第四讲：高阶递归论

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Kleene's $\mathcal{O}$ (I)

Let $\langle , \rangle : \omega \times \omega \to \omega$ be a recursive bijection. Define an arithmetical set $P \subseteq 2^\omega$ such that $x \in P$ if and only if x satisfies the following conditions:

(i) $\langle 1, 2 \rangle \in x$;

(ii) $(\forall m)(\forall n)(\langle m, n \rangle \in x \to \langle n, 2^n \rangle \in x)$;

(iii) $(\forall e)((\forall n)\Phi_e(n) \downarrow \wedge (\forall n)\langle \Phi_e(n), \Phi_e(n+1) \rangle \in x) \to (\forall n)(\langle \Phi_e(n), 3 \cdot 5^e \rangle \in x))$;

(iv) $(\forall n)(\forall m)(\forall k)(\langle n, m \rangle \in x \wedge \langle m, k \rangle \in x \to \langle n, k \rangle \in x)$.

Kleene's $\mathscr{O}$ (II)

Definition

$<_o = \bigcap_{x \in P} x$.
$\mathscr{O} = \{n \mid (\exists m)(n <_o m)\}$.

Proposition

For each $n \in \mathscr{O}$, $<_o$ is a well ordering over $\{m \mid m <_o n\}$.

ω_1^{CK}

For each $n \in \mathscr{O}$, we use $|n|$ to denote the order typer of $(\{m \mid m <_o n\}, <_o)$.

Definition

ω_1^{CK} = the least ordinal α greater than $|n|$ for every $n \in \mathscr{O}$.

Operators on $\mathscr{O}$ (I)

Let g be a a recursive function such that

$$\Phi_{g(e,m,n)}(k) = \Phi_e(m, \Phi_n(k))$$

for all e, k, m, n.
By the s-m-n Theorem, there is a recursive function h such that

$$\Phi_{h(e)}(m,n) = \begin{cases} m & n = 1, \\ 2^{\Phi_e(m,k)} & n = 2^k, \\ 3 \cdot 5^{g(e,m,k)} & n = 3 \cdot 5^k, \\ 7 & \text{Otherwise.} \end{cases}$$

By the Recursion Theorem, there is a c such that $\Phi_{h(c)} = \Phi_c$. Define

$$m +_o n = \Phi_c(m, n).$$

Operators on $\mathscr{O}$ (I)

$$m +_o n = \begin{cases} m & n = 1, \\ 2^{m +_o k} & n = 2^k, \\ 3 \cdot 5^{g(c,m,k)} & n = 3 \cdot 5^k, \\ 7 & \text{Otherwise.} \end{cases}$$

Proposition

The function $+_o$ satisfies the following properties: For all m, n,

(i) $m, n \in \mathscr{O} \Leftrightarrow m +_o n \in \mathscr{O}$.

(ii) $m, n \in \mathscr{O} \implies |m +_o n| = |m| + |n|$.

(iii) $m, n \in \mathscr{O}$ *and* $n \neq 1 \implies m <_o m +_o n$.

(iv) $m \in \mathscr{O}$ *and* $k <_o n \Leftrightarrow m +_o k <_o m +_o n$.

(v) $m \in \mathscr{O} \wedge n = k \in \mathscr{O} \Leftrightarrow m +_o n = m +_o k \in \mathscr{O}$.

Uniformity of $\mathscr{O}$ (I)

Theorem

There are two recursive functions f and g such that for all $n \in \mathscr{O}$,

(i) $W_{f(n)} = \{m \mid m <_o n\}$,

(ii) $W_{g(n)} = \{\langle i, j\rangle \mid i <_o j <_o n\}$.

Proof.

By the Recursion Theorem, there is a recursive function f so that

$$W_{f(n)} = \begin{cases} \emptyset & n = 1, \\ \{k\} \cup W_{f(k)} & n = 2^k, \\ \bigcup\{W_{f(\Phi_k(m))} \mid m \in \omega\} & n = 3 \cdot 5^k, \\ W_0 & \text{Otherwise.} \end{cases}$$

□

Uniformity of $\mathscr{O}$ (II)

Proof.

By the Recursion Theorem, there is a g so that

$$W_{g(n)} = \begin{cases} \emptyset & n = 1, \\ \{\langle i, k\rangle | i <_o k\} \cup W_{g(k)} & n = 2^k, \\ \bigcup\{W_{g(\Phi_k(m))} \mid m \in \omega\} & n = 3 \cdot 5^k, \\ W_0 & \text{Otherwise.} \end{cases}$$

□

Σ^0_1-boundedness

Proposition

There is a recursive function g such that for all e:

(i) $g(e) \in \mathcal{O} \Leftrightarrow W_e \subseteq \mathcal{O}$.

(ii) $g(e) \in \mathcal{O} \Leftrightarrow |n| < |g(e)|$ *for every* $n \in W_e$.

Proof.

Let r be a recursive function enumerating all r.e. sets. Let $g(e)$ be the sum of $\Phi_{r(e)}$. □

WF_0 and WO_0

Fix an effective enumeration $\{R_e\}_{e\in\omega}$ of r.e. binary relations over its domain which is a subset of ω.

Definition

Let $WF_0 = \{e \mid R_e$ is a well-founded partial order over its domain$\}$ and $WO_0 = \{e \mid R_e$ is a well-ordering over its domain$\}$.

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Proposition

Both WF_0 and WO_0 are Π^1_1

Proof.

$e \in WF_0$ if and only if R_e is partial order and there is no $f \in \omega^\omega$ so that $\forall n R_e(f(n+1), f(n))$.
Similarly for WO_0. □

Π^1_1-completeness (I)

Theorem

- *A is Π^1_1 if and only if there is a recursive function f so that $\forall e e \in A \leftrightarrow f(e) \in WF_0$;*
- *A is Π^1_1 if and only if there is a recursive function f so that $\forall e e \in A \leftrightarrow f(e) \in WO_0$.*

Proof.

The direction from right to left is clear.
We prove that if A is Π^1_1, then there is a recursive function f so that $e \in A \implies f(e) \in WO_0$ and $e \notin A \implies f(e) \notin WF_0$. There is a recursive relation R so that $\forall e e \notin A \leftrightarrow \exists f \forall n R(e, f \upharpoonright n)$. We define Kleene-Brouwer order over $\omega^{<\omega}$ by letting $\sigma <_{KB} \tau$ if and only if either $\tau \prec \sigma$ or σ is in the left of τ. □

Π^1_1-completeness (II)

Proof.

For any e, let $T_e = \{\sigma \in \omega^{<\omega} \mid \forall n \leq |\sigma| R(e, \sigma \upharpoonright n)\}$. Then there is a recursive function f so that $(T_e, <_{KB}) = R_{f(e)}$ is a linear order. It is clear that $e \in A \implies f(e) \in WO_0$ and $e \notin A \implies f(e) \notin WF_0$. □

So neither WF_0 nor WO_0 is Σ^1_1.

Note that T_e can be viewed as a tree and so $e \notin A$ if and only if there is an infinite path through T_e.

Π^1_1-completeness of $\mathscr{O}$

Theorem

There is a recursive function f so that $\forall e e \in WF_0 \leftrightarrow f(e) \in \mathscr{O}$.

Proof.

There is a recursive function h such that $R_{h(e,k)}(m,n) \Leftrightarrow R_e(m,n) \wedge R_e(m,k) \wedge R_e(n,k)$. By the recursion theorem, there is a recursive function p so that

$$W_{p(e)} = \begin{cases} \emptyset & R_e = \emptyset \\ \{g(p(h(e,k))) \mid k \in \omega\} & \text{Otherwise} \end{cases}$$

, where g is a recursive function as before. Let $f(e) = g(p(e))$. If $e \in WF_0$, then by induction, $f(e) \in \mathscr{O}$. If $f(e) \in \mathscr{O}$, then for any k, $g(p(h(e,k))) \in \mathscr{O}$ which recovers the well-founded relation R_e.

□

ω_1^{CK} again

Corollary

ω_1^{CK} is the least non-recursive ordinal.

Proof.

By the proof, every recursive ordinal is less than ω_1^{CK}.
If $n \in \mathscr{O}$, then $(\{m \mid m <_o n\}, <_o)$ is partial recursive relation. Define $(k, s) \prec (m, t)$ if either k is in the domain of $\{m \mid m <_o n\}$ at stage s but m is not at stage t; or else neither k is in the domain of $\{m \mid m <_o n\}$ at stage s nor m is not at stage t and $\langle k, s\rangle < \langle m, t\rangle$; or else but $k \neq m$ and $k <_o m$; or else $s < t$. Then $\prec$ is a recursive well oder over $\omega \times \omega$ and extend the order $<_o$. □

Σ^1_1-boundedness

Theorem

- *If $A \subseteq \mathscr{O}$ is Σ^1_1, then there is some $n \in \mathscr{O}$ so that $\forall m \in A|m| < |n|$.*
- *If R is a Σ^1_1 well order over ω, then the order type of R is less that ω_1^{CK}.*

Proof.

(1) Otherwise, $e \in WO_0$ if and only if there is $n \in A$ and an order preserving function f from the domain R_e to $\{m \mid m <_o n\}$. Then WO_0 would be Σ^1_1.
(2) Otherwise, $e \in WO_0$ if and only if there is n and an order embedding function f from the domain R_e to R. Then WO_0 would be Σ^1_1. □

H-sets

Definition

- The sequence $\{H_n\}_{n\in\mathcal{O}}$ is defined by transfinite induction over $\mathcal{O}$ as follows.

$$H_1 = \emptyset,$$

$$H_{2^n} = (H_n)', \text{the Turing jump of } H_n$$

$$H_{3\cdot 5^e} = \{\langle m, n\rangle | m \in H_{\Phi_e(n)}\}.$$

- A member of $\{H_n\}_{n\in\mathcal{O}}$ is called an H-set.
- A real y is *hyperarithmetic* if $y \leq_T H_n$ for some $n \in \mathcal{O}$.

Uniformity

Theorem

There is a recursive function f so that $m <_o n$ implies $H_m = \Phi^{H_n}_{f(m,n)}$.

Proof.

Note that $x = \Phi^{x'}_{e_0}$ for some fixed e_0. Then by effective transfinite induction. □

Π^0_2-singletons

Theorem

Every H-set is a Π^0_2-singleton.

Proof.

By effective transfinite induction, we may define a Π^0_2 relation $H \subset \omega \times 2^\omega$ so that for any $n \in \mathcal{O}$, H_n is the unique real x so that $H(n, x)$. □

Kleene's theorem

Theorem

$\Delta^1_1 = HYP$.

Proof.

If $x \in HYP$, then there is Π^0_2-singleton y so that $x \leq_T y$. So x is Δ^1_1. If x is Δ^1_1, then there is a recursive function f so that $\forall n n \in x \leftrightarrow f(n) \in \mathcal{O}$. Then there is some $m \in \mathcal{O}$ so that $\forall n |f(n)| < |m|$. It is sufficient to prove that $\mathcal{O}_m = \{i \in \mathcal{O} \mid |i| < |m|\}$ is hyperarithmetic. This can be done by an effective transfinite induction. □

Relativization

All the definitions and theorems can be relativized. Note that HYP^x is defined via $\mathcal{O}^x$ not $\mathcal{O}$.

Definition

x is hyperarithmetic reducible to y, or $x \leq_h y$, if there is some $n \in \mathcal{O}^y$, $x = \Phi_e^{H_n^y}$.

$x \leq_h y$ is a partial order and Π_1^1-relation.

$\hat{H}$-sets

Hyperarithmetic reduction is not a uniform reduction since it is possible that $\omega_1^x > \omega_1^{\mathrm{CK}}$.

Definition

For $n \in \mathscr{O}$, we define $\hat{H}_n^x$ as H^x-sets..

Theorem

If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then every real hyperarithmetic in x is Turing below some $\hat{H}^x$-set.

Some Π^1_1-sets.

Proposition

1. $\{\mathcal{O}\}$ *is* Π^1_1;
2. $\{x \mid \mathcal{O} \leq_h x\}$ *is* Π^1_1.

Proof.

(1). By the definition.

(2). $x \geq_h \mathcal{O}$ if and only if there is some $n \in \mathcal{O}^x$ and some e so that $\Phi_e^{H_n^x} \in \{\mathcal{O}\}$. □

Spector's unique theorem

Theorem

There is a recursive function f so that $|n| < |m|$ implies $H_n = \Phi_{f(n,m)}^{H_m}$

Proof.

By an effective transfinite induction. □

Corollary

$|n| = |m|$ *implies* $H_n \equiv_T H_m$.

A wild Π^0_1-set.

Proposition

There is a nonempty Π^0_1 set $P \subset \omega^\omega$ containing no hyperarithmetic member.

Proof.

Clearly the set $A = \{x \mid x \notin HYP\} \subset 2^\omega$ is Σ^1_1. So there is a nonempty Π^0_1 set $P \subset 2^\omega \times \omega$ so that $A = \{x \mid \exists y(x, y) \in P\}$. Then P contains no hyperarithmetic real. □

Dichotomy of Π^1_1-set

Theorem

If $A \subset \omega$ is Π^1_1, then either A is hyperarithmetic or $\mathcal{O} \leq_h A$.

Proof.

Since A is Π^1_1, there is a recursive function f so that $n \in A \leftrightarrow f(n) \in \mathcal{O}$. Let B be the range of f over A. If B is bounded, then $A \in HYP$. Otherwise, $n \in \mathcal{O}$ if and only if there is some $m \in B$ so that $|n| < |m|$. So $\mathcal{O}$ is hyperarithmetic in A. □

By the proof above, we have that $x \geq_h \mathcal{O}$ if and only if $\omega_1^x > \omega_1^{\mathrm{CK}}$.

Above ω_1^{CK}

Theorem

$x \geq_h \mathcal{O}$ if and only if $\omega_1^x > \omega_1^{\mathrm{CK}}$.

Proof.

The direction from left to right is clear.
Suppose $\omega_1^x > \omega_1^{\mathrm{CK}}$. Then fix an x-well ordering $\prec$ over ω. Then $n \in \mathcal{O}$ if and only if there is some f which is isomorphism from $<_o n$ to an initial segment of $\prec$. So $\mathcal{O}$ is Σ_1^1 in x. □

Gandy's basis

Theorem (Gandy)

If $A \subseteq \omega^\omega$ is Σ^1_1, then A contains a real x so that $x \leq_h \mathcal{O}$ and $\omega_1^x = \omega_1^{\mathrm{CK}}$.

Proof.

Let $B = \{x \oplus y \mid x \in A \wedge y \not\leq_h x\}$. Then B is a nonempty Σ^1_1 set and so contains a real $x \oplus y \leq_h \mathcal{O}$. Then $x \in A$ and $\leq_h \mathcal{O}$. Since $y \not\leq_h x$, we have that $x <_h \mathcal{O}$. Then $\omega_1^x = \omega_1^{\mathrm{CK}}$. □

Countable Σ_1^1-set

Theorem

If $A \subset \omega^\omega$ is a nonempty countable Σ_1^1-set, then every real in A is hyperarithmetic.

Proof.

Otherwise, A must have a perfect subset. □

Incomparable h-degrees

Theorem (Spector)

There are two $\leq_h$ incomparable degrees.

Proof.

By the previous results, the set $\{x \notin HYP \mid \omega_1^x = \omega_1^{\mathrm{CK}}\}$ is an uncountable Σ_1^1 set. So it has a member $x \not\leq_h \mathscr{O}$ and so $\leq_h$-incomparable with $\mathscr{O}$. $\square$

Gandy topology

Definition

Gandy topology is a topology with all the Σ^1_1-sets as basic open sets.

Theorem

If $\{U_n\}$ is a sequence dense open sets in Gandy topology, then $\bigcap_n U_n \neq \emptyset$.

Proof.

By Tree representation of Σ^1_1-sets. □

Kreisel's basis theorem

Theorem (Kreisel)

If x is nonhyparithmetic and A is a nonempty Σ^1_1 set, then there is some $g \in A$ so that $g \not\geq_h x$.

Proof.

By the property of Gandy-topology. □

A Π^1_1-path through $\mathcal{O}$

Theorem (Spector)

There is a Π^1_1-path through $\mathcal{O}$.

Proof.

Let $\mathcal{O}^*$ be the intersection of all hyperarithmetic sets with the property defining $\mathcal{O}$. Then $\mathcal{O}^*$ is a Σ^1_1-set. Let $n \in \mathcal{O}^* \setminus \mathcal{O}$, then $\{m \in \mathcal{O}^* \mid m <_{o^*} n\}$ is a path through $\mathcal{O}$. □

Some open questions

Question

- *For any countable set of hyperarithmetic degrees, must there exist an minimal upper bound ?*
- *For any real* x *and* $\Delta^1_1(x)$ *set* A *of reals which contains a member* $y >_h x$*, is it true that for any real* z*, there is some* $z_0 \in A$ *so that* $z_0 \geq_h z$*?*

Exercise

1. Every Δ^0_2-real is a Π^0_2-singleton.
2. Every Π^0_2-singleton in 2^ω is Turing equivalent to a Π^0_1-singleton in ω^ω.
3. A real x is Δ^1_1 if and only if it is a Σ^1_1-singleton.
4. If A is a nonempty Σ^1_1-set, then it contains member x with $\mathcal{O}^x \leq_T \mathcal{O}$.

Further readings

Recursion Theory: Computational Aspects of Definability, Chong and Yu, 2015.

Higher recursion theory, Gerald Sacks, 1990.