Strengthening the Knaster property induces the forcing axiom

Yinhe Peng

Academy of Mathematics and Systems Science
Chinese Academy of Sciences

2025 Fudan Conference on Mathematical Logic, August 8 2025

Outline

Partially ordered sets

A poset (partially ordered set) $\mathcal{P}$ is ccc if every uncountable subset of $\mathcal{P}$ contains 2 elements with a common lower bound.

Partially ordered sets

A poset (partially ordered set) $\mathcal{P}$ is ccc if every uncountable subset of $\mathcal{P}$ contains 2 elements with a common lower bound.

Cohen forcing: $Fn(\omega, 2)$ is the collection of finite partial maps from ω to 2, ordered by extension as functions.

The product of κ many Cohen posets with finite support: $Fn(\kappa \times \omega, 2)$ or $Fn(\kappa, 2)$.

Partially ordered sets

A poset (partially ordered set) $\mathcal{P}$ is ccc if every uncountable subset of $\mathcal{P}$ contains 2 elements with a common lower bound.

Cohen forcing: $Fn(\omega, 2)$ is the collection of finite partial maps from ω to 2, ordered by extension as functions.

The product of κ many Cohen posets with finite support: $Fn(\kappa \times \omega, 2)$ or $Fn(\kappa, 2)$.

- ccc forcing preserves cardinals.
- $Fn(\kappa, 2)$ is ccc.

(Cohen) $L^{Fn(\kappa,2)} \models 2^\omega = \kappa$ for $\kappa \geq \omega_1$.

Productivity

In general, ccc is not productive: If S is a Suslin tree, then S is ccc and S^2 is not ccc.

A poset (partially ordered set) $\mathcal{P}$ has property K_n (K for Knaster), for $n \geq 2$, if every uncountable subset of $\mathcal{P}$ has an uncountable subset that is n-linked.

A subset is n-linked if every n-element subset has a common lower bound.

Property K_n is stronger than ccc and is productive.

$Fn(\kappa, 2)$ has property K_n.

Stronger forcing properties

$\mathcal{P}$ has *precaliber* ω_1 if every uncountable subset of $\mathcal{P}$ has an uncountable centered subset. A subset X of $\mathcal{P}$ is *centered* if every finite subset of X has a common lower bound.

$\mathcal{P}$ is *σ-n-linked* if $\mathcal{P}$ is a countable union of n-linked subsets.

$\mathcal{P}$ is *σ-centered* if $\mathcal{P}$ is a countable union of centered subsets.

Stronger forcing properties

$\mathcal{P}$ has *precaliber* ω_1 if every uncountable subset of $\mathcal{P}$ has an uncountable centered subset. A subset X of $\mathcal{P}$ is *centered* if every finite subset of X has a common lower bound.

$\mathcal{P}$ is σ-*n-linked* if $\mathcal{P}$ is a countable union of n-linked subsets.

$\mathcal{P}$ is σ-*centered* if $\mathcal{P}$ is a countable union of centered subsets.

- Property K_n (precaliber ω_1) is closed under finite support product.
- σ-n-linked (σ-centered) is closed under finite support product of length $\leq 2^\omega$.

$Fn(\kappa, 2)$ has precaliber ω_1, and is σ-centered if $\kappa \leq 2^\omega$.

Forcing properties

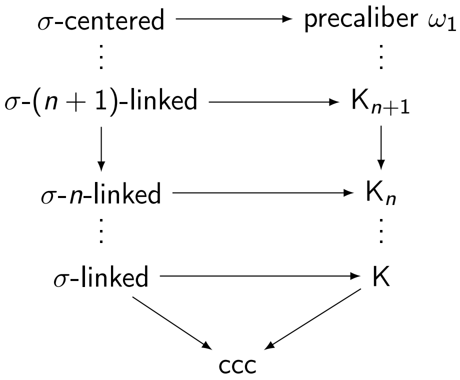

Figure: Forcing properties

Arrows denote implications. No other implication is ZFC provable except for properties connected by (combinations of) arrows.

Consequences of MA_{ω_1}

Assume MA_{ω_1}. The following statements hold.

(M1) The union of ω_1 meager (null) sets is meager (null).

Consequences of MA_{ω_1}

Assume MA_{ω_1}. The following statements hold.

(M1) The union of ω_1 meager (null) sets is meager (null).

(M2) (Juhasz) If X is an infinite ccc compact Hausdorff space, then X cannot be covered by ω_1 nowhere dense sets.

Consequences of MA_{ω_1}

Assume MA_{ω_1}. The following statements hold.

(M1) The union of ω_1 meager (null) sets is meager (null).

(M2) (Juhasz) If X is an infinite ccc compact Hausdorff space, then X cannot be covered by ω_1 nowhere dense sets.

(M3) (Kunen) Every $X \subseteq \omega_2^2$ equals $\bigcap_{n<\omega} \bigcup_{m>n} C_m \times D_m$ for some $C_n, D_n \subseteq \omega_2$.

Consequences of MA_{ω_1}

Assume MA_{ω_1}. The following statements hold.

(M1) The union of ω_1 meager (null) sets is meager (null).

(M2) (Juhasz) If X is an infinite ccc compact Hausdorff space, then X cannot be covered by ω_1 nowhere dense sets.

(M3) (Kunen) Every $X \subseteq \omega_2^2$ equals $\bigcap_{n<\omega} \bigcup_{m>n} C_m \times D_m$ for some $C_n, D_n \subseteq \omega_2$.

(M4) (Rowbottom) ccc is productive.

Consequences of MA_{ω_1}

Assume MA_{ω_1}. The following statements hold.

(M1) The union of ω_1 meager (null) sets is meager (null).

(M2) (Juhasz) If X is an infinite ccc compact Hausdorff space, then X cannot be covered by ω_1 nowhere dense sets.

(M3) (Kunen) Every $X \subseteq \omega_2^2$ equals $\bigcap_{n<\omega} \bigcup_{m>n} C_m \times D_m$ for some $C_n, D_n \subseteq \omega_2$.

(M4) (Rowbottom) ccc is productive.

(M5) Every ccc poset has precaliber ω_1 and is σ-centered if the poset has size $\leq \omega_1$.

A space is ccc if it has no uncountable pairwise disjoint open subsets.

(M2) $\leftrightarrow$ (M5) $\leftrightarrow$ MA_{ω_1}.

Forcing axioms

$\mathcal{P}$ has property FA_{ω_1} (or say $\mathrm{FA}_{\omega_1}(\mathcal{P})$) if

- For every collection $\mathscr{D}$ of ω_1 many dense subsets of $\mathcal{P}$, there is a directed subset of $\mathcal{P}$ meeting every member of $\mathscr{D}$.

For a property Φ stronger than ccc, $\mathrm{MA}_{\omega_1}(\Phi)$ is the assertion that if $\mathcal{P}$ has property Φ, then $\mathcal{P}$ has property FA_{ω_1}.

MA_{ω_1} is $\mathrm{MA}_{\omega_1}(\text{ccc})$.

For posets of size $\leq \omega_1$

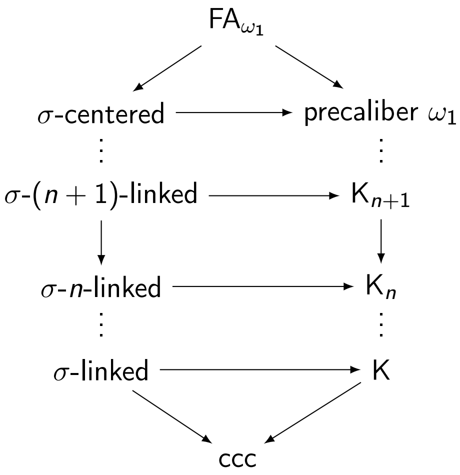

Figure: Forcing properties

How different are these properties

Theorem 1 (Todorcevic-Velickovic)

If every ccc poset has precaliber ω_1, then MA_{ω_1} holds.

How different are these properties

Theorem 1 (Todorcevic-Velickovic)

If every ccc poset has precaliber ω_1, then MA_{ω_1} holds.

Definition 1

$\mathrm{P}(\Phi \to \Psi)$ *denotes the property that every poset with property* Φ *has property* Ψ.

How different are these properties

Theorem 1 (Todorcevic-Velickovic)

If every ccc poset has precaliber ω_1, then MA_{ω_1} holds.

Definition 1

$\mathrm{P}(\Phi \to \Psi)$ denotes the property that every poset with property Φ has property Ψ.

$\mathrm{P}(\Phi \to \mathrm{FA}_{\omega_1})$ is the same as $\mathrm{MA}_{\omega_1}(\Phi)$.

(Todorcevic-Velickovic) P(ccc$\to$precaliber ω_1) is equivalent to P(ccc$\to$FA$_{\omega_1}$).

Natural questions

Question 1

Does $\mathrm{P}(\text{ccc}\rightarrow\mathrm{K}_n)$ (denoted by $\mathscr{K}_n$ in the literature) imply MA_{ω_1} for some $n \geq 2$?

Natural questions

Question 1

Does $\mathrm{P}(\text{ccc}{\to}\mathrm{K}_n)$ (denoted by $\mathscr{K}_n$ in the literature) imply MA_{ω_1} for some $n \geq 2$?

Question 2

For $n \geq 2$, are $\mathrm{P}(\mathrm{K}_n \to \mathrm{K}_m)$ all different for different $m > n$?

$\mathscr{K}_3$

Consequences of $\mathscr{K}_3$:

1. (Todorcevic) $\mathfrak{b} > \omega_1$.
2. (Todorcevic) Every Aronszajn tree is special.
3. (Todorcevic-Velickovic) Every ccc poset of size ω_1 is σ-linked.
4. (Todorcevic-Velickovic) $2^\omega = 2^{\omega_1}$.
5. (Moore) $add(\mathscr{N}) > \omega_1$.

Properties (1) and (2) above are consequences of the weaker property $\mathscr{K}_2$. However, even some weak consequences of MA_{ω_1}, e.g., every ladder system can be uniformized, seems to require a possibly stronger property $\mathscr{K}_4$.

P(ccc→K$_3$)

P(ccc→K$_3$) (P(ccc→precaliber ω_1)) is equivalent to

- For a coloring $c : [\omega_1]^3 \to 2$ ($c : [\omega_1]^{<\omega} \to 2$), if an uncountable 0-homogeneous subset of c can be forced by a ccc poset, then it already exists.

First attempt

An iteration method, of minimizing the damage to a strong coloring, is introduced to distinguish MA and MA(powerfully ccc).

Theorem 2 (2025)

MA(powerfully ccc) does not imply MA.

It is likely that the method can be generalized to distinguish MA_{ω_1} and $\mathscr{K}_3$.

First attempt

An iteration method, of minimizing the damage to a strong coloring, is introduced to distinguish MA and MA(powerfully ccc).

Theorem 2 (2025)

MA(powerfully ccc) does not imply MA.

It is likely that the method can be generalized to distinguish MA_{ω_1} and $\mathscr{K}_3$.

Fix a strong ccc coloring $\pi : [\omega_1]^{<\omega} \to 2$. Iteratively force

$\mathscr{K}_3$

while preserve that

π has no uncountable 0-homogeneous subset.

Difficulty

The procedure encounters difficulty:

1. There may be a ccc coloring $c : [\omega_1]^3 \to 2$ such that
   - $\mathcal{H}_0^c \subseteq \mathcal{H}_0^\pi$

   where $\mathcal{H}_0^c$ ($\mathcal{H}_0^\pi$) is the poset consisting of finite 0-homogeneous subsets of c (π) ordered by reverse inclusion.

Difficulty

The procedure encounters difficulty:

1. There may be a ccc coloring $c : [\omega_1]^3 \to 2$ such that
   - $\mathcal{H}_0^c \subseteq \mathcal{H}_0^\pi$

   where $\mathcal{H}_0^c$ ($\mathcal{H}_0^\pi$) is the poset consisting of finite 0-homogeneous subsets of c (π) ordered by reverse inclusion.

An uncountable 0-homogeneous subset of c would induce an uncountable 0-homogeneous subset of π. So we cannot add an uncountable 0-homogeneous subset of c.

Only option: Destroy ccc of $\mathcal{H}_0^c$.

Difficulties

But the difficulty becomes major if the coloring c has the following stronger property.

1. $\mathcal{H}_0^c$ is σ-linked.

Positive side

$\mathscr{K}_3$ together with $\mathrm{MA}_{\omega_1}(\sigma$-centered) implies the existence of a coloring c with properties (I) and (II).

- $\mathscr{K}_3$+$\mathrm{MA}_{\omega_1}(\sigma$-centered) implies MA_{ω_1}.

For posets of size $\leq \omega_1$

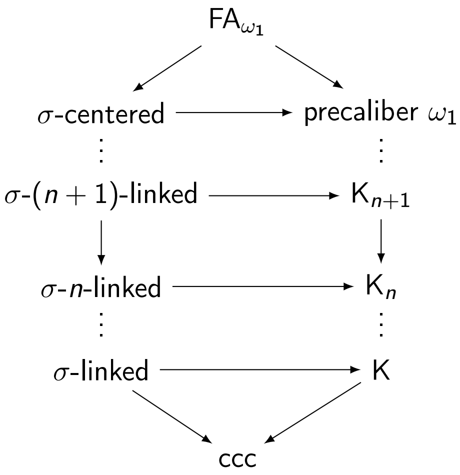

Figure: Forcing properties

$\mathrm{MA}_{\omega_1}(\sigma\text{-centered})$

Bell's Theorem: $\mathrm{MA}_\kappa(\sigma\text{-centered})$ is equivalent to $\mathfrak{p} > \kappa$.

Rothberger's Theorem: $\mathfrak{p} > \omega_1$ iff $\mathfrak{t} > \omega_1$; or Malliaris-Shelah's Theorem: $\mathfrak{p} = \mathfrak{t}$.

Theorem 3

$\mathscr{K}_3 + \mathfrak{t} > \omega_1$ implies MA_{ω_1}.

$\mathfrak{t} > \omega_1$

Fix a tower $\mathcal{T} = \{t_\alpha : \alpha < \omega_1\}$, i.e., every $t_\alpha \subseteq \omega$ and $t_\beta \setminus t_\alpha$ is finite whenever $\alpha < \beta$.

Say $t \subseteq \omega$ fills the tower $\mathcal{T}$ if $t \setminus t_\alpha$ is finite for all α.

$\mathfrak{t} > \omega_1$ is the assertion that every tower of size ω_1 is filled.

One way to fill the tower $\mathcal{T}$ is to find $\Gamma \in [\omega_1]^{\omega_1}$ such that $\bigcap_{\alpha \in \Gamma} t_\alpha$ is infinite. Then $t = \bigcap_{\alpha \in \Gamma} t_\alpha$ will fill the tower $\mathcal{T}$.

$\mathfrak{t} > \omega_1$

Want: $\bigcap_{\alpha \in \Gamma} t_\alpha$ is infinite.

Attempt: Partition ω into infinitely many intervals and expect that $\bigcap_{\alpha \in \Gamma} t_\alpha$ intersects every interval.

(Todorcevic) $\mathscr{K}_3$ implies $\mathfrak{b} > \omega_1$.

So find $I \in [\omega]^\omega$ (going to a sub-tower) such that

- for every $\alpha < \omega_1$ and every $n < \omega$, $t_\alpha \cap I_n \neq \emptyset$ where $I_n = [I(n), I(n+1))$.

Difficulty

The next difficulty: the forcing that adds $\Gamma \in [\omega_1]^{\omega_1}$ satisfying

$$(\bigcap_{\alpha\in\Gamma} t_\alpha) \cap I_n \neq \emptyset \text{ for all } n$$

is not even ccc.

Difficulty

The next difficulty: the forcing that adds $\Gamma \in [\omega_1]^{\omega_1}$ satisfying

$$(\bigcap_{\alpha\in\Gamma} t_\alpha) \cap I_n \neq \emptyset \text{ for all } n$$

is not even ccc.

Solution: ccc colorings on $[\omega_1]^3$: $c : [\omega_1]^3 \to 2$ by $c(\alpha, \beta, \gamma) = 0$ iff

$$e(\alpha, \beta) = e(\alpha, \gamma) \to t_\alpha \cap t_\beta \cap I_{e(\alpha,\beta)} \neq \emptyset$$

where e is a arbitrary coherent finite-to-one function.

Make the poset ccc

The coloring c is ccc and an uncountable 3-linked (or 0-homogeneous) subset induces ccc of the following poset:

- IV There are $\Sigma \in [\omega_1]^{\omega_1}$ and $A \in [\omega]^\omega$ such that

$$\mathcal{P} = \{F \in [\Sigma]^{<\omega} : (\bigcap_{\alpha \in F} t_\alpha) \cap I_n \neq \emptyset \text{ for all } n \in A\}$$

ordered by reverse inclusion is ccc.

Difficulty

$\mathcal{P}$ forces $\Gamma' \in [\Sigma]^{\omega_1}$ such that

$$(\bigcap_{\alpha\in\Gamma'} t_\alpha) \cap I_n \neq \emptyset \text{ for all } n \in A.$$

Goal: $(\bigcap_{\alpha\in\Gamma'} t_\alpha) \cap I_n \neq \emptyset$.

Need: $(\bigcap_{\alpha\in F} t_\alpha) \cap I_n \neq \emptyset$ for all $F \in [\Gamma']^{|I_n|}$.

$\mathscr{K}_3$: $(\bigcap_{\alpha\in F} t_\alpha) \cap I_n \neq \emptyset$ for all $F \in [\Gamma']^3$.

Note that $|I_n| \to \infty$ as $n \to \infty$. So strengthening $\mathscr{K}_3$ to $\mathscr{K}_n$ for some fixed n does not resolve the problem.

Now the difficulty is to find, by ccc coloring on triples, $\Gamma \in [\Sigma]^{\omega_1}$ such that for $n \in A$,

- $\bigcap_{\alpha\in\Gamma}(t_\alpha \cap I_n) \neq \emptyset$.

attempt

Natural idea: Find a coloring on $[\omega_1]^3$ such that a pair of ordinals $\{\alpha, \beta\}$ would guess an interval

- $I_{n_{\alpha\beta}}$ and
- a natural number $m_{\alpha\beta} \in I_{n_{\alpha\beta}}$

while for the third ordinal γ, $m_{\alpha\beta} \in t_\gamma$.

attempt

Natural idea: Find a coloring on $[\omega_1]^3$ such that a pair of ordinals $\{\alpha, \beta\}$ would guess an interval

- $I_{n_{\alpha\beta}}$ and
- a natural number $m_{\alpha\beta} \in I_{n_{\alpha\beta}}$

while for the third ordinal γ, $m_{\alpha\beta} \in t_\gamma$.

However, all such attempts failed. It is still open if there is a ccc coloring to fulfill above purpose.

Correct structure

$\langle \mathcal{I}_n^\Gamma = \{t_\alpha \cap I_n : \alpha \in \Gamma\} : n \in A \rangle$ for $\Gamma \subseteq \Sigma$.

$\pi : [\omega_1]^3 \to 2$ by for $\alpha < \beta < \gamma$, $\pi(\alpha, \beta, \gamma) = 0$ iff

$$e(\alpha, \beta) = e(\alpha, \gamma) \to (t_\beta \cap I_{e(\alpha,\beta)} \subseteq t_\gamma \vee t_\gamma \cap I_{e(\alpha,\beta)} \subseteq t_\beta).$$

π is ccc and an uncountable 0-homogeneous subset induces $\Gamma \in [\Sigma]^{\omega_1}$ and $B \in [A]^\omega$ such that

- for every $n \in B$, $(\mathcal{I}_n^\Gamma, \subseteq)$ is linearly order and hence has non-empty intersection.

Solution

Theorem 4

$\mathcal{K}_3$ implies MA_{ω_1}.

For posets of size $\leq \omega_1$

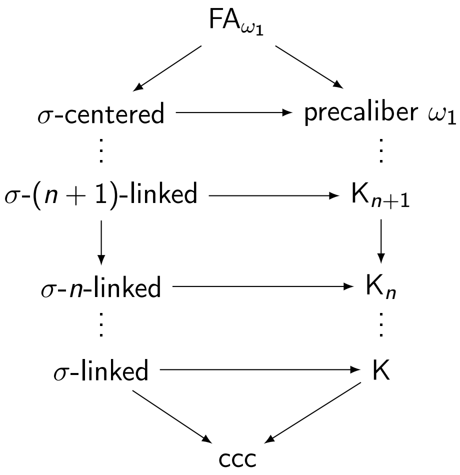

Figure: Forcing properties

K_3 and precailber ω_1

The following are equivalent.

- P(ccc$\to K_3$).
- P(ccc$\to$precaliber ω_1).
- For a coloring $c : [\omega_1]^3 \to 2$, if some ccc poset adds an uncountable 0-homogeneous subset of c, then c already has an uncountable 0-homogeneous subset.
- For a coloring $\pi : [\omega_1]^{<\omega} \to 2$, if some ccc poset adds an uncountable 0-homogeneous subset of π, then π already has an uncountable 0-homogeneous subset.

It seems to indicate that a ccc coloring on finite sets may be coded by a ccc coloring on triples.

It seems to indicate that a ccc coloring on finite sets may be coded by a ccc coloring on triples.

Then we look into the general case and find out that on the contrary, to code an $(n+1)$-ary ccc coloring by an n-ary ccc coloring is so difficulty that the attempt will not succeed unless we reach the corresponding forcing axiom.

It seems to indicate that a ccc coloring on finite sets may be coded by a ccc coloring on triples.

Then we look into the general case and find out that on the contrary, to code an $(n+1)$-ary ccc coloring by an n-ary ccc coloring is so difficulty that the attempt will not succeed unless we reach the corresponding forcing axiom.

More precisely, we have the following result.

Theorem 5

Assume $n \geq 2$. $\mathrm{P}(\mathrm{K}_n \to \mathrm{K}_{n+1})$ is equivalent to $\mathrm{MA}_{\omega_1}(\mathrm{K}_n)$ (or $\mathrm{P}(\mathrm{K}_n \to \mathrm{FA}_{\omega_1})$).

Strategy to prove $P(K_n \to K_{n+1})$ implies $MA_{\omega_1}(K_n)$

1. $P(K_n \to K_{n+1})$ implies $P_{\omega_1}(K_n \to \sigma\text{-linked})$.
2. $P(K_n \to K_{n+1})$ implies $P_{\omega_1}((K_n \wedge \sigma\text{-}i\text{-linked}) \to \sigma\text{-}(i+1)\text{-linked})$ for $2 \leq i < n$.
3. $P(\sigma\text{-}n\text{-linked} \to K_{n+1})$ implies $P(\sigma\text{-}n\text{-linked} \to \text{precaliber } \omega_1)$.
4. $P(\sigma\text{-}n\text{-linked} \to \text{precaliber } \omega_1)$ implies $P_{\omega_1}(\sigma\text{-}n\text{-linked} \to \sigma\text{-centered})$.
5. $P(\sigma\text{-}n\text{-linked} \to K_{n+1})$ implies $\mathfrak{t} > \omega_1$.

σ-n-linked

(★3)-(★5) show

Theorem 6

Assume $n \geq 2$. $\mathrm{P}(\sigma\text{-}n\text{-linked} \to \mathrm{K}_{n+1})$ *is equivalent to* $\mathrm{MA}_{\omega_1}(\sigma\text{-}n\text{-linked})$.

K_n to σ-n-linked

Theorem 7

For $n \geq 2$, $\mathrm{P}_{\omega_1}(\mathrm{K}_n \to \sigma\text{-}n\text{-linked})$ does not imply $\mathrm{MA}_{\omega_1}(\mathrm{K}_n)$.

Theorem 8

$\mathrm{P}_{\omega_1}(\text{precaliber } \omega_1 \to \sigma\text{-centered})$ *implies* $\mathrm{MA}_{\omega_1}(\text{precaliber } \omega_1)$.

A dashed arrow from Φ to Ψ denotes that $\mathrm{P}(\Phi \to \Psi)$ is as strong as $\mathrm{MA}_{\omega_1}(\Phi)$.

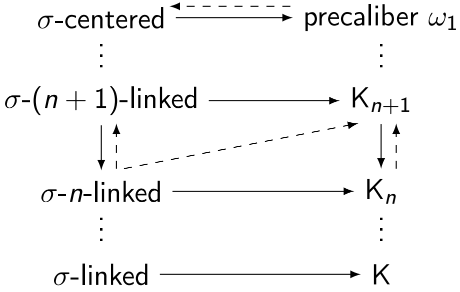

Figure: Forcing properties' strength

By the results summarized above, the diagram is complete.

The remaining properties

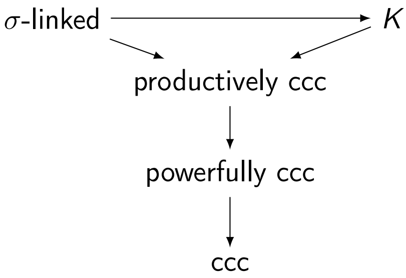

Figure: Weak forcing properties

Open problems

The following 3 statements are equivalent.

- P(ccc→powerfully ccc).
- P(ccc→productively ccc).
- $\mathcal{C}^2$, i.e., the product of any two ccc posets is ccc.

Also, P(ccc→K) is the well-known property $\mathscr{K}_2$.

Question 3

Does $\mathcal{C}^2$ or $\mathscr{K}_2$ imply MA_{ω_1}? Or $\mathfrak{t} > \omega_1$?

Thank you!