\begin{array}{ll}
(\mathcal{L}) & \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid B_i\varphi \mid (P_{i,s} * r) \mid (P_{i,s} * P_{j,t}) \\
(\mathcal{L}^+) & \psi ::= \varphi \mid [i,s]\psi
\end{array}
$$

where $p \in \mathsf{P}$, $i, j \in \mathsf{A}$, $s, t \in \mathsf{S}$, $r \in [0,1]$, *and* $* \in \{\leq, <, =, >, \geq\}$.

$$
\begin{array}{lll}
M, w \models p & \iff & p \in V(w) \\
M, w \models \neg\psi & \iff & M, w \not\models \psi \\
M, w \models \psi \to \chi & \iff & \text{if } M, w \models \psi, \text{ then } M, w \models \chi \\
M, w \models B_i\psi & \iff & \text{for all } u \in W, \text{ if } C(i) \subseteq E(w,u), \text{ then } M, u \models \psi \\
M, w \models P_{i,s} * r & \iff & C(i)(s) * r \\
M, w \models P_{i,s} * P_{j,t} & \iff & C(i)(s) * C(j)(t) \\
M, w \models [i,s]\psi & \iff & \text{for all } (i,s)\text{-variant } C' \text{ of } C, (W, E, C', V), w \models \psi
\end{array}
$$

Skill Assessment

SAP Given a frame (W, R), a valuation function V, a world $w \in W$, and an $\mathcal{L}^+$-formula φ, find all the capability functions $C : A \to FS$ such that $(W, R, C, V), w \models \varphi$.

ISAP Given a frame (W, R), a valuation function V, a world $w \in W$, an $\mathcal{L}^+$-formula φ, an agent $i \in A$, and a partial capability function $C \restriction_{A \setminus \{i\}}: (A \setminus \{i\}) \to FS$ for agents other than i, find all the $C \restriction_{\{i\}}$ such that $(W, R, C, V), w \models \varphi$.[1]

CVP Given a frame (W, R), a valuation function V, a world $w \in W$, an $\mathcal{L}^+$-formula φ, and a set Σ of capability functions, is it true that $C \in \Sigma$ iff $(W, R, C, V), w \models \varphi$?

ICVP Given a frame (W, R), a valuation function V, a world $w \in W$, an $\mathcal{L}^+$-formula φ, an agent $i \in A$, a partial capability function $C \restriction_{A \setminus \{i\}}: (A \setminus \{i\}) \to FS$ for agents other than i, and a set Σ of partial capability functions restricted to the domain $\{i\}$, is it true that $C \restriction_{\{i\}} \in \Sigma$ iff $(W, R, C, V), w \models \varphi$?


you

Computational Complexity of the Model Checking Problem

- Logics without quantifiers: in P
- Logics with quantifiers: PSPACE complete
  - Hardness: reducing the Undirected Edge Geography (UEG) problem

Upper Bound

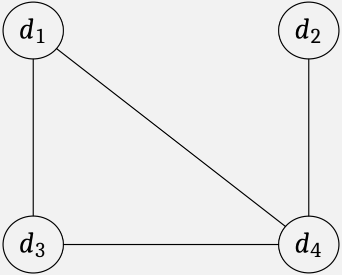
We only need to consider one new skill in addition to those that already appear

Algorithm Function $Val((W, E, C, \beta), \varphi)$:

```
Initialize: temVal ← ∅
Initialize: S_1 ← (⋃_{w,v∈W} E(w,v)) ∪ (⋃_{a appears in φ} C(a))
Initialize: S_2 ← S_1 ∪ {s}                                  ▷ Here s ∈ S is new for S_1
if ... then ...
else if φ = ⊞_a ψ then
    for all t ∈ W do
        Initialize: n ← true
        for all S ⊆ S_2 do
            if S ≠ ∅ and t ∉ Val((W, E, C^{a+S}, β), ψ) then n ← false
        if n = true then tmpVal ← tmpVal ∪ {t}
    return tmpVal
                ▷ Returns {t ∈ W | ∀S ⊆ S_1 : t ∈ Val((W, E, C^{a+S}, β), ψ)}
else if ... then ...
```

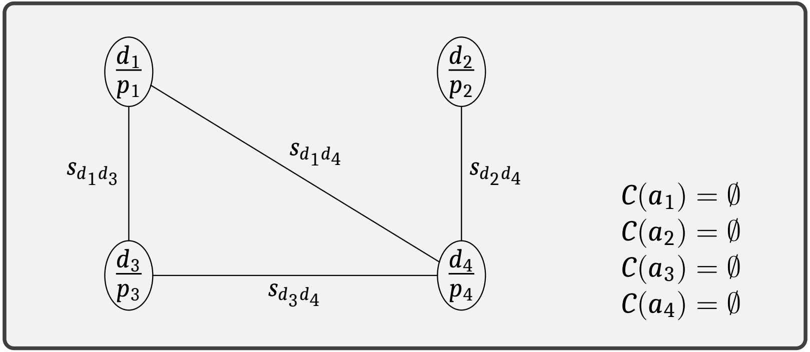
Example: UEG Game on (G, d_1)

$$G = \Big(\{d_1, d_2, d_3, d_4\}, \{(d_1, d_3), (d_1, d_4), (d_2, d_4), (d_3, d_4)\}\Big)$$



Model $M_G = (W, E, C, \beta)$

$W = \{d_1, \ldots, d_4\}$



- $E(d_m, d_k) = \{s_{d_m d_k}\}$ whenever 

- $C(a_1) = C(a_2) = C(a_3) = C(a_4) = \emptyset$ (a_i is the player who performs the i's move)
- $V(d_j) = \{p_j\}$ for $1 \leq j \leq 4$

Formula φ_G

For i's move in the UEG game:

$$\psi_i := \neg K_{a_i}\bot \wedge \left(K_{a_i}p_1 \vee K_{a_i}p_2 \vee K_{a_i}p_3 \vee K_{a_i}p_4\right)$$

$$\chi_1 := \bot$$

$$\chi_2 := (\hat{K}_{a_1}p_1 \wedge K_{a_2}p_1) \vee (\hat{K}_{a_1}p_2 \wedge K_{a_2}p_2) \vee (\hat{K}_{a_1}p_3 \wedge K_{a_2}p_3) \vee (\hat{K}_{a_1}p_4 \wedge K_{a_2}p_4)$$

$$\begin{aligned}\chi_3 := {} & (\hat{K}_{a_1}p_1 \wedge K_{a_2}p_1) \vee (\hat{K}_{a_1}p_2 \wedge K_{a_2}p_2) \vee (\hat{K}_{a_1}p_3 \wedge K_{a_2}p_3) \vee (\hat{K}_{a_1}p_4 \wedge K_{a_2}p_4) \\ & \vee(\hat{K}_{a_1}p_1 \wedge K_{a_3}p_1) \vee (\hat{K}_{a_1}p_2 \wedge K_{a_3}p_2) \vee (\hat{K}_{a_1}p_3 \wedge K_{a_3}p_3) \vee (\hat{K}_{a_1}p_4 \wedge K_{a_3}p_4) \\ & \vee(\hat{K}_{a_2}p_1 \wedge K_{a_3}p_1) \vee (\hat{K}_{a_2}p_2 \wedge K_{a_3}p_2) \vee (\hat{K}_{a_2}p_3 \wedge K_{a_3}p_3) \vee (\hat{K}_{a_2}p_4 \wedge K_{a_3}p_4)\end{aligned}$$

$$\chi_i := \bigvee_{1 \leq j < i} \left((\hat{K}_{a_j}p_1 \wedge K_{a_i}p_1) \vee (\hat{K}_{a_j}p_2 \wedge K_{a_i}p_2) \vee (\hat{K}_{a_j}p_3 \wedge K_{a_i}p_3) \vee (\hat{K}_{a_j}p_4 \wedge K_{a_i}p_4)\right)$$

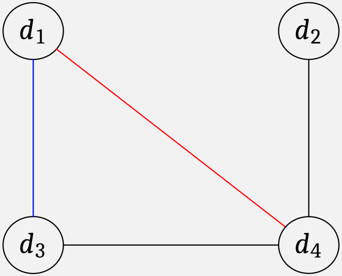
$$\varphi_G := \diamondplus_{a_1}(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4))))$$

The following are equivalent

- Player 1 has a winning strategy in (G, d_1)
- $M_G, d_1 \models \varphi_G$

Player 1's Move for Step 1

$$G = \Big(\{d_1, d_2, d_3, d_4\}, \{(d_1, d_3), (d_1, d_4), (d_2, d_4), (d_3, d_4)\}\Big)$$



- Player 1 chooses blue: will win
- Player 1 chooses red: can loose

First Step in the Model Checking

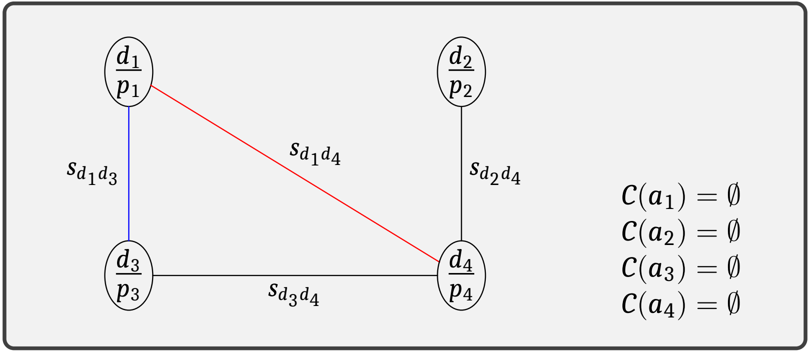
$M_G, d_1 \models \varphi_G$, where φ_G is:

$$\diamondplus_{a_1}\left(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))\right)$$

After some upskilling for a_1, true in d_1 are:

- $\psi_1 = \neg K_{a_1}\bot \wedge (K_{a_1}p_1 \vee K_{a_1}p_2 \vee K_{a_1}p_3 \vee K_{a_1}p_4)$
- $\neg\chi_1 = \neg\bot$
- $K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))$

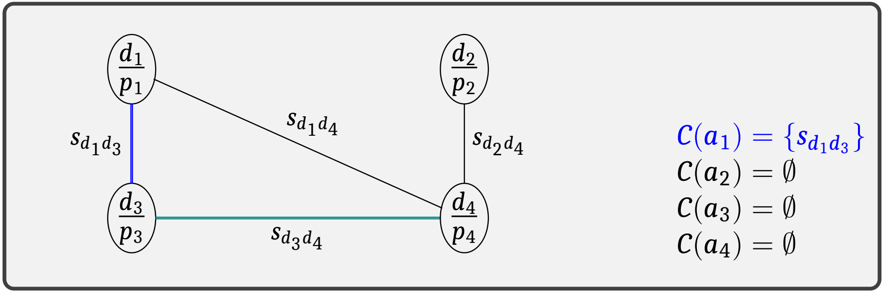
Step 1: Model Checking



$$M_G, d_1 \models (+\{s_{d_1d_3}\})_{a_1}\Big(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3} (\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))\Big)$$

$$M_G, d_1 \not\models (+\{s_{d_1d_4}\})_{a_1}\Big(\psi_1 \wedge \neg\chi_1 \wedge K_{a_1} \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \oplus_{a_3} (\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))\Big)$$

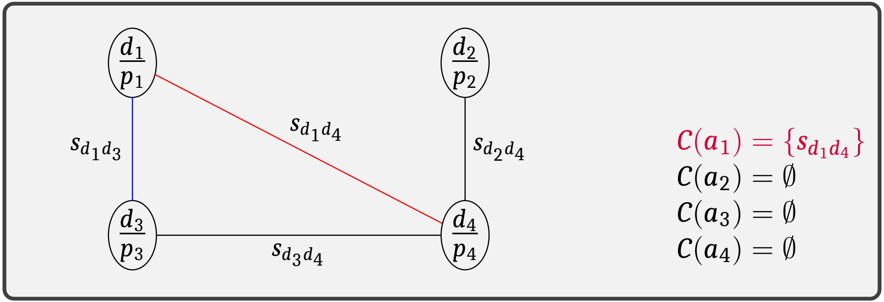
Step 2: Blue Case



$M_G, d_3 \models \boxplus_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2}\diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3}\boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$

- $M_G, d_3 \models (+\{s_{d_1d_3}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2}\diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3}\boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$
- $M_G, d_3 \models (+\{s_{d_3d_4}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2}\diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3}\boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$
- $M_G, d_3 \models (+\{s_{d_1d_4}\})_{a_2}(\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2}\diamondplus_{a_3}(\psi_3 \wedge \neg\chi_3 \wedge K_{a_3}\boxplus_{a_4}(\neg\psi_4 \vee \chi_4)))$ (or any other combinations)

Step 2: Red Case



$M_G, d_4 \not\models \boxplus_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3} (\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))$

- $M_G, d_4 \not\models (+\{s_{d_2 d_4}\})_{a_2} (\neg\psi_2 \vee \chi_2 \vee \hat{K}_{a_2} \diamondplus_{a_3} (\psi_3 \wedge \neg\chi_3 \wedge K_{a_3} \boxplus_{a_4} (\neg\psi_4 \vee \chi_4)))$