# 第二讲：递归可枚举度理论

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# Post's problem and program

Keep in mind, the theory of r.e. degrees is nonsense. The treasure of the area is the technique.

**Question (Post)**

*Whether there exists an incomplete nonrecursive r.e. set?*

**Question (Post)**

*Finding out a property of an r.e. set A which guarantees A to be nonrecursive and incomplete.*

# Simple and immune sets

**Definition**

- An infinite set $x \subset \omega$ is *immune* if it has no infinite recursive subset.
- An r.e. set $x$ is *simple* if $\omega \setminus x$ is immune.

Note that a simple set is not recursive.
Why simple?

# A simple complete set

**Theorem**

*Every non-zero r.e. Turing degree contains a simple set.*

**Proof.**

Let $A$ be a non-recursive infinite r.e. set and $f$ be a recursive 1-1 function so that $A$ is the range of $f$. Define $S = \{s \mid \exists t > s f(t) < f(s)\}$. Clearly $S$ is r.e.
If $B$ is an infinite recursive set for which $B \cap S = \emptyset$. Then for any $n$, $n \notin A$ if and only if $\exists s \notin B \forall m \leq s(n < f(s) \wedge n \neq f(m))$. □

# A simple incomplete set

**Theorem**

*There is a simple set $A$ so that $x <_T \emptyset'$.*

We need to construct an r.e. set $A$ to meet the following requirements:

$$P_e : |W_e| = \infty \implies W_e \cap A \neq \emptyset;$$

$$N_e : \forall s \exists t \geq s \Phi_e(A_t)(e)[t] \downarrow \to \Phi_e(A)(e) \downarrow .$$

The priority is $P_i < N_i < P_{i+1}$.

For $N_e$, we need to set up a restriction to protect our predication to which all the requirements has the lower priority have to respect.

Every requirement can be injured at most finitely many times.

Then $A$ is a low simple set.

# Friedberg–Muchnik's solution

**Theorem (Friedberg; Muchnik)**

*There are two r.e. sets $A$ and $B$ which are Turing incomparable.*

**Proof.**

We need to construct an r.e. sets $A$ and $B$ to meet the following requirements:

$$P_e : A \neq \Phi_e(B);$$

$$Q_e : B \neq \Phi_e(A).$$

The priority is $P_e < Q_e < P_{e+1}$. □

# Sacks's splitting theorem.

**Theorem (Sacks)**

*For any nonrecursive r.e. set $A$ and $C$, there are two r.e. sets $B_0$ and $B_1$ for which $A = B_0 \cup B_1$, $B_0 \cap B_1 = \emptyset$ and $C \not\leq_T B_0, B_1$.*

**Proof.**

We need to construct an r.e. splitting $B$ and $B_1$ of $A$ to meet the following requirements:

$$P_e : C \neq \Phi_e(B_0);$$

$$Q_e : C \neq \Phi_e(B_1).$$

The priority is $P_e < Q_e < P_{e+1}$. □

Actually both $B_0$ and $B_1$ are low.

# Downward density.

**Theorem (Sacks)**

*For any non-recursive r.e. set $A$, there is a nonrecursive r.e. set $B <_T A$.*

We need to construct an r.e. set $B$ to meet the following requirements:

$$P_e : |W_e| = \infty \implies W_e \cap B \neq \emptyset;$$

$$Q_e : A \neq \Phi_e(B).$$

The priority is $P_e < Q_e < P_{e+1}$.
We use a permitting argument. If some $n$ enters into $B$ at stage $s$, then there must be some stage $t > s$ and $m \leq n$ so that $A_t(m) \neq A_{t+1}(m)$.

# Tree argument

We start from $\lambda$, and for each requirement, we attribute several outcomes. The outcomes are the nodes of the trees. There is an order for the outcomes.
The node in the tree respect the restriction of set by the node in the left and higher.
The restriction of a node will be initialized once a node of the left is visited.
The left-most infinite path for which is visited infinitely many times is called true-path.

# Shoenfield conjecture

**Conjecture (Shoenfield)**

*Every finite upper semi-lattice embedded into r.e. degrees can be extended.*

# Lachlan's theorem (I)

**Theorem**

*There are two non recursive r.e. sets $A_0$ and $A_1$ so that every set $B \leq_T A_0, A_1$ is recursive.*

**Proof.**

We need to construct r.e. sets $A_0$ and $A_1$ to meet the following requirements:

$$P_e^i : |W_e| = \infty \implies W_e \cap A_i \neq \emptyset;$$

$$N_{i,j} : \Phi_i(A_0) = \Phi_j(A_1) = B \implies B \text{ is recursive.}$$

The priority is $P_e^0 < P_e^1 < Q_e < N_{\langle i,j \rangle} < P_{e+1}^0$.

□

# Lachlan's theorem (II)

For $P_e^i$, there are two finite outcomes: $0 > 1$. 0 denotes that $W_e$ is finite; and 1 denotes $W_e$ is infinite.
For $N_{i,j}$, there is a finite outcome $f$ and an infinite outcome $\infty$. $\infty < f$. $f$ denotes $\lim_s l_s^{i,j} < \infty$; and $\infty$ denotes that $\overline{\lim}_s l_s^{i,j} = \infty$, where $l_s^{i,j} = \min\{l \mid \Phi_i(A_0)(l)(s) \neq \Phi_1(A_0)(l)(s)\}$.
For the outcome $f$, the restriction is to protect $l^{i,j}$. For the outcome $\infty$, no restriction.

# References

1. Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets (Perspectives in Mathematical Logic), Robert Soare.

# Exercise

1. Every non-zero Turing degree contains an immune set.
2. The $\Sigma_1$ theory in the partial ordering language of r.e. degrees is decidable.
3. For any non-zero r.e. degree $\mathbf{a}$, there are two incomparable r.e. degrees below it.
4. $(\mathbf{R}, <) \not\equiv (\mathbf{D}_{\leq \mathbf{0}'}, <)$, which can be witnessed by a $\Sigma_2$-sentence but not $\Sigma_1$.