Projective Fraïssé limits and profinite groups

Shichang Song

Beijing Jiaotong University

7 August 2025 @ Shanghai

joint work with Sulin Hu

(Injective) Fraïssé classes

A class $\mathcal{K}$ of finite structures in a fixed (countable) signature is called a Fraïssé class if it satisfies the following properties:

- (HP) Hereditary property.
- (JEP) Joint embedding property.
- (AP) Amalgamation property.
- It is countable, up to isomorphism.
- It contains arbitrarily large finite structures.

Joint embedding property

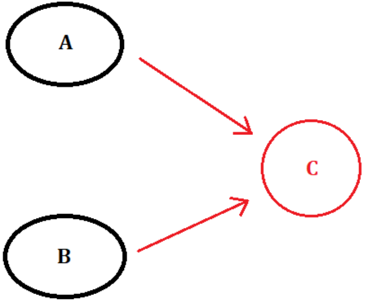

Figure: JEP

Amalgamation property

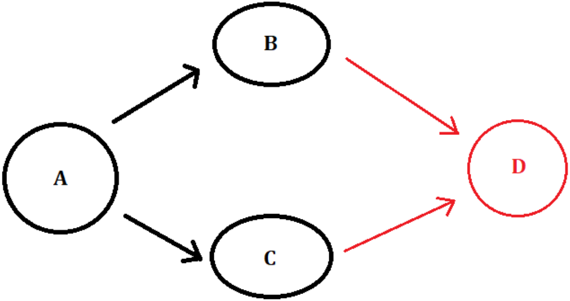

Figure: AP

(Injective) Fraïssé limits

Theorem (Fraïssé, 1954)

For every Fraïssé class $\mathcal{K}$, there is a countably infinite structure $\mathbb{K}$, unique up to isomorphism, such that

(a) $\mathbb{K}$ *is locally finite.*

(b) $\mathbb{K}$ *is ultrahomogeneous, i.e., every isomorphism between finite substructures of $\mathbb{K}$ extends to an automorphism of $\mathbb{K}$.*

(c) $\mathrm{Age}(\mathbb{K}) = \mathcal{K}$, *where $\mathrm{Age}(\mathbb{K})$ is the class of all finite structures embeddable in $\mathbb{K}$.*

Remark

- $\mathbb{K}$ *is called the Fraïssé limit of $\mathcal{K}$.*
- *A countably infinite structure satisfying (a) and (b) is called a Fraïssé structure.*

Examples

- The class of finite linear orderings is a Fraïssé class, and its Fraïssé limit is $(\mathbb{Q}, <)$.
- The class of finite graphs is a Fraïssé class, and its Fraïssé limit is the random graph, also called the Rado graph $\mathcal{R}$.
- The class of finite metric spaces with rational distances is a Fraïssé class, and its Fraïssé limit is the rational Urysohn space $\mathbb{U}_0$.
- The class of finite groups is a Fraïssé class, and its Fraïssé limit is Philip Hall's universal locally finite group $\mathbb{H}$.

Examples

- The class of finite linear orderings is a Fraïssé class, and its Fraïssé limit is $(\mathbb{Q}, <)$.
- The class of finite graphs is a Fraïssé class, and its Fraïssé limit is the random graph, also called the Rado graph $\mathcal{R}$.
- The class of finite metric spaces with rational distances is a Fraïssé class, and its Fraïssé limit is the rational Urysohn space $\mathbb{U}_0$.
- The class of finite groups is a Fraïssé class, and its Fraïssé limit is Philip Hall's universal locally finite group $\mathbb{H}$.

Examples

- The class of finite linear orderings is a Fraïssé class, and its Fraïssé limit is $(\mathbb{Q}, <)$.
- The class of finite graphs is a Fraïssé class, and its Fraïssé limit is the random graph, also called the Rado graph $\mathcal{R}$.
- The class of finite metric spaces with rational distances is a Fraïssé class, and its Fraïssé limit is the rational Urysohn space $\mathbb{U}_0$.
- The class of finite groups is a Fraïssé class, and its Fraïssé limit is Philip Hall's universal locally finite group $\mathbb{H}$.

Examples

- The class of finite linear orderings is a Fraïssé class, and its Fraïssé limit is $(\mathbb{Q}, <)$.
- The class of finite graphs is a Fraïssé class, and its Fraïssé limit is the random graph, also called the Rado graph $\mathcal{R}$.
- The class of finite metric spaces with rational distances is a Fraïssé class, and its Fraïssé limit is the rational Urysohn space $\mathbb{U}_0$.
- The class of finite groups is a Fraïssé class, and its Fraïssé limit is Philip Hall's universal locally finite group $\mathbb{H}$.

The class of finite groups is a Fraïssé class

Let $\mathcal{FG}$ be the class of finite groups. Then, $\mathcal{FG}$ has the following properties:

- (HP) Easy.
- (JEP) Let $A, B \in \mathcal{FG}$. Take $C = A \times B \in \mathcal{FG}$.
- (AP) This is proved in Section 3 of Neumann, B. H., *Permutational products of groups*, J. Aust. Math. Soc. Ser. A, **1** (1959), 299–310.

The class of finite groups is a Fraïssé class

Let $\mathcal{FG}$ be the class of finite groups. Then, $\mathcal{FG}$ has the following properties:

- (HP) Easy.
- (JEP) Let $A, B \in \mathcal{FG}$. Take $C = A \times B \in \mathcal{FG}$.
- (AP) This is proved in Section 3 of Neumann, B. H., *Permutational products of groups*, J. Aust. Math. Soc. Ser. A, **1** (1959), 299–310.

The class of finite groups is a Fraïssé class

Let $\mathcal{FG}$ be the class of finite groups. Then, $\mathcal{FG}$ has the following properties:

- (HP) Easy.
- (JEP) Let $A, B \in \mathcal{FG}$. Take $C = A \times B \in \mathcal{FG}$.
- (AP) This is proved in Section 3 of Neumann, B. H., *Permutational products of groups*, J. Aust. Math. Soc. Ser. A, **1** (1959), 299–310.

The class of finite groups is a Fraïssé class

Let $\mathcal{FG}$ be the class of finite groups. Then, $\mathcal{FG}$ has the following properties:

- (HP) Easy.
- (JEP) Let $A, B \in \mathcal{FG}$. Take $C = A \times B \in \mathcal{FG}$.
- (AP) This is proved in Section 3 of Neumann, B. H., *Permutational products of groups*, J. Aust. Math. Soc. Ser. A, **1** (1959), 299–310.

Definition

A Polish group is a topological group that is also a Polish space, *i.e.*, it is homeomorphic to a separable complete metric space.

- S_∞ is a Polish group under pointwise convergence topology.
- $\mathrm{Aut}(K) \leqslant S_\infty$ is a closed subgroup, where K is a countable structure.
- Every closed subgroup $G \leqslant S_\infty$ is of the form $G = \mathrm{Aut}(\mathbb{K})$, where $\mathbb{K}$ is a Fraïssé structure, *i.e.*, $\mathbb{K}$ is locally finite and ultrahomogeneous.

Questions

1. When does $\mathrm{Aut}(\mathbb{K})$ have a dense conjugacy class?
2. When does $\mathrm{Aut}(\mathbb{K})$ have a comeager (equivalent to dense G_δ) conjugacy class?
   Truss call every element with comeager conjugacy class a generic element.
3. When does $\mathrm{Aut}(\mathbb{K})$ have ample generics?
   A Polish group G has ample generics if there is a comeager orbit in its diagonal conjugacy action on G^n for every n.

Remark

These questions are related to the JEP and AP.

Questions

1. When does $\mathrm{Aut}(\mathbb{K})$ have a dense conjugacy class?
2. When does $\mathrm{Aut}(\mathbb{K})$ have a comeager (equivalent to dense G_δ) conjugacy class?
   Truss call every element with comeager conjugacy class a generic element.
3. When does $\mathrm{Aut}(\mathbb{K})$ have ample generics?
   A Polish group G has ample generics if there is a comeager orbit in its diagonal conjugacy action on G^n for every n.

Remark

These questions are related to the JEP and AP.

Questions

1. When does $\mathrm{Aut}(\mathbb{K})$ have a dense conjugacy class?
2. When does $\mathrm{Aut}(\mathbb{K})$ have a comeager (equivalent to dense G_δ) conjugacy class?
   Truss call every element with comeager conjugacy class a generic element.
3. When does $\mathrm{Aut}(\mathbb{K})$ have ample generics?
   A Polish group G has ample generics if there is a comeager orbit in its diagonal conjugacy action on G^n for every n.

Remark

These questions are related to the JEP and AP.

Questions

1. When does $\mathrm{Aut}(\mathbb{K})$ have a dense conjugacy class?
2. When does $\mathrm{Aut}(\mathbb{K})$ have a comeager (equivalent to dense G_δ) conjugacy class?
   Truss call every element with comeager conjugacy class a generic element.
3. When does $\mathrm{Aut}(\mathbb{K})$ have ample generics?
   A Polish group G has ample generics if there is a comeager orbit in its diagonal conjugacy action on G^n for every n.

Remark

These questions are related to the JEP and AP.

Questions

1. When does $\mathrm{Aut}(\mathbb{K})$ have a dense conjugacy class?
2. When does $\mathrm{Aut}(\mathbb{K})$ have a comeager (equivalent to dense G_δ) conjugacy class?
   Truss call every element with comeager conjugacy class a generic element.
3. When does $\mathrm{Aut}(\mathbb{K})$ have ample generics?
   A Polish group G has ample generics if there is a comeager orbit in its diagonal conjugacy action on G^n for every n.

Remark

These questions are related to the JEP and AP.

Questions

1. When does $\mathrm{Aut}(\mathbb{K})$ have a dense conjugacy class?
2. When does $\mathrm{Aut}(\mathbb{K})$ have a comeager (equivalent to dense G_δ) conjugacy class?
   Truss call every element with comeager conjugacy class a generic element.
3. When does $\mathrm{Aut}(\mathbb{K})$ have ample generics?
   A Polish group G has ample generics if there is a comeager orbit in its diagonal conjugacy action on G^n for every n.

Remark

These questions are related to the JEP and AP.

Definition of $\mathcal{K}_p$

- Let $\mathcal{K}$ be a Fraïssé class. Define $\mathcal{K}_p$ as the class of

$$\{\langle A, \psi\colon B \to C\rangle | A, B, C \in \mathcal{K}, B \subseteq A, C \subseteq A, \psi \text{ is an isomorphism}\}$$

- We call

$$f\colon \langle A, \psi\colon B \to C\rangle \to \langle D, \varphi\colon E \to F\rangle$$

is an embedding in $\mathcal{K}_p$ if $f\colon A \to D$ is an embedding such that $f(B) \subseteq E$, $f(C) \subseteq F$, and $f \circ \psi \subseteq \varphi \circ f$.

- Once we have embeddings, we may define JEP and AP.

Definition of $\mathcal{K}_p$

- Let $\mathcal{K}$ be a Fraïssé class. Define $\mathcal{K}_p$ as the class of

  $$\{\langle A, \psi\colon B \to C\rangle | A, B, C \in \mathcal{K}, B \subseteq A, C \subseteq A, \psi \text{ is an isomorphism}\}$$

- We call

  $$f\colon \langle A, \psi\colon B \to C\rangle \to \langle D, \varphi\colon E \to F\rangle$$

  is an embedding in $\mathcal{K}_p$ if $f\colon A \to D$ is an embedding such that $f(B) \subseteq E$, $f(C) \subseteq F$, and $f \circ \psi \subseteq \varphi \circ f$.

- Once we have embeddings, we may define JEP and AP.

Definition of $\mathcal{K}_p$

- Let $\mathcal{K}$ be a Fraïssé class. Define $\mathcal{K}_p$ as the class of

  $$\{\langle A, \psi\colon B \to C\rangle | A, B, C \in \mathcal{K}, B \subseteq A, C \subseteq A, \psi \text{ is an isomorphism}\}$$

- We call

  $$f\colon \langle A, \psi\colon B \to C\rangle \to \langle D, \varphi\colon E \to F\rangle$$

  is an embedding in $\mathcal{K}_p$ if $f\colon A \to D$ is an embedding such that $f(B) \subseteq E$, $f(C) \subseteq F$, and $f \circ \psi \subseteq \varphi \circ f$.
- Once we have embeddings, we may define JEP and AP.

Theorem (Kechris and Rosendal)

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a dense conjugacy class in $\mathrm{Aut}(\mathbb{K})$.*
2. *$\mathcal{K}_p$ satisfies the JEP.*

Theorem (Kechris and Rosendal)

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a dense conjugacy class in* $\mathrm{Aut}(\mathbb{K})$.
2. $\mathcal{K}_p$ *satisfies the JEP.*

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is cofinal under embeddability and $\mathcal{L}$ satisfies the AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $e\colon A \to B$ such that $\forall f\colon B \to B_1$ and $\forall g\colon B \to B_2$, where $B_1, B_2 \in \mathcal{C}$, $\exists C \in \mathcal{C}, r\colon B_1 \to C, s\colon B_2 \to C$ with $r \circ f \circ e = s \circ g \circ e$.

Remark (Truss)

Obviously, CAP $\Rightarrow$ WAP.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is cofinal under embeddability and $\mathcal{L}$ satisfies the AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $e\colon A \to B$ such that $\forall f\colon B \to B_1$ and $\forall g\colon B \to B_2$, where $B_1, B_2 \in \mathcal{C}$, $\exists C \in \mathcal{C}, r\colon B_1 \to C, s\colon B_2 \to C$ with $r \circ f \circ e = s \circ g \circ e$.

Remark (Truss)

Obviously, CAP $\Rightarrow$ WAP.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is cofinal under embeddability and $\mathcal{L}$ satisfies the AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $e\colon A \to B$ such that $\forall f\colon B \to B_1$ and $\forall g\colon B \to B_2$, where $B_1, B_2 \in \mathcal{C}$, $\exists C \in \mathcal{C}, r\colon B_1 \to C, s\colon B_2 \to C$ with $r \circ f \circ e = s \circ g \circ e$.

Remark (Truss)

Obviously, CAP $\Rightarrow$ WAP.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is cofinal under embeddability and $\mathcal{L}$ satisfies the AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $e\colon A \to B$ such that $\forall f\colon B \to B_1$ and $\forall g\colon B \to B_2$, where $B_1, B_2 \in \mathcal{C}$, $\exists C \in \mathcal{C}, r\colon B_1 \to C, s\colon B_2 \to C$ with $r \circ f \circ e = s \circ g \circ e$.

Remark (Truss)

Obviously, CAP $\Rightarrow$ WAP.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is cofinal under embeddability and $\mathcal{L}$ satisfies the AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $e\colon A \to B$ such that $\forall f\colon B \to B_1$ and $\forall g\colon B \to B_2$, where $B_1, B_2 \in \mathcal{C}$, $\exists C \in \mathcal{C}, r\colon B_1 \to C, s\colon B_2 \to C$ with $r \circ f \circ e = s \circ g \circ e$.

Remark (Truss)

Obviously, CAP $\Rightarrow$ WAP.

Theorem (Kechris and Rosendal)

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager conjugacy class in $\mathrm{Aut}(\mathbb{K})$.*
2. *$\mathbb{K}$ has a generic automorphism.*
3. *$\mathcal{K}_p$ satisfies the JEP and WAP.*

Remark

Thus, if $\mathcal{K}_p$ satisfies the JEP and CAP, then $\mathbb{K}$ has a generic automorphism.

Theorem (Kechris and Rosendal)

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager conjugacy class in* $\mathrm{Aut}(\mathbb{K})$.
2. $\mathbb{K}$ *has a generic automorphism.*
3. $\mathcal{K}_p$ *satisfies the JEP and WAP.*

Remark

Thus, if $\mathcal{K}_p$ satisfies the JEP and CAP, then $\mathbb{K}$ has a generic automorphism.

Theorem (Kechris and Rosendal)

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager diagonal conjugacy class in $\mathrm{Aut}(\mathbb{K})^n$ for every n.*
2. $\mathrm{Aut}(\mathbb{K})$ *has ample generics.*
3. $\mathcal{K}_p^n$ *satisfies the JEP and WAP for every n.*

Remark

Thus, $\mathbb{K}$ has ample generic automorphisms if $\mathcal{K}_p^n$ has JEP and CAP for every n.

Theorem (Kechris and Rosendal)

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager diagonal conjugacy class in $\mathrm{Aut}(\mathbb{K})^n$ for every n.*
2. $\mathrm{Aut}(\mathbb{K})$ *has ample generics.*
3. $\mathcal{K}_p^n$ *satisfies the JEP and WAP for every n.*

Remark

Thus, $\mathbb{K}$ has ample generic automorphisms if $\mathcal{K}_p^n$ has JEP and CAP for every n.

Fact

- *(Hodges-Hodkinson-Lascar-Shelah, 1993)* $\mathrm{Aut}(\mathcal{R})$ *has ample generics.*
- *(Solecki, 2005)* $\mathrm{Iso}(\mathbb{U}_0)$ *has ample generics.*
- *(S., 2019)* $\mathrm{Aut}(\mathbb{H})$ *has ample generics.*

Fact

- *(Hodges-Hodkinson-Lascar-Shelah, 1993)* $\mathrm{Aut}(\mathcal{R})$ *has ample generics.*
- *(Solecki, 2005)* $\mathrm{Iso}(\mathbb{U}_0)$ *has ample generics.*
- *(S., 2019)* $\mathrm{Aut}(\mathbb{H})$ *has ample generics.*

Fact

- *(Hodges-Hodkinson-Lascar-Shelah, 1993)* $\mathrm{Aut}(\mathcal{R})$ *has ample generics.*
- *(Solecki, 2005)* $\mathrm{Iso}(\mathbb{U}_0)$ *has ample generics.*
- *(S., 2019)* $\mathrm{Aut}(\mathbb{H})$ *has ample generics.*

Consequences of ample generics

Theorem (Kechris and Rosendal)

Suppose that G is a Polish group with ample generics. Then

1. *G has the small index property, i.e., every subgroup of index less than* $2^{\aleph_0}$ *is open.*
2. *every homomorphism* π *from G to a Polish group H is automatically continuous.*
3. *G has a unique Polish group topology.*
4. *G is not the union of a countable chain of non-open subgroups.*

Topological L-structures

Fix a language L consisting of relations $\{R_i\}_{i \in I}$ and functions $\{f_j\}_{j \in J}$. A *topological L-structure* is a zero-dimensional, compact, second countable space A satisfying:

- closed sets $R_i^A \subseteq A^{m_i}$ for each i
- continuous functions $f_j^A \colon A^{n_j} \to A$ for each j

Epimorphisms

Let A and B be two topological L-structures. An *epimorphism* $\phi\colon A \to B$ is a surjective continuous function satisfying:

- $f_j^B(\phi(x_1), \cdots, \phi(x_{n_j})) = \phi(f_j^A(x_1, \cdots, x_{n_j}))$ for each j
- for each i,

$$(y_1, \cdots, y_{m_i}) \in R_i^B$$

$$\Updownarrow$$

$$\exists x_1, \cdots, x_{m_i} \in A(\phi(x_k) = y_k (1 \leq k \leq m_i) \text{ and } (x_1, \cdots, x_{m_i}) \in R_i^A)$$

Projective Fraïssé classes

The class Δ of finite topological L-structures is called a projective Fraïssé class if it satisfies the following properties:

- (Joint Projection Property (JPP)) for $A, B \in \Delta$, there is $C \in \Delta$ and epimorphisms from C onto A and onto B.
- (Projective AP) for $A, B, C \in \Delta$ and epimorphism $\phi_1 : B \to A$ and $\phi_2 : C \to A$, there is $D \in \Delta$ and epimorphisms $\psi_1 : D \to B$ and $\psi_2 : D \to C$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$.
- It is countable, up to isomorphism.

Joint Projection property

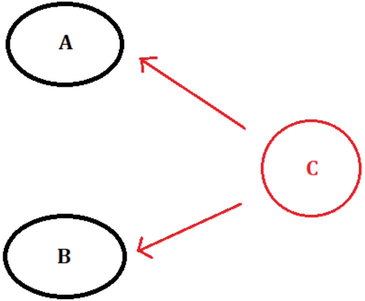

Figure: JPP

Projective Amalgamation property

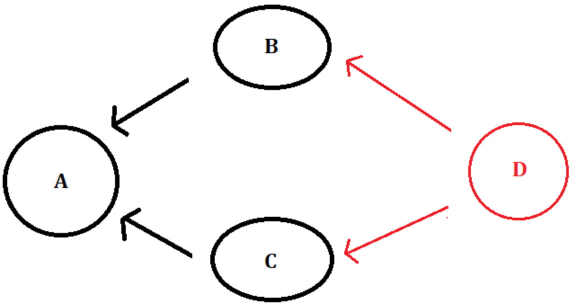

Figure: Projective AP

Projective Fraïssé limits

Theorem (Irwin-Solecki, 2006)

For every projective Fraïssé class Δ, there is a topological L-structure $\mathbb{D}$, unique up to isomorphism, such that

(L1) *(projective universality) for $D \in \Delta$ there is an epimorphism from $\mathbb{D}$ to D*

(L2) *for finite discrete topological space A and continuous function $f\colon \mathbb{D} \to A$ there is $D \in \Delta$, an epimorphism $\phi\colon \mathbb{D} \to D$, and a function $f'\colon D \to A$ such that $f = f' \circ \phi$*

(L3) *(projective ultrahomogeniety) for $D \in \Delta$ and epimorphisms $\phi_1\colon \mathbb{D} \to D$ and $\phi_2\colon \mathbb{D} \to D$ there is an isomorphism $\psi\colon \mathbb{D} \to \mathbb{D}$ such that $\phi_2 = \phi_1 \circ \psi$*

Remark

$\mathbb{D}$ *is called the projective Fraïssé limit of* Δ.

Examples

- The class of finite points is a projective Fraïssé class, and its projective Fraïssé limit is the Cantor set.
- The class of finite reflexive linear graphs is a projective Fraïssé class, and its projective Fraïssé limit is $(\mathbb{P}, R^{\mathbb{P}})$ whose quotient $\mathbb{P}/R^{\mathbb{P}}$ is the pseudoarc.
- The pseudoarc is the unique hereditarily indecomposable chainable continuum, where a continuum is a compact connected metric space.

Examples

- The class of finite points is a projective Fraïssé class, and its projective Fraïssé limit is the Cantor set.
- The class of finite reflexive linear graphs is a projective Fraïssé class, and its projective Fraïssé limit is $(\mathbb{P}, R^{\mathbb{P}})$ whose quotient $\mathbb{P}/R^{\mathbb{P}}$ is the pseudoarc.
- The pseudoarc is the unique hereditarily indecomposable chainable continuum, where a continuum is a compact connected metric space.

Examples

- The class of finite points is a projective Fraïssé class, and its projective Fraïssé limit is the Cantor set.
- The class of finite reflexive linear graphs is a projective Fraïssé class, and its projective Fraïssé limit is $(\mathbb{P}, R^{\mathbb{P}})$ whose quotient $\mathbb{P}/R^{\mathbb{P}}$ is the pseudoarc.
- The pseudoarc is the unique hereditarily indecomposable chainable continuum, where a continuum is a compact connected metric space.

Theorem (Hu-S.)

The class of finite groups is a projective Fraïssé class, and its projective Fraïssé limit $\mathbb{F}$ is the free profinite group on a countably infinite set converging to 1.

Proof: The JPP is $A \times B$ and the projective AP is $B \times_A C$.

Theorem (Hu-S.)

The class of finite groups is a projective Fraïssé class, and its projective Fraïssé limit $\mathbb{F}$ is the free profinite group on a countably infinite set converging to 1.

Proof: The JPP is $A \times B$ and the projective AP is $B \times_A C$.

Profinite groups

- A *profinite group* is the inverse limit of an inverse system of discrete finite groups.
- A profinite group is a topological group that is zero-dimensional, compact and second countable if the system is countable.
- Let G be a group. Then its normal subgroups with finite indices induce an inverse system. Its inverse limit is the *profinite completion* of G, denoted by $\hat{G}$.

Definition

Let G be a profinite group. We call G the *free profinite group on a countably infinite set converging to 1*, if there is a countable $B = \{b_n \mid n \in \mathbb{N}\} \subseteq G$ satisfying:

- b_n is converging to 1_G.
- for every profinite H and every $f\colon B \to H$ with $f(b_n)$ converging to 1_H, there is a unique continuous extension $\phi\colon G \to H$.

Remark

The free profinite group on a countably infinite set converging to 1 is unique up to isomorphism, which is the projective Fraïssé limit $\mathbb{F}$.

Let F_ω denote the free group on a countably infinite set.

Fact

- *(Perin-Sklinos, 2012) Nonabelian free groups are strongly ω-homogeneous, and thus F_ω is strongly ω-homogeneous.*
- *F_ω is projectively universal among countable groups, that is, for every countable group G, there is a surjective homomorphism $\varphi\colon F_\omega \to G$.*
- *(Ding, 2012) There is a topology on F_ω such that its completion is projectively universal Polish group, that is, every Polish group is isomorphic to a quotient of it.*

Projectively homogeneity

Theorem (Hu-S.)

F_ω is projectively homogeneous among finite groups, that is, for all finite groups G, and all surjective homomorphisms $\varphi\colon F_\omega \to G$ and $\psi\colon F_\omega \to G$, there is an automorphism θ of F_ω such that $\psi = \varphi \circ \theta$.

Remark

F_ω is a countable projectively universal and projectively homogeneous group.

Projectively homogeneity

Theorem (Hu-S.)

F_ω is projectively homogeneous among finite groups, that is, for all finite groups G, and all surjective homomorphisms $\varphi\colon F_\omega \to G$ and $\psi\colon F_\omega \to G$, there is an automorphism θ of F_ω such that $\psi = \varphi \circ \theta$.

Remark

F_ω is a countable projectively universal and projectively homogeneous group.

Profinite completion

- The profinite completion of F_ω is not second countable, and thus
$$\hat{F}_\omega \ncong \mathbb{F}$$

- Instead of profinite topology, we may revise it to get a "restricted" profinite topology.
- Fix a generating set $\{e_1, e_2, \cdots, e_n, \cdots\}$ for F_ω. Let $S = \{N \trianglelefteq F_\omega \mid N$ is of finite index and contains all but finite e_n's$\}$.
- Let $\bar{F}_\omega$ denote the inverse limit of F_ω of an inverse system S,
$$\bar{F}_\omega := \varprojlim_{N \in S} F_\omega / N.$$

Profinite completion

- The profinite completion of F_ω is not second countable, and thus
$$\hat{F}_\omega \ncong \mathbb{F}$$
- Instead of profinite topology, we may revise it to get a "restricted" profinite topology.
- Fix a generating set $\{e_1, e_2, \cdots, e_n, \cdots\}$ for F_ω. Let $S = \{N \trianglelefteq F_\omega \mid N$ is of finite index and contains all but finite e_n's$\}$.
- Let $\bar{F}_\omega$ denote the inverse limit of F_ω of an inverse system S,
$$\bar{F}_\omega := \varprojlim_{N \in S} F_\omega / N.$$

Profinite completion

- The profinite completion of F_ω is not second countable, and thus
$$\hat{F}_\omega \ncong \mathbb{F}$$
- Instead of profinite topology, we may revise it to get a "restricted" profinite topology.
- Fix a generating set $\{e_1, e_2, \cdots, e_n, \cdots\}$ for F_ω. Let $S = \{N \trianglelefteq F_\omega \mid N$ is of finite index and contains all but finite e_n's$\}$.
- Let $\bar{F}_\omega$ denote the inverse limit of F_ω of an inverse system S,
$$\bar{F}_\omega := \varprojlim_{N \in S} F_\omega / N.$$

Profinite completion

- The profinite completion of F_ω is not second countable, and thus

$$\hat{F}_\omega \ncong \mathbb{F}$$

- Instead of profinite topology, we may revise it to get a "restricted" profinite topology.
- Fix a generating set $\{e_1, e_2, \cdots, e_n, \cdots\}$ for F_ω. Let $S = \{N \trianglelefteq F_\omega \mid N$ is of finite index and contains all but finite e_n's$\}$.
- Let $\bar{F}_\omega$ denote the inverse limit of F_ω of an inverse system S,

$$\bar{F}_\omega := \varprojlim_{N \in S} F_\omega / N.$$

Proposition

$$\bar{F}_\omega \cong \mathbb{F}$$

Profinite rigidity

- Let H be a countable projectively universal and projectively homogeneous group.
- Let $\widetilde{H}$ denote the intersection of all finite-indexed normal subgroups of H, that is,

$$\widetilde{H} = \bigcap \{K \mid K \trianglelefteq H, [H:K] < \infty\}.$$

- Then $H/\widetilde{H}$ is *residually finite*.
- Free groups are residually finite.
- The restricted profinite completion $\overline{H/\widetilde{H}} \cong \mathbb{F}$.

Profinite rigidity

- Let H be a countable projectively universal and projectively homogeneous group.
- Let $\widetilde{H}$ denote the intersection of all finite-indexed normal subgroups of H, that is,

$$\widetilde{H} = \bigcap\{K \mid K \trianglelefteq H, [H : K] < \infty\}.$$

- Then $H/\widetilde{H}$ is *residually finite*.
- Free groups are residually finite.
- The restricted profinite completion $\overline{H/\widetilde{H}} \cong \mathbb{F}$.

Profinite rigidity

- Let H be a countable projectively universal and projectively homogeneous group.
- Let $\widetilde{H}$ denote the intersection of all finite-indexed normal subgroups of H, that is,

$$\widetilde{H} = \bigcap \{K \mid K \trianglelefteq H, [H : K] < \infty\}.$$

- Then $H/\widetilde{H}$ is *residually finite*.
- Free groups are residually finite.
- The restricted profinite completion $\overline{H/\widetilde{H}} \cong \mathbb{F}$.

Profinite rigidity

- Let H be a countable projectively universal and projectively homogeneous group.
- Let $\widetilde{H}$ denote the intersection of all finite-indexed normal subgroups of H, that is,

$$\widetilde{H} = \bigcap \{K \mid K \trianglelefteq H, [H : K] < \infty\}.$$

- Then $H/\widetilde{H}$ is *residually finite*.
- Free groups are residually finite.
- The restricted profinite completion $\overline{H/\widetilde{H}} \cong \mathbb{F}$.

Profinite rigidity

- Let H be a countable projectively universal and projectively homogeneous group.
- Let $\widetilde{H}$ denote the intersection of all finite-indexed normal subgroups of H, that is,

$$\widetilde{H} = \bigcap \{K \mid K \trianglelefteq H, [H:K] < \infty\}.$$

- Then $H/\widetilde{H}$ is *residually finite*.
- Free groups are residually finite.
- The restricted profinite completion $\overline{H/\widetilde{H}} \cong \mathbb{F}$.

Profinite rigidity

Question

Let G be a countable, residually finite, projectively universal and projectively homogeneous group. Then the restricted profinite completion

$$\overline{G} \cong \overline{F_\omega} \cong \mathbb{F}.$$

Is $G \cong F_\omega$?

Profinite rigidity open question

A finitely generated residually finite group G is *profinitely rigid in the absolute sense* if whenever a finitely generated residually finite group H satisfying $\hat{G} \cong \hat{H}$, then $G \cong H$.

Question (Remeslennikov 1979)

Is every nonabelian free group profinitely rigid in the absolute sense?

Profinite rigidity open question

A finitely generated residually finite group G is *profinitely rigid in the absolute sense* if whenever a finitely generated residually finite group H satisfying $\hat{G} \cong \hat{H}$, then $G \cong H$.

Question (Remeslennikov 1979)

Is every nonabelian free group profinitely rigid in the absolute sense?

Profinite rigidity

The profinite rigidity fails in the case of abelian groups.

Theorem

Let F^{ab}_ω denote the free abelian group on a countably infinite set and let $\mathbb{F}^{ab}$ denote the free profinite abelian group on a countably infinite set converging to 1. Then there is a countable projectively universal and projectively homogeneous abelian group G whose restricted profinite completion is isomorphic to $\mathbb{F}^{ab}$, but $G \ncong F^{ab}_\omega$.

Definition of $\mathcal{K}^p$

- Let $\mathcal{K}$ be a projective Fraïssé class and let $\mathbb{K}$ be its projective Fraïssé limit. Let s be a binary relation symbol. Define $\mathcal{K}^p$ as the class of

$$\{(A, s^A) \mid A \in \mathcal{K} \text{ and } \exists f \in \mathrm{Aut}(\mathbb{K}), \exists \phi\colon \mathbb{K} \to A$$

$$\text{such that } \phi\colon (\mathbb{K}, f) \to (A, s^A) \text{ is an epimorphism}\}$$

- Naturally, we define epimorphisms in $\mathcal{K}_p$.
- Once we have epimorphisms, we may define JPP and Projective AP.

Definition of $\mathcal{K}^p$

- Let $\mathcal{K}$ be a projective Fraïssé class and let $\mathbb{K}$ be its projective Fraïssé limit. Let s be a binary relation symbol. Define $\mathcal{K}^p$ as the class of

$$\{(A, s^A) \mid A \in \mathcal{K} \text{ and } \exists f \in \mathrm{Aut}(\mathbb{K}), \exists \phi\colon \mathbb{K} \to A$$

$$\text{such that } \phi\colon (\mathbb{K}, f) \to (A, s^A) \text{ is an epimorphism}\}$$

- Naturally, we define epimorphisms in $\mathcal{K}_p$.
- Once we have epimorphisms, we may define JPP and Projective AP.

Definition of $\mathcal{K}^p$

- Let $\mathcal{K}$ be a projective Fraïssé class and let $\mathbb{K}$ be its projective Fraïssé limit. Let s be a binary relation symbol. Define $\mathcal{K}^p$ as the class of

$$\{(A, s^A) \mid A \in \mathcal{K} \text{ and } \exists f \in \mathrm{Aut}(\mathbb{K}), \exists \phi \colon \mathbb{K} \to A$$

$$\text{such that } \phi \colon (\mathbb{K}, f) \to (A, s^A) \text{ is an epimorphism}\}$$

- Naturally, we define epimorphisms in $\mathcal{K}_p$.
- Once we have epimorphisms, we may define JPP and Projective AP.

Theorem (Kwiatkowska)

Let $\mathcal{K}$ be a Projective Fraïssé class with Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a dense conjugacy class in $\mathrm{Aut}(\mathbb{K})$.*
2. *$\mathcal{K}^p$ satisfies the JPP.*

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is coinitial AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $\phi\colon B \twoheadrightarrow A$ such that $\forall \phi_1 : C_1 \twoheadrightarrow B$ and $\forall \phi_2 : C_2 \twoheadrightarrow B$, where $C_1, C_2 \in \mathcal{C}$, $\exists D \in \mathcal{C}$, $\phi_3 : D \twoheadrightarrow C_1, \phi_4 : D \twoheadrightarrow C_2$ with $\phi \circ \phi_1 \circ \phi_3 = \phi \circ \phi_2 \circ \phi_4$.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is coinitial AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $\phi\colon B \twoheadrightarrow A$ such that $\forall \phi_1\colon C_1 \twoheadrightarrow B$ and $\forall \phi_2\colon C_2 \twoheadrightarrow B$, where $C_1, C_2 \in \mathcal{C}$, $\exists D \in \mathcal{C}, \phi_3\colon D \twoheadrightarrow C_1, \phi_4\colon D \twoheadrightarrow C_2$ with $\phi \circ \phi_1 \circ \phi_3 = \phi \circ \phi_2 \circ \phi_4$.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is coinitial AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $\phi\colon B \twoheadrightarrow A$ such that $\forall \phi_1 : C_1 \twoheadrightarrow B$ and $\forall \phi_2 : C_2 \twoheadrightarrow B$, where $C_1, C_2 \in \mathcal{C}$, $\exists D \in \mathcal{C}, \phi_3 : D \twoheadrightarrow C_1, \phi_4 : D \twoheadrightarrow C_2$ with $\phi \circ \phi_1 \circ \phi_3 = \phi \circ \phi_2 \circ \phi_4$.

Weaker APs

- A class $\mathcal{C}$ satisfies the CAP if it has a subclass $\mathcal{L}$ which is coinitial AP.
- A class $\mathcal{C}$ satisfies the WAP if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{C}$ and $\phi\colon B \twoheadrightarrow A$ such that $\forall \phi_1\colon C_1 \twoheadrightarrow B$ and $\forall \phi_2\colon C_2 \twoheadrightarrow B$, where $C_1, C_2 \in \mathcal{C}$, $\exists D \in \mathcal{C}, \phi_3\colon D \twoheadrightarrow C_1, \phi_4\colon D \twoheadrightarrow C_2$ with $\phi \circ \phi_1 \circ \phi_3 = \phi \circ \phi_2 \circ \phi_4$.

Theorem (Kwiatkowska)

Let $\mathcal{K}$ be a projective Fraïssé class with projective Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager conjugacy class in* $\mathrm{Aut}(\mathbb{K})$.
2. $\mathbb{K}$ *has a generic automorphism.*
3. $\mathcal{K}^p$ *satisfies the JPP and WAP.*

Remark

Thus, if $\mathcal{K}^p$ satisfies the JEP and CAP, then $\mathbb{K}$ has a generic automorphism.

Theorem (Kwiatkowska)

Let $\mathcal{K}$ be a projective Fraïssé class with projective Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager conjugacy class in $\mathrm{Aut}(\mathbb{K})$.*
2. *$\mathbb{K}$ has a generic automorphism.*
3. *$\mathcal{K}^p$ satisfies the JPP and WAP.*

Remark

Thus, if $\mathcal{K}^p$ satisfies the JEP and CAP, then $\mathbb{K}$ has a generic automorphism.

Theorem (Kwiatkowska)

Let $\mathcal{K}$ be a projective Fraïssé class with projective Fraïssé limit $\mathbb{K}$. Then TFAE:

1. *there is a comeager diagonal conjugacy class in $\mathrm{Aut}(\mathbb{K})^n$ for every n.*
2. $\mathrm{Aut}(\mathbb{K})$ *has ample generics.*
3. $\mathcal{K}_n^p$ *satisfies the JEP and WAP for every n.*

Fact

- *(Kwiatkowska, 2012)* $\mathrm{Homeo}(2^{\mathbb{N}})$ *has ample generics.*
- *(Kwiatkowska, 2014)* $\mathrm{Aut}(\mathbb{P})$ *has ample generics.*

Fact

- *(Kwiatkowska, 2012)* $\mathrm{Homeo}(2^{\mathbb{N}})$ *has ample generics.*
- *(Kwiatkowska, 2014)* $\mathrm{Aut}(\mathbb{P})$ *has ample generics.*

Fact

- *(Kwiatkowska, 2012)* $\mathrm{Homeo}(2^{\mathbb{N}})$ *has ample generics.*
- *(Kwiatkowska, 2014)* $\mathrm{Aut}(\mathbb{P})$ *has ample generics.*

Theorem (Hu-S.)

$\mathrm{Aut}(\mathbb{F})$ *has ample generics.*

Profinite systems

Let G be a profinite group. We associate a *profinite system* $S(G)$ consisting of all finite quotients of G together with all the epimorphisms, which is an inverse system whose inverse limit is G.

$$G = \varprojlim G/N,$$

where $N \trianglelefteq G$ is of finite index.

Chatzidakis considered such profinite system as an L-structure for suitable L.

Profinite systems

Let G be a profinite group. We associate a *profinite system* $S(G)$ consisting of all finite quotients of G together with all the epimorphisms, which is an inverse system whose inverse limit is G.

$$G = \varprojlim G/N,$$

where $N \trianglelefteq G$ is of finite index.

Chatzidakis considered such profinite system as an L-structure for suitable L.

L-structures

Let L be the language $\{\leq, C, P, 1\}$. The structure on $S(G)$ is defined as follows:

- The universe is $\{gN \mid N \trianglelefteq G \text{ is of finite index, and } g \in G\}$
- $gN \leq hM$ iff $N \subseteq M$
- $1 = gG$
- $P(g_1N_1, g_2N_2, g_3N_3)$ iff $N_1 = N_2 = N_3$ and $g_1g_2N_1 = g_3N_3$
- $C(gN, hM)$ iff $N \subseteq M$ and $gM = hM$

Basically, P is the group multiplication on the finite quotients G/N, C is the group epimorphisms π_{MN}, and 1 is the trivial quotient of G.

Many-sorted structures

The class of profinite systems $S(G)$ is not an elementary class. Further, Chatzidakis considered profinite systems as ω-sorted structures, where she view L as a many-sorted language indexed by the positive integers. She defines that gN is of sort n iff $|G/N| \leq n$. As many-sorted structures, the class of profinite systems is an elementary class.

Duality

Let $S(G)$ and $S(H)$ are two profinite systems where G and H are profinite groups. Then every embedding

$$\varphi\colon S(G) \to S(H)$$

induces an epimorphism

$$\hat{\varphi}\colon H \to G.$$

Also, every epimorphism

$$\psi\colon H \to G$$

induces an embedding

$$\check{\psi}\colon S(G) \to S(H).$$

Proposition

The class of finite profinite systems is a Fraïssé class, and its Fraïssé limit is $S(\mathbb{F})$.

Theorem (Cherlin-van den Dries-Macintyre, Chatzidakis)

The theory of $S(\mathbb{F})$ is ω-categorical, ω-stable, and $S(\mathbb{F})$ is a saturated model.

Theorem (Hodges, Hodkinson, Lascar, and Shelah)

If M is a countable ω-stable ω-categorical structure, then $\mathrm{Aut}(M)$ has the small index property. Also, $\mathrm{Aut}(M)$ is not the union of a countable chain of proper subgroups.

Corollary

$\mathrm{Aut}(\mathbb{F})$ *has the small index property.*

Remark

Not every ω-stable ω-categorical structure has ample generic automorphisms.

Theorem (Hodges, Hodkinson, Lascar, and Shelah)

If M is a countable ω-stable ω-categorical structure, then $\mathrm{Aut}(M)$ has the small index property. Also, $\mathrm{Aut}(M)$ is not the union of a countable chain of proper subgroups.

Corollary

$\mathrm{Aut}(\mathbb{F})$ *has the small index property.*

Remark

Not every ω-stable ω-categorical structure has ample generic automorphisms.

Theorem (Hodges, Hodkinson, Lascar, and Shelah)

If M is a countable ω-stable ω-categorical structure, then $\mathrm{Aut}(M)$ has the small index property. Also, $\mathrm{Aut}(M)$ is not the union of a countable chain of proper subgroups.

Corollary

$\mathrm{Aut}(\mathbb{F})$ *has the small index property.*

Remark

Not every ω-stable ω-categorical structure has ample generic automorphisms.

Future plan

- Cherlin-van den Dries-Macintyre introduced the notion of cologic. What can cologic help us to study projective Fraïssé limits? Especially, the pseudoarc?
- Give an explicit description of a generic automorphism of $\mathbb{F}$.

Future plan

- Cherlin-van den Dries-Macintyre introduced the notion of cologic. What can cologic help us to study projective Fraïssé limits? Especially, the pseudoarc?
- Give an explicit description of a generic automorphism of $\mathbb{F}$.

Thanks!!

Thanks for your attention!