第一讲：递归论基础

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Baire and Cantor space

Baire space is a topology space over ω^ω and Cantor space is over 2^ω. They both are "Euclid spaces" for logicians. Note that Cantor space is compact (Konig lemma).

We identify $x \subseteq \omega$ with its characteristic function $x \in 2^\omega$.

Lebesgue measure over 2^ω.

$x \oplus y(2n) = x(n)$ and $x \oplus y(2n+1) = y(n)$.

Arithmetical hierarchy

Definition

1. The set A of bounded formulas in the arithmetic language is the smallest set of the formulas containing the Boolean combination of all atomic formulas and so that
   - If $\varphi \in A$, then $\exists u < v \varphi \in A$;
   - If $\varphi \in A$, then $\neg\varphi \in A$.
2. A formula φ is Σ_0^0 (or Σ_0^0) if it is bounded.
3. A formula φ is Σ_{n+1}^0 if there is a Π_n^0 formula ψ so that φ is of the form $\exists \overrightarrow{u} \psi$.
4. A formula φ is Π_{n+1}^0 if there is a Σ_n^0 formula ψ so that φ is of the form $\forall \overrightarrow{u} \psi$.

Second order arithmetical hierarchy

Definition

1. A formula φ is Σ^1_1 if there is a Π^0_1 formula ψ so that φ is of the form $\exists f \in \omega^\omega \forall n \psi(f \upharpoonright n, m)$.
2. A formula φ is Σ_{n+1} if there is a Π^1_n formula ψ so that φ is of the form $\exists \overrightarrow{u} \psi$.
3. A formula φ is Π_{n+1} if there is a Σ^1_n formula ψ so that φ is of the form $\forall \overrightarrow{u} \psi$.

Definable sets

Definition

Given a collection of formulas Γ, a set $x \subseteq \omega$ (or $A \subseteq \omega^\omega$) is called Γ-*definable*, if there is a formula $\varphi \in \Gamma$ so that $x = \{n \mid \varphi(n)\}$ (or $A = \{x \mid \varphi(x)\}$).

A set $x \subset \omega$ is Δ^0_n (or Δ^1_n) if it is both Σ^0_n (Σ^1_n) and Π^0_n (Π^1_n).

Some basic facts

Proposition

- *All the arithmetical definable sets are both* Σ_1^1 *and* Π_1^1*-definable.*
- *The hierarchy does not collapse.*

Recursive functions

Definition

A function $f\colon \omega^\omega \to \omega^\omega$ is (partial-) recursive if there is a (partial-) recursive function $\Phi : \omega^{<\omega} \to \omega^{<\omega}$ so that

- $\sigma \preceq \tau$ implies $\Phi(\sigma) \preceq \Phi(\tau)$; and
- $\forall x \bigcup_n \Phi(x \upharpoonright n) = f(x)$.

The definition can be generalized to multiple dimensions in an obvious way.

Turing reduction and Turing degrees

Definition

We say that $x \leq_T y$ for $x, y \in 2^\omega$, or x is Turing reducible to y (or x is y-recursive), if there is a partial recursive function Φ so that $\Phi(y) = x$. If Φ is recursive, then we say that $x \leq_{tt} y$, or truth table reducible to y.

We say that $x \equiv_T y$ if $x \leq_T y$ and $y \leq_T x$. The equivalence class $[x]_T$ is called as a Turing degree.
We use $\mathbf{0}$ to denote the Turing degree of recursive sets and $\mathbf{0}'$ to denote the Turing degree of Halting problem.

Basic facts on Turing degrees

Proposition

- $\mathbf{x} \vee \mathbf{y}$, *the least upper bound of Turing degrees* $\mathbf{x}$ *and* $\mathbf{y}$, *is the degree of* $x \oplus y$.
- *Any countable set of Turing degrees has an upper bound.*

Δ^0_2-reals (I)

Proposition

TFAE

(1) $x \subseteq \omega$ is Δ^0_2 ;

(2) $x \leq_T \emptyset'$;

(3) There is a recursive function $f\colon \omega^2 \to 2$ so that $\forall n x(n) = \lim_s f(s, n)$.

Proof.

(1) implies (2): By the fact that every r.e. set is Turing reducible to $\emptyset'$.

□

Δ^0_2-reals (II)

(2) implies (3). Then there is a partial recursive function Φ so that for every n, there is some m_n so that $\Phi(\emptyset' \upharpoonright m_n) = x \upharpoonright n$. Define $f(s, n)$ to be the value of $\Phi(\emptyset'[s] \upharpoonright s)(n)$, where $\emptyset'[s]$ is $\emptyset'$ enumerated at stage s.
(3) implies (1). x is Σ^0_2 since $x(n) = 1$ if and only if $\exists s \forall t \geq s f(t, n) = 1$. x is also Π^0_2 since $x(n) = 1$ if and only if $\forall s \exists t \geq s f(t, n) = 1$.

Relativized heirarchy

Both the arithmetical and second order arithmetical hierarchy can be relativized.

x' is halting problem relative to x. So $x <_T x'$.

A set of reals is open if and only if it is $\Sigma^0_1(x)$ for some x.

All the arithmetical definable sets are Borel.

Proposition

Every continuous function $f : \omega^\omega \to \omega^\omega$ is a recursive function relativized to some x.

Relativization principle

If a result is proved, then the relativization follows.

For example, a set of numbers is recursive if it is Σ^0_1 and Π^0_1. Then for any x, a set of numbers is x-recursive if it is $\Sigma^0_1(x)$ and $\Pi^0_1(x)$.
And there is an r.e. set $U \subseteq \omega$ so that for any r.e. set W, there is some e so that $W = \{n \mid \langle e, n\rangle \in U\}$. Then this also holds for any x-r.e. set.

Lowness and highness

Definition

- A real x is low if $x' \equiv_T \emptyset'$; A real x is high if $x' \equiv_T \emptyset''$.
- A real x is generalized low if $x' \equiv_T x \oplus \emptyset'$.

Upward closure.

Theorem (Sacks; de Leeuw, Moore, Shannon, and Shapiro)

Given a non-recursive real $x \in 2^\omega$, the set $\{y \in 2^\omega \mid x \leq_T y\}$ is null.

Proof.

The set $\{y \in 2^\omega \mid x \leq_T y\}$ is Borel and so must be measurable. Further more $\{y \in 2^\omega \mid x \leq_T y\} = \bigcup_e A_e$, where $A_e = \{y \mid \Phi_e(y) = x\}$. Since each A_e is Borel, there must be some e so that A_e has positive measure. By Lebesgue density, there must be some σ so that $A_e \cap [\sigma] = \{y \succ \sigma \mid \Phi_e(y) = x\}$ has measure greater than $\frac{3}{4}2^{-|\sigma|}$. To decide $x(n)$, find sufficiently many τ's so that $\{\tau \succ \sigma \mid \Phi_e(\tau)(n)\}$ share a common value over n and has measure greater than $2^{-|\sigma|-1}$. Then the value is $x(n)$. □

Kleene-Post theorem (I)

Theorem (Kleene-Post)

There are two reals x and y Turing reducible to $\emptyset'$ so that they are Turing incomparable.

We $\emptyset'$-recursively build two sequences $\sigma_0 \prec \sigma_1 \cdots$ and $\tau_0 \prec \tau_1 \cdots$ so that $x = \bigcup_s \sigma_s$ and $y = \bigcup_s \tau_s$ with the required property as follows.

At stage $s+1$,

Kleene-Post theorem (II)

Substep (1). If there are $\sigma', \sigma'' \succ \sigma_s$ and some $n_0 > |\tau_s|$ so that $\Phi_s(\sigma')(n_0) \neq \Phi_s(\sigma'')(n_0)$. Let $\tau' \succ \tau_s$ so that $\Phi_s(\sigma')(n_0) \neq \tau'(n_0)$. Go to next step. Otherwise, go to next substep.
Substep (2). If there are $\tau'', \tau''' \succ \tau'$ and some $n_1 > |\sigma'|$ so that $\Phi_s(\tau'')(n_1) \neq \Phi_s(\tau''')(n_1)$. Let $\tau_{s+1} = \tau''$ and $\sigma_{s+1} \succ \sigma'$ so that $\Phi_s(\tau_{s+1})(n_1) \neq \sigma_{s+1}(n_1)$. Otherwise, go to next step.

Jump inversion theorem

Theorem (Friedberg)

For any real $x \geq_T \emptyset'$, there is a real g so that $g' \equiv_T g \oplus \emptyset' \equiv_T x$.

Proof.

We build a sequence $\sigma_0 \prec \sigma_1 \cdots$ as follows:

At step $s+1$, check whether there is some $\tau \succ \sigma_s^\frown x(s)$ so that $\Phi_s(\tau)(s)\downarrow$.

If yes, let σ_{s+1} be the shortest such τ. Otherwise, let $\sigma_{s+1} = \sigma_s^\frown x(s)$.

Let $g = \bigcup_s \sigma_s$.

The construction can be decoded by both $g \oplus \emptyset'$ and x. So $g' \equiv_T g \oplus \emptyset' \equiv_T x$. □

On Π^0_1-sets

Proposition

Given a Π^0_1-set $A \subset \omega^\omega$, there is a recursive tree $T \subseteq \omega^{<\omega}$ so that $A = [T] = \{x \in \omega^\omega \mid \forall n x \upharpoonright n \in T\}$.

Proof.

The complement of A is a Σ^0_1 set.
At stage 0, $T_s = 2^{<\omega}$.
At stage $s+1$, if we find some σ enters the complement of A, then cut all of $\tau \succeq \sigma$ with length at least $s+1$ from T_s.
Then $T = \lim_s T_s$ is recursive and $[T] = A$. □

So, by compactness, if x is an isolated point in a Π^0_1 set $A \subset 2^\omega$, then x is recursive.

Kreisel's basis theorem

Theorem (Kreisel)

If $A \subseteq 2^\omega$ is a nonempty Π^0_1-set, then there must be some real $x \in A$ with $x \leq_T \emptyset'$.

Proof.

By the compactness, define x to be the leftmost infinite path through the recursive tree. □

Low basis theorem (I)

Theorem (Jockusch and Soare)

If $A \subseteq 2^\omega$ is a nonempty Π^0_1-set, then there must be some real $x \in A$ with $x' \leq_T \emptyset'$.

We $\emptyset'$-recursively build a sequence recursive infinite trees $T = T_0 \supseteq T_1 \cdots$ and finite segments $\sigma_0 \prec \sigma_1 \cdots$ from T_s as follows:
At stage $s+1$, check whether there are infinitely many $\tau \succ \sigma_s$ in T_s so that $\Phi_s(\tau)(s)$ is undefined.
Case (1). Yes. Let $\sigma_{s+1} = \sigma_s$ and
$T_{s+1} = \{\tau \mid \tau$ is comparable with $\sigma_s \wedge \Phi_s(\tau)(s)\uparrow\}$.

Low basis theorem (II)

Case (2). No. By compactness, find some n so that for each $\sigma \in T_s$ with length n extending σ_s, $\Phi_s(\sigma)(s)\downarrow$. $\emptyset'$-recursively find $\sigma_{s+1} \in T_s$ extending with length n so that $T_{s+1} = \{\tau \succ \sigma_{s+1} \mid \tau \in T_s\}$ is infinite.

$x = \bigcup_s \sigma_s$ is an infinite path through T with required property.

Domination property (1)

Given two $f, g \in \omega^\omega$, we say that f dominates g if $\exists m \forall n \geq m f(n) \geq g(n)$.

Theorem (Kucera)

If x is a nonrecursive Δ^0_2 real, then there is function $f \leq_T x$ that is not dominated by any recursive function.

Since x is Δ^0_2, there is a recursive function $\hat{x}$ so that $\forall n \lim_s \hat{x}(s, n) = x(n)$. Define $f(n)$ to be least stage s so that for each $m \leq n$, $\hat{x}(s, m) = x(m)$.

f is x-recursive. Suppose that f is dominated by a recursive increasing function g. Define $g^{(i+1)} = g(g^{(i)})$ for each $i \in \omega$.

Domination property (2)

Then for any n, there must be some i so that for any $m \leq n$ and $\forall s \in [g^{(i)}(m), g^{(i+1)}(m))\hat{x}(s, m) = \hat{x}(s+1, m)$. Then $\hat{x}(g^{(i)}(m), m) = x(m)$ for each $m \leq n$.
Then x must be recursive, a contradiction.

HiF basis theorem (I)

A real x is hyperimmune-free if each function $f \leq_T x$ is dominated by a recursive function.

Theorem (Jockusch and Soare)

Given a nonempty Π^0_1 set A, there is a hyperimmune-free real $x \in A$.

We build a sequence recursive infinite trees $T = T_0 \supseteq T_1 \cdots$ and finite segments $\sigma_0 \prec \sigma_1 \cdots$ from T_s as follows:
At stage $s+1$, check whether there is some n so that there are infinitely many $\tau \succ \sigma_s$ in T_s so that $\Phi_s(\tau)(n)$ is undefined.
Case (1). Yes. Let $T_{s+1} = \{\tau \mid \tau \text{ is comparable with } \sigma_s \wedge \Phi_s(\tau)(n) \uparrow\}$ and $\sigma_{s+1} \succ \sigma_s$ has infinitely many extensions in T_{s+1} with length at least $s+1$.
Case (2). No. For each n, recursively find some m so that for each $\sigma \in T_s$ with length m, $\Phi_s(\sigma)(n) \downarrow$. Let $g_s(n)$ be the sum of all of the values. Define $T_{s+1} = T_s$ and $\sigma_{s+1} \succ \sigma_s$ has infinitely many

HiF basis theorem (II)

Case (2). No. For each n, recursively find some m so that for each $\sigma \in T_s$ with length m, $\Phi_s(\sigma)(n)\downarrow$. Let $g_s(n)$ be the sum of all of the values. Define $T_{s+1} = T_s$ and $\sigma_{s+1} \succ \sigma_s$ has infinitely many extensions in T_{s+1} with length at least $s+1$.
Let $x = \bigcup_s \sigma_s$ be an infinite path through T.

Recursion theorem and *DNR*–functions

Theorem (Kleene)

For any x-recursive function $f:\omega \to \omega$, there is some e so that $\Phi_e^x = \Phi_{f(e)}^x$, where $\{\Phi_e^x\}_e$ is an x-effective enumeration of x-partial recursive functions.

Definition

A function f is diagonally non-recursive (or *DNR*-) function if there is no e so that $\Phi_e = \Phi_{f(e)}$.

Clearly there is a *DNR*-function.

DNR–function

Proposition (Jockusch)

Given a real x, there is a DNR-function $f \leq_T x$ if and only if there is some $g \leq_T x$ so that $\forall e g(e) \neq \Phi_e(e)$.

Proof.

Given f be a *DNR*-function. Let d be a recursive function so that $\Phi_{d(e)} = \Phi_{\Phi_e(e)}$. Then let $g = f(d)$. Then for all e, $\Phi_{f(d(e))} \neq \Phi_{d(e)} = \Phi_{\Phi_e(e)}$. So $g(e) \neq \Phi_e(e)$.
Let θ be a partial recursive function so that $\theta(e) \in W_e$ if $W_e \neq \emptyset$. Let q be a recursive function so that $\Phi_{q(e)} = \theta(e)$ and $W_{f(e)} = \{g(q(e))\}$. We prove that $W_{f(e)} \neq W_e$ for all e. Otherwise, $W_{f(e_0)} = W_{e_0}$. Then $\Phi_{q(e_0)}(q(e_0)) = \theta(e_0) = g(q(e_0))$.

□

Some properties of DNR–functions

Theorem (Arslanov)

If x is an r.e. real and is of DNR-degree, then $x \equiv_T \emptyset'$.

Lemma

The set $\{x \in 2^\omega \mid x \in DNR\}$ is Π^0_1.

So there is a hif DNR real and a low DNR-real.

Minimal degrees (I)

Theorem (Spector)

There is a minimal degree $\mathbf{x}$*. I.e. a non-recursive real* x *so that for any real* $y <_T x$*,* y *is recursive.*

We build a sequence recursive perfect tree $T_0 = 2^{<\omega} \supseteq T_1 \supset \cdots$ with roots $\sigma_0 = \emptyset \prec \sigma_1 \prec \cdots$ respectively as follows:

At stage $s+1$,

Case(1). There is some $\tau \succeq \sigma_s$ in T_s and some n so that $\forall \nu \succeq \tau(\nu \in T_s \to \Phi_s(\nu)(n)\uparrow)$. Let $\sigma_{s+1} = \tau$ and $T_{s+1} = \{\nu \mid \nu \text{ is comparable with } \sigma_{s+1} \wedge \nu \in T_s\}$.

Minimal degrees (II)

Case(2). There is some $\tau \succeq \sigma_s$ in T_s and some n so that $\forall m \geq n \forall \nu_0 \succeq \tau \forall \nu_1 \succeq \tau (\nu_0 \in T_s \wedge \nu_1 \in T_s \wedge \Phi_s(\nu_0)(m) \downarrow \wedge \Phi_s(\nu_1)(m) \downarrow \rightarrow \Phi_s(\nu_0)(m) = \Phi_s(\nu_0)(m))$. Let $\sigma_{s+1} = \tau$. Then one can build a recursive perfect subtree T_{s+1} of T_s so that for any infinite path z through T_{s+1}, $\Phi_s(x)(m)$ is defined everywhere. And actually they all must be recursive.

Case (3). Otherwise. Then one can build a recursive perfect subtree T_{s+1} of T_s so that for different infinite paths z_0, z_1 through T_{s+1}, both $\Phi_s(z_0)$ and $\Phi_s(z_0)$ are defined everywhere but they are different functions. So for every $z \in [T_s]$, $z \leq_T \Phi_s(z)$. Let σ_{s+1} be the root of T_{s+1}.

Set $x = \bigcup_s \sigma_s$.

References

1. Turing Computability: Theory and Applications, Robert Soare.

Exercise

1. $x \leq_{tt} y$ if and only if there is a recursive function Φ and a recursive function $f : \omega \to \omega$ so that $\forall n \Phi(y \upharpoonright f(n)) = x \upharpoonright n$.
2. Given a non-recursive real $x \in 2^{\omega}$, the set $\{y \in 2^{\omega} \mid x \leq_T y\}$ is meager.
3. Low basis theorem fails for a Π^0_1 subset of ω^{ω}.
4. There is a hyperimmune-free *DNR*-real $x \leq_T \emptyset''$.
5. If $x \in 2^{\omega}$ is *DNR*, then every nonempty Π^0_1 subset of 2^{ω} contains a member Turing below x.
6. How many minimal degrees are there?
7. Given a countable set of reals $\{x_n\}_{n \in \omega}$, there is a some real z Turing above all of them but there is no real $y <_T z$ so that y is Turing above all of them.