Several topics on $*$-local weak o-minimality

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Section 1

Introduction

Intermediate Tameness

Wild	Intermediate	Tame
$(\mathbb{R}, <, +, \cdot, \mathbb{N})$	Weak o-minimality Local o-minimality D-minimality	o-minimality $(\mathbb{R}, <, +, \cdot)$

Basic definitions

Notation

$\mathcal{M} = (M, <, \dots)$ is an expansion of dense linear order without endpoints.

Definition

- $\mathcal{M}$ is *definable complete* if every nonempty definable subset of M has a supremum and infimum in $M \cup \{\pm\infty\}$.
- $\mathcal{M}$ is *o-minimal* if every univariate definable set is a union of a finite set and finitely many open intervals.
- $\mathcal{M}$ is *weakly o-minimal* if every univariate definable set is a union of finitely many convex sets.
- $\mathcal{M}$ is a *locally o-minimal structure* if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I such that $a \in I$ and $X \cap I$ is a union of a finite set and finitely many open intervals.

Relations between these concepts and preservation under elementary equivalence

- O-minimality $\Rightarrow$ weak o-minimality $\Rightarrow$ local o-minimality
- O-minimality $=$ weak o-minimality $+$ definable completeness

Is it preserved under elementary equivalence?

definable completeness	yes
o-minimality	yes
weak o-minimality	no
local o-minimality	yes

Facts on local o-minimality

Suppose that $\mathcal{M}$ is definably complete and locally o-minimal.

- **Local monotonicity:** Let $f : I \to M$ be a univariate definable function. There exists a discrete closed definable set D such that $f|_{I \setminus D}$ is continuous and locally monotone.
- **Tame behavior of dimension function:**
  - $\dim(X_1 \cup X_2) = \max\{\dim X_1, \dim X_2\}$.
  - **(Addition property)** If $f : X \to Y$ is surjective and has equidimensional fibers, $\dim X = \dim Y + \dim f^{-1}(y)$ for $y \in Y$.
  - **(Continuity property)** every definable function $f : X \to M$ is continuous except the definable subset of smaller dimension.

On the other hand, in a non-definably complete locally o-minimal structure,

- There exists a definable function which is discontinuous everywhere.
- Addition property fails.

Facts on weak o-minimality

Suppose that $\mathcal{M}$ is weakly o-minimal.

- Monotonicity theorem without continuity property holds.
- $\dim(X_1 \cup X_2) = \max\{\dim X_1, \dim X_2\}$.
- (Wencel) Addition property is equivalent to univariate $*$-continuity property (which is defined later).

In weakly o-minimal structures, dimension function does not necessarily behave tamely, but it is known when dimension function behaves tamely.

Motivation

- Definably complete local o-minimality possesses tame topological properties:
  - Local monotonicity theorem
  - Tame behavior of dimension function
  - and so on...
- Weak o-minimality possesses a little bit wilder but somewhat tame topological properties.
- Does a good subclass of local o-minimality without definable completeness wider than weak o-minimality possess topological properties as tame as weak o-minimality?

My answer: *-local weak o-minimality

Results to be introduced in this talk

1. An equivalent condition for addition formula of dimension to hold in $*$-locally weakly o-minimal structures;
2. Nonvaluational expansion of divisible Abelian group of finite burden (combinatorical concept) defining no 'wild' set is $*$-locally weakly o-minimal (topological concept);
3. Another characterization of $*$-local weak o-minimality by bounded 1-types (optional).

Definition (definable Dedekind completion) (1)

Definition

A *gap* is a pair (A, B) of nonempty subsets of M such that

- $M = A \cup B$;
- $a < b$ for all $a \in A$ and $b \in B$;
- A does not have a largest element and B does not have a smallest element.

We say that the gap is *definable* if A (equivalently, B) is definable.

Set $\overline{M} = M \cup \{\text{definable gaps in } M\}$. We can naturally extend the order $<$ in M to an order in $\overline{M}$, which is denoted by the same symbol $<$. The linearly ordered set $(\overline{M}, <)$ is called the *definable Dedekind completion* of $(M, <)$.

Definition (definable Dedekind completion) (2)

Notation

For an open interval $I = (b_1, b_2)$, where $b_1, b_2 \in M \cup \{\pm\infty\}$, we set

$$\overline{I} = \{x \in \overline{M} \mid b_1 < x < b_2\}.$$

Throughout, we use the overlined notations to represent Dedekind completions and their subsets defined above.

Definition

For definable $\emptyset \neq X \subseteq M$, define $\sup X \in \overline{M} \cup \{+\infty\}$ as follows:

- $\sup X = +\infty$ when, for any $a \in M$, there exists $x \in X$ with $x > a$.
- Assume that $\exists z \in M$ s.t. $x < z$ for every $x \in X$. Set $B = \{y \in M \mid \forall x \in X \ y > x\}$ and $A = M \setminus B$.
  - If B has a smallest element m, we set $\sup X = m$.
  - If A has a largest element m', we set $\sup X = m'$.
  - Finally, if (A, B) is a definable gap, we set $\sup X = (A, B) \in \overline{M}$.

Definition (definable function)

Definition

Let X be a definable subset of M^n. A *definable function* $F : X \to \overline{M}$ is defined as follows:
Let $\pi : M^{n+1} \to M^n$ be the coordinate projection forgetting the last coordinate.
There exists a definable subset Y of M^{n+1} such that $\pi(Y) = X$ and $F(x) = \sup Y_x$ for $x \in X$, where $Y_x := \{y \in M \mid (x, y) \in Y\}$.

A definable function $F : X \to \overline{M} \cup \{\pm\infty\}$ is a pair of a decomposition $X = X_{\overline{M}} \cup X_{+\infty} \cup X_{-\infty}$ into definable sets and a definable function $f : X_{\overline{M}} \to \overline{M}$.

Definition ($*$-local weak o-minimality)

Definition

$\mathcal{M}$ is a *locally o-minimal structure* if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I such that $a \in I$ and $X \cap I$ is a union of a finite set and finitely many open intervals.

$\mathcal{M}$ is an *almost weakly o-minimal structure* if every bounded definable subset of M is a union of finitely many convex sets.

$\mathcal{M}$ is a *$*$-locally weakly o-minimal structure* if, for every definable subset X of M and for every point $a \in \overline{M}$, there exists an open interval I such that $a \in \overline{I}$ and $X \cap I$ is a union of finitely many convex sets.

Fact

- *almost weak o-minimality $\Rightarrow$ $*$-local weak o-minimality $\Rightarrow$ local o-minimality.*
- **-local weak o-minimality is preserved under elementary equivalence.*

Section 2

Addition formula of dimension function

Dimension and addition formula

Definition

Let X be a nonempty definable subset of M^n. Recall that M^0 is a singleton with the trivial topology.

$$\dim X = \max\{d \mid \exists \pi : M^n \to M^d \text{:coord. proj. s.t. } \operatorname{int}(\pi(X)) \neq \emptyset\}.$$

We set $\dim(X) = -\infty$ if $X = \emptyset$.

Definition (Addition property)

$\dim$ possesses the *addition property* if the following holds:
Let $\varphi : X \to Y$ be a definable surjective map whose fibers are equi-dimensional; that is, the dimensions of the fibers $\varphi^{-1}(y)$ are constant. We have

$$\dim X = \dim Y + \dim \varphi^{-1}(y)$$

for all $y \in Y$.

Wencel's equivalent condition

Definition

$\mathcal{M}$ enjoys the *univariate $*$-continuity property* if, for every definable function $f : I \to \overline{M}$ from a nonempty open interval I, there exists a nonempty open subinterval J of I such that the restriction of f to J is continuous.

Theorem (Wencel, 2010)

Suppose $\mathcal{M}$ is weakly o-minimal.
$\dim$ *possesses the addition property if and only if $\mathcal{M}$ enjoys univariate $*$-continuity property.*

Dimensionally wild set

A definable set X of M^2 is called *dimensionally wild* if the following conditions are satisfied:

(i) X has an empty interior;

(ii) $\pi(X)$ has a nonempty interior, where π denotes the projection onto the first coordinate;

(iii) $X_x := \{y \in M \mid (x, y) \in X\}$ has a nonempty interior for every $x \in \pi(X)$.

A dimensionally wild set violates the addition property.

Structure theorem of locally o-minimal structures

Theorem (Structure theorem of locally o-minimal structures)

Consider a locally o-minimal structure $\mathcal{M} = (M, <, \ldots)$. At least one of the following two assertions holds:

(1) *$\mathcal{M}$ does not possess the univariate $*$-continuity property and has a dimensionally wide definable set.*

(2) *Let $f : I \to \overline{M}$ be an arbitrary definable function defined on an arbitrary open interval I. The interval I is decomposed into four definable sets X_+, X_-, X_c, X_d satisfying the following conditions:*

  (i) *X_d is discrete and closed.*

  (ii) *X_c is open and the restriction of f to X_c is locally constant;*

  (iii) *X_- is open and the restriction of f to X_- is locally strictly decreasing;*

  (iv) *X_+ is open and the restriction of f to X_+ is locally strictly increasing.*

We do not know whether this theorem is a dichotomy. i.e. We do not know whether there exists a structure satisfying (1) and (2) simultaneously.

$*$-locally weakly o-minimal case

Theorem (F, 2024)

Suppose $\mathcal{M}$ is $*$-locally weakly o-minimal.
$\dim$ *possesses the addition property if and only if $\mathcal{M}$ enjoys univariate $*$-continuity property.*

Strategy of proof
If part: We can prove the addition property in the same manner as definably complete locally o-minimal case using strong local monotonicity theorem in this case.
Only if part: Suppose that univariate $*$-continuity property is violated. We use structure theorem of locally o-minimal structures. We consider two separate cases where conditions (1) and (2) hold, respectively. If (2) holds, we can find a definable monotone function which is discontinuous everywhere. We can construct a dimensionally wild set from it. □

If $\mathcal{M}$ possesses univariate $*$-continuity property, ...

Theorem (F, 2024)

Suppose $\mathcal{M}$ is a $*$-locally weakly locally o-minimal structure possessing univariate $*$-continuity property. The following assertions hold:

(1) *Let X and Y be definable subsets of M^n. We have* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

(2) *Let $f : X \to M^n$ be a definable map. We have* $\dim(f(X)) \leq \dim X$.

(3) *Let $f : X \to M^n$ be a definable map. The notation $\mathcal{D}(f)$ denotes the set of points at which the map f is discontinuous. The inequality* $\dim(\mathcal{D}(f)) < \dim X$ *holds true.*

(4) *Let X be a definable set. The notation ∂X denotes the frontier of X defined by* $\partial X = \mathrm{cl}(X) \setminus X$. *We have* $\dim(\partial X) < \dim X$.

If $\mathcal{M}$ possesses univariate $*$-continuity property, ... (Cont'd)

Theorem (F, 2024)

Suppose $\mathcal{M}$ is a $*$-locally weakly locally o-minimal structure possessing univariate $*$-continuity property. The following assertions hold:

(5) *A definable set X is of dimension d if and only if the nonnegative integer d is the maximum of nonnegative integers e such that there exist an open box B in M^e and a definable injective continuous map $\varphi : B \to X$ homeomorphic onto its image.*

(6) *Let X be a definable subset of M^n. There exists a point $x \in X$ such that we have $\dim(X \cap B) = \dim(X)$ for any open box B containing the point x.*

Proof is long, but we can prove it in the same manner as definably complete locally o-minimal case.

Section 3

Expansions of OAGs of finite burden

Definition (burden)

We fix a complete first-order theory T. Let $p(\overline{x})$ be a partial type and κ be a cardinal.

An *inp-pattern of depth κ in $p(\overline{x})$* is a sequence $(\phi_\alpha(\overline{x};\overline{y}) \mid \alpha < \kappa)$ of formulas, a sequence $(k_\alpha \mid \alpha < \kappa)$ of positive integers and a sequence $(\overline{b}_i^\alpha \mid \alpha < \kappa, i < \omega)$ of tuples from some model $\mathcal{M}$ of T such that:

- $\{\phi_\alpha(\overline{x};\overline{b}_i^\alpha) \mid i < \omega\}$ is k_α-inconsistent for every $\alpha < \kappa$;
- $\{\phi_\alpha(\overline{x};\overline{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \kappa\}$ is consistent with $p(\overline{x})$ for all map $\eta : \kappa \to \omega$.

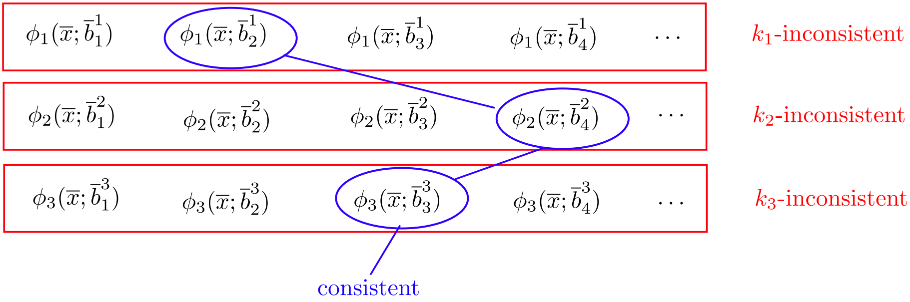
The partial type $p(\overline{x})$ has *burden* $< \kappa$ if there is no inp-pattern of depth κ in $p(\overline{x})$. If the least κ such that the burden of $p(\overline{x})$ is less than κ is a successor cardinal with $\kappa = \lambda^+$, then we say that *the burden of $p(\overline{x})$* is λ. If the burden of the partial type $x = x$ in a single free variable x exists and is equal to κ, we say that *the burden of T* is κ.

Fact

If T is NIP, the dp-rank of any partial type in T is equal to its burden.

Conceptual figure of burden

- $\{\phi_\alpha(\overline{x}; \overline{b}_i^\alpha) \mid i < \omega\}$ is k_α-inconsistent for every $\alpha < \kappa$;
- $\{\phi_\alpha(\overline{x}; \overline{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \kappa\}$ is consistent with $p(\overline{x})$ for all map $\eta : \kappa \to \omega$.



Definition (open core)

Definition

The *open core* of $\mathcal{M}$ is a reduct of $\mathcal{M}$ generated by definable open sets.

Proposition (F, 2021)

A definably complete expansion $\mathcal{M}$ of an ordered group has a locally o-minimal open core if and only if every definable closed subset of M with empty interior is discrete.

Motivational Fact

Proposition (Dolich & Goodrick, 2017)

In a definably complete expansion $\mathcal{M}$ of an ordered group such that $\mathrm{Th}(\mathcal{M})$ is strong, if $X \subset M$ is definable and nowhere dense, then X is discrete.

$$+$$

Proposition [F,2021]

$$\Downarrow$$

Every definably complete expansion $\mathcal{M}$ of an ordered group such that $\mathrm{Th}(\mathcal{M})$ is strong defining no nonempty subset X of M which is dense and codense in a definable open subset U of M with $X \subseteq U$ is a locally o-minimal.

The theory T is *strong* if, for any finite tuple of variables $\overline{x}$, every inp-pattern in the partial type $\overline{x} = \overline{x}$ has finite depth.

Main theorem

From now on, we assume that $\mathcal{M}$ is an sufficiently saturated expansion of an ordered divisible Abelian group.

Definition

$\mathcal{M}$ is *nonvaluational* (n.v. for short) if, for every nonempty definable subsets A, B of M with $A < B$ and $A \cup B = M$, $\inf\{b - a \mid a \in A, b \in B\} = 0$.

Theorem (F, 2025)

Consider a nonvaluational $\mathcal{M} = (M, <, +, \ldots)$ of finite burden defining no nonempty subset X of M which is dense and codense in a definable open subset U of M with $X \subseteq U$. Then, $\mathcal{M}$ is $*$-locally weakly o-minimal.

Roughly speaking, blue part $\fallingdotseq$ (structure = open core)

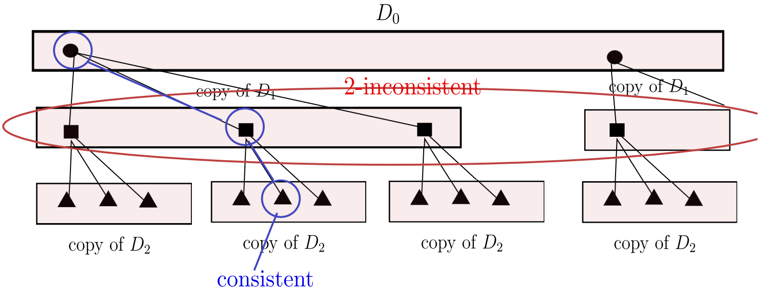
Discrete case (1)

Lemma (Dolich & Goodrick, 2017)

Suppose that there is an infinite family of infinite definable discrete sets D_i and $\varepsilon_i > 0$ for $i \in \mathbb{N}$ s.t.:

1. $D_i \subseteq (0, \varepsilon_i/3)$ *and*
2. *If* $x \in D_i$, *then* $(x - \varepsilon_i, x + \varepsilon_i) \cap D_i \subseteq \{x\}$.

Then $\mathrm{Th}(\mathcal{M})$ *is not strong.*



Lemma (F, 2025)

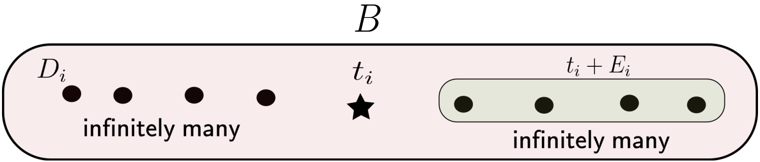
Let $\alpha \in \overline{M}$ be an n.v. element. Let D_i be infinite definable discrete sets and $\varepsilon_i > 0$ for $i \in \mathbb{N}$ satisfying the following conditions:

(1) $D_{i+1} \subseteq (\alpha, \alpha + \varepsilon_i/3)$;

(2) *If* $x \in D_i$, *then* $(x - \varepsilon_i, x + \varepsilon_i) \cap D_i = \{x\}$;

Then $\mathrm{Th}(\mathcal{M})$ *is not strong.*

α is a definable gap (A, B). We construct infinite definable discrete sets E_i and $t_i \in M$ such that $E_i \subseteq (0, \varepsilon_i/3)$ and $t_i + E_i \subseteq D_i$.



Technical definition (convex component)

Definition

Let $X \subseteq M$ be a set.
A *convex component* of X is a maximal convex subset of X. Every convex component C of X is definable if X is definable. In fact,

$$C = \{c\} \cup \{x \in X \mid x > c \wedge \forall y \ (c < y < x \rightarrow y \in X)\} \\ \cup \{x \in X \mid x < c \wedge \forall y \ (x < y < c \rightarrow y \in X)\},$$

where c is an arbitrary point in C.

Open case (1)

Lemma

Suppose $\mathcal{M}$ is n.v. Let $N \geq 2$ be a natural number. Let $\{X_n\}_{n=1}^N$ be a family of definable subsets of M having infinitely many maximal convex subsets. Let $\{\varepsilon_n\}_{n=1}^N$ be a decreasing family of positive elements in M. Suppose the following condition is satisfied:

(1) $X_{n+1} \subseteq (0, \varepsilon_n/3)$,

(2) *If $x \in X_n$, then $(x - 2\varepsilon_n, x + 2\varepsilon_n) \cap X_n \subseteq D(X_n, x)$.*

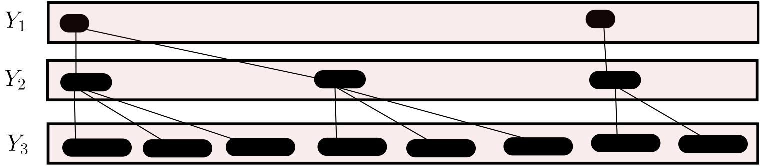
Then $\mathrm{Th}(\mathcal{M})$ is of burden $\geq N - 1$.

$D(X_n, x)$ is the convex component of X_n containing the point x.

Open case (2)

$D(X_n, x)$ is the convex component of X_n containing the point x.
$l(X_n, x)$ is the length of $D(X_n, x)$

1. First reduce to the case in which $l(X_n, x)$ is very small $< \varepsilon_N/(2N)$.
2. Let $Y_1 := X_1$ and $Y_n := \{y + d \mid y \in Y_{n-1},\ d \in X_n\}$ for $n > 1$.
   (a) For $n > 1$, $d \in X_n$ and $y \in Y_{n-1}$,
   $$D(Y_n, y + d) = D(Y_{n-1}, y) + D(X_n, d);$$
   (b) $l(Y_n, x) < n\varepsilon_N/N$ and
   (c) $(x - \varepsilon_n, x + \varepsilon_n) \cap Y_n \subseteq D(Y_n, x)$ for $n > 0$ and $x \in Y_n$.



Open case (3)

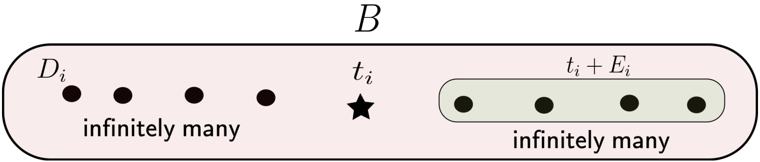
Lemma

Suppose $\mathcal{M}$ is n.v. Let $N \geq 2$ be a natural number and $\alpha \in \overline{M}$. Let $\{X_n\}_{n=1}^N$ be a family of definable subsets of M having infinitely many maximal convex subsets. Let $\{\varepsilon_n\}_{n=1}^N$ be a decreasing family of positive elements in M. Suppose the following condition is satisfied:

(1) $X_{n+1} \subseteq (\alpha, \alpha + \varepsilon_n/3)$,

(2) *If* $x \in X_n$, *then* $(x - 2\varepsilon_n, x + 2\varepsilon_n) \cap X_n \subseteq D(X_n, x)$.

Then $\mathrm{Th}(\mathcal{M})$ *is of burden* $\geq N - 1$.



No accumulations and No Cantor-like sets

Corollary

Suppose $\mathrm{Th}(\mathcal{M})$ is strong. Let D be an infinite discrete definable subset of M. Then, for every n.v. $a \in \overline{M}$, there exists an open interval I such that $a \in \overline{I}$ and $D \cap I$ is a finite set.

Corollary

Suppose $\mathcal{M}$ is n.v. and $\mathrm{Th}(\mathcal{M})$ is of finite burden. Let U be a definable open subset of M having infinitely many maximal convex subsets. Then, for every $a \in \overline{M}$, there exists an open interval I with $a \in \overline{I}$ such that $U \cap I$ is a union of finitely many open convex set.

In addition, $\mathcal{M}$ does not define a nowhere dense subset of M having no isolated points (Cantor-like set).

Main theorem, revisited

Theorem (F, 2025)

Consider a nonvaluational $\mathcal{M} = (M, <, +, \ldots)$ of finite burden defining no nonempty subset X of M which is dense and codense in a definable open subset U of M with $X \subseteq U$. Then, $\mathcal{R}$ is $*$-locally weakly o-minimal.

Every unary definable set Y with empty interior is partitioned as $Y = Y_1 \cup Y_2 \cup Y_3$ satisfying the following conditions:

- Y_1, Y_2 and Y_3 are definable;
- Y_1 is either empty or has a definable open subset V of M such that $Y_1 \subseteq V$ and Y_1 is dense and codense in V;
- Y_2 is either empty or a nowhere dense subset of M having no isolated points (Cantor-like set);
- Y_3 is either empty or discrete.

This partition and corollaries imply the theorem.

Section 4

Characterization by bounded 1-types

Kulpeshov's characterization of weak o-minimality

Definition

A complete 1-type $p(x) \in S_1^{\mathcal{M}}(M)$ is *convex* if the set of realizations of $p(x)$ is a convex set in any elementary extension of $\mathcal{M}$.

Fact (Kulpeshov, 1998)

$\mathcal{M}$ is weakly o-minimal if and only if every complete type $p(x) \in S_1^{\mathcal{M}}(M)$ is convex.

Definition (bounded types)

Definition

A partial 1-type $p(x)$ in $\mathcal{M}$ is *bounded* if '$a < x < b$' $\in p(x)$ for some $a, b \in M$.

Suppose a complete 1- type $p(x) \in S_1^{\mathcal{M}}(M)$ is bounded. Put $B := \{b \in M \mid \text{'}x > b\text{'} \in p(x)\}$ and $C := \{c \in M \mid \text{'}x < c\text{'} \in p(x)\}$.
We have three possibilities:

(1) (Non-gap case) $p(x)$ is one of the following three forms:
   (a) $\mathcal{M} \models p(m)$ for some $m \in M$;
   (b) Either B has a largest element m or C has a smallest element m;

(2) (definable gap case) (B, C) is a definable gap.

(3) (Non-definable gap case) (B, C) is a non-definable gap.

If condition (n) is satisfied for $1 \leq n \leq 3$, we say that $p(x)$ is *of class* n.

Characterization of $*$-local weak o-minimality by bounded types

Theorem (F, 2025)

(a) *$\mathcal{M}$ is locally o-minimal if and only if every bounded complete 1-type over M of class 1 is convex.*

(b) *$\mathcal{M}$ is $*$-locally weakly o-minimal if and only if every bounded complete 1-type over M of class 1 and 2 is convex.*

(c) *$\mathcal{M}$ is almost weakly o-minimal if and only if every bounded complete 1-type over M is convex.*

ご清聴
ありがとうございました

Thank you!