Building Structures From Reals

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Fundamental insight of computability theory:

Complexity is measured by definability!

One can use the computability-theoretic properties of reals (elements of 2^ω, ω^ω, $\mathbb{R}$, $\dots$) to build structures.

Goal of this talk

Two examples of real numbers building *interesting* structures:

- in classical mathematics, especially fractal geometry
- in set theory, with a connection to topology

Part I: Projection Theorems in Fractal Geometry

Limits of Provability

A regularity property is a property of sets of reals (i.e. elements of $\mathbb{R}$) which describe a "nice" structural behaviour.

Definition

A set $A \subseteq \mathbb{R}$ has the perfect set property if it is either countable or if it contains a perfect subset (i.e. a copy of Cantor space 2^ω).

For example, no set with the Perfect Set Property can be a counterexample to the Continuum Hypothesis. It is regular.

Question

Which sets satisfy these regularity properties?
Can they be classified?

Turing Computability

Work over $\omega = \{0, 1, 2, \ldots\}$. Main idea: successful computations take finite time and finite resources.

Definition

A set $A \subseteq \omega$ is computable if there exists a program P which halts in finite time and outputs

$$P(n) = \begin{cases} \text{yes} & \text{if } n \in A \\ \text{no} & \text{if } n \notin A. \end{cases}$$

Turing's insight: overcome finite-time-restriction through oracles:

Definition

A program P is an oracle program for $A \subseteq \omega$ if it can ask at any point whether "$n \in A$". Write P^A. A set A computes B if there exists a program P^A which computes B. Write $B \leq_T A$.

Sets of reals

Not only sets of numbers can be analysed, but also sets of reals.
Topologically, we get the Borel hierarchy:

$\underset{\sim}{\mathbf{\Sigma}}^0_1 =$ open sets $\qquad \underset{\sim}{\mathbf{\Pi}}^0_1 =$ closed sets

$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha =$ union of $\underset{\sim}{\mathbf{\Pi}}^0_\beta$-sets $\qquad \underset{\sim}{\mathbf{\Pi}}^0_\alpha =$ intersection of $\underset{\sim}{\mathbf{\Sigma}}^0_\beta$-sets

$\underset{\sim}{\mathbf{\Delta}}^0_\alpha = \underset{\sim}{\mathbf{\Sigma}}^0_\alpha \cap \underset{\sim}{\mathbf{\Pi}}^0_\alpha$

where $\beta < \alpha < \omega_1$.

$$\begin{array}{ccccccccccccc}
 & & \underset{\sim}{\mathbf{\Sigma}}^0_1 & & & & \underset{\sim}{\mathbf{\Sigma}}^0_2 & & & & \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1} & & \\
 & \subset & & \supset & & \subset & & & & \subset & & \supset & \\
\underset{\sim}{\mathbf{\Delta}}^0_1 & & & & \underset{\sim}{\mathbf{\Delta}}^0_2 & & & \cdots & \underset{\sim}{\mathbf{\Delta}}^0_{\alpha+1} & & & & \underset{\sim}{\mathbf{\Delta}}^0_{\alpha+2} \quad \cdots \\
 & \subset & & \supset & & \subset & & & & \subset & & \supset & \\
 & & \underset{\sim}{\mathbf{\Pi}}^0_1 & & & & \underset{\sim}{\mathbf{\Pi}}^0_2 & & & & \underset{\sim}{\mathbf{\Pi}}^0_{\alpha+1} & &
\end{array}$$

Superscript 0 indicates first-orderness—this can be made explicit via Turing computability!

Consistency and Provability

The Borel hierarchy can be extended to the right: there exists a set that is not Borel (Souslin). Continuous images of Borel sets are called $\boldsymbol{\Sigma}^1_1$—this gives the projective hierarchy.

$$
\begin{array}{ccccccc}
 & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 \\
 & \subseteq & & \supseteq & & \subseteq & \\
\text{Borel} = \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & \cdots \\
 & \subseteq & & \supseteq & & \subseteq & \\
 & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2
\end{array}
$$

(Think of $\boldsymbol{\Sigma}^1_1$ as computably enumerable with real witnesses.)

Note: The projective hierarchy is well-ordered!
This helps with provability of regularity properties:

Question

Which (projective) pointclasses satisfy regularity properties?

Some Axioms of Set Theory

ZF = Zermelo-Fränkel set theory

Some axioms give more sets:

AC = Axiom of Choice

- "every non-empty set has a choice function"

+ equivalent: every set can be well-ordered, Zorn's lemma, every vector space has a basis

− at the cost of definable structure: Vitali set, Banach-Tarski

Some axioms give more structure:

AD = Axiom of Determinacy

- "every two-player game on $\mathbb{R}$ has a winning strategy"

+ every regularity property expressible as games holds for all sets

− incompatible with the Axiom of Choice

Best of both worlds:

$(V{=}L)$ = Axiom of Constructibility

- "every set is constructible" (think "definable")
- proves the Axiom of Choice, the generalised continuum hypothesis, and much more

In $(V{=}L)$, we get *both* lots of sets (through AC) *and* a lot of structure (through definability of every set)!

This gives us the ideal environment to find optimal definable counterexamples.

The “usual” pattern for regularity properties

Axioms	Behaviour
ZFC	
ZFC	
ZF + DC + AD	
ZFC + $(V{=}L)$	

$$
\begin{array}{ccccccc}
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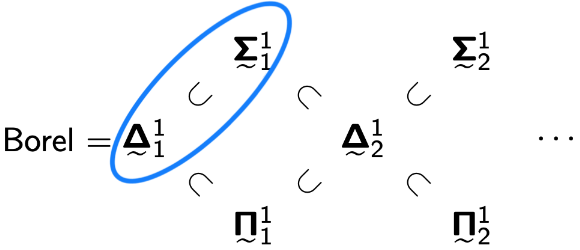
\end{array}
$$

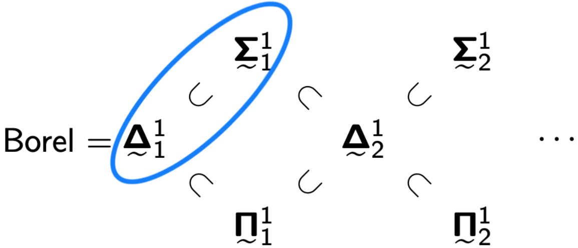
The “usual” pattern for regularity properties

Axioms	Behaviour
ZFC	PSP holds for all $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$ sets (Souslin)
ZFC	
ZF + DC + AD	
ZFC + (V=L)	



The “usual” pattern for regularity properties

Axioms	Behaviour
ZFC	PSP holds for all $\boldsymbol{\Sigma}^1_1$ sets (Souslin)
ZFC	PSP fails for some set (Bernstein)
ZF + DC + AD	
ZFC + $(V{=}L)$	



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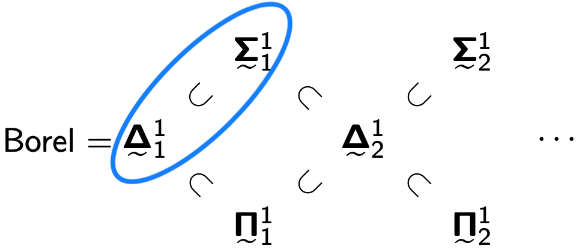
ZFC + $(V{=}L)$	



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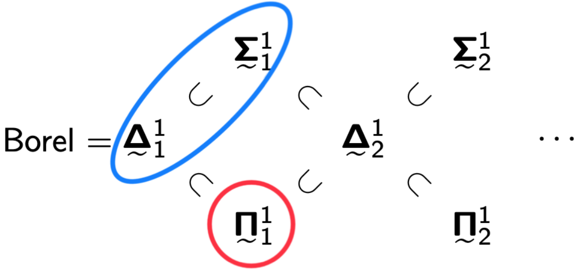
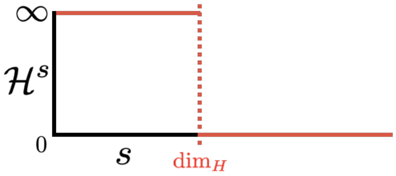
ZFC	PSP fails for some set (Bernstein)
ZF + DC + AD	PSP holds for all sets (Mycielski, Swierczkowski)
ZFC + $(V{=}L)$	PSP fails for some $\utilde{\boldsymbol{\Pi}}^1_1$ set (Gödel)



A Projection Theorem for Fractals

The s-dimensional Hausdorff outer measure $\mathcal{H}^s$ is a generalisation of Lebesgue outer measure; its coverings are given a weight:

- if s is too large, $\mathcal{H}^s$ is zero.
- if s is too small, $\mathcal{H}^s$ is infinite.



Example

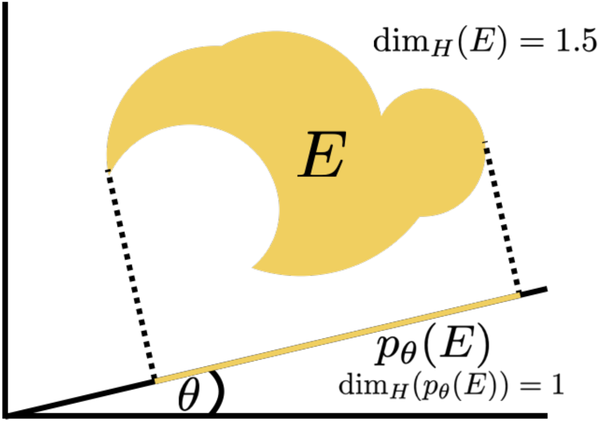
- $\dim_H\left([0,1]^2\right) = 2$
- $\dim_H(\text{middle-third Cantor set}) = \log(2)/\log(3)$

Every set of reals has a Hausdorff dimension.

$\dim_H$ is a classical object of study in geometric measure theory.

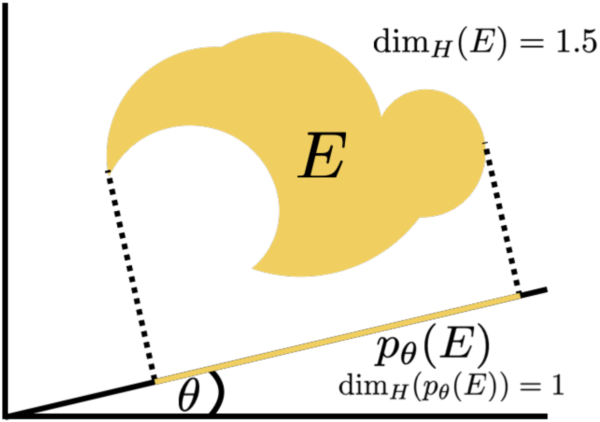
Definition

A set $A \subseteq \mathbb{R}^2$ has the Marstrand property if for almost every angle θ we have $\dim_H(\mathrm{proj}_\theta(A)) = \min\{1, \dim_H(A)\}$.



Theorem (Marstrand, 1954)

Every $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$ set has the Marstrand property.



Can we prove *more* in ZFC?

Axioms	Behaviour
ZFC	
ZFC + CH	
ZF + DC + AD	
ZFC + $(V{=}L)$	

$$
\begin{array}{ccccccccccc}
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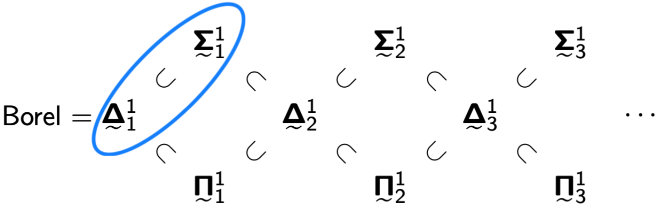
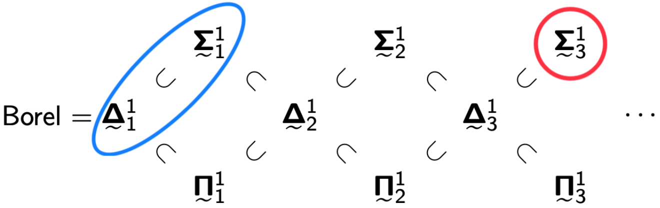
 & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & & & \boldsymbol{\Pi}^1_3
\end{array}
$$

Axioms	Behaviour
ZFC	MP holds for all $\utilde{\boldsymbol{\Sigma}}^1_1$ sets (Marstrand, 1954)
ZFC + CH	
ZF + DC + AD	
ZFC + $(V{=}L)$	



Axioms	Behaviour
ZFC	MP holds for all $\utilde{\mathbf{\Sigma}}^1_1$ sets (Marstrand, 1954)
ZFC + CH	MP fails for some set (Davies, 1979)
ZF + DC + AD	
ZFC + $(V = L)$	



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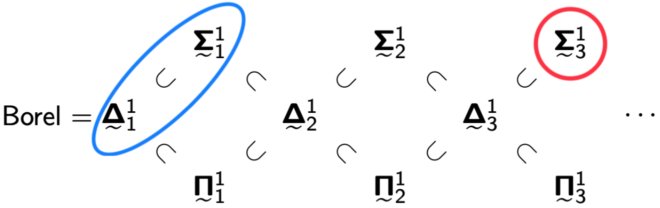
ZF + DC + AD	MP holds for all sets (Stull, 2021)
ZFC + $(V{=}L)$	



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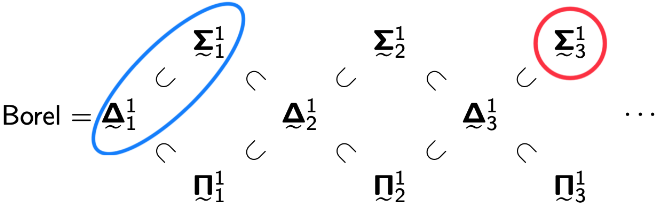
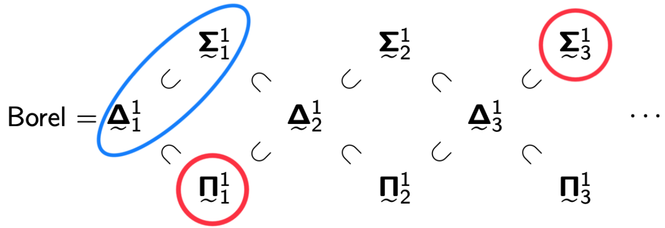


Completing the Picture for MP

Theorem (R.)

($V{=}L$) *There exists a* $\boldsymbol{\Pi}^1_1$ *set* $E \subseteq \mathbb{R}^2$ *for which*

$$\dim_H(E) = 1 \text{ yet for every } \theta \text{ we have } \dim_H(\mathrm{proj}_\theta(E)) = 0.$$



How do we construct such a set? By recursion!

From Points to Sets

Theorem (Lutz and Lutz, 2018)

If $A \subseteq \mathbb{R}^2$ *then*

$$\dim_H(A) = \min_{Z \in 2^\omega} \sup_{x \in A} \liminf_{n \to \infty} \frac{K^Z(x{\upharpoonright}_n)}{n}.$$

From (the complexity of) points one can measure the complexity of sets—hence it's called the point-to-set principle.

Lemma

Every countable set has Hausdorff dimension 0*.*

Proof.

Suppose $A = \{x_i \mid i < \omega\}$. Let $Z = \bigoplus_i x_i$. Let P compute $x_i{\upharpoonright}_n$ on input (i, n). For fixed i, the pair (i, n) has a description of length $\log(n) + c$, which vanishes $/n$ as $n \to \infty$. □

The $\underset{\sim}{\mathbf{\Pi}}^1_1$-recursion theorem

Theorem (Erdős, Kunen and Mauldin; A. Miller; Vidnyánszky)

$(V{=}L)$ If at every step of the recursion there exist arbitrarily $\leq_T$-complex witnesses, the constructed set is $\underset{\sim}{\mathbf{\Pi}}^1_1$.

The idea:

1. Well-order the set of conditions $\{c_\alpha \mid \alpha < \omega_1\}$.
2. If $A_\alpha \subseteq \mathbb{R}$ is a partial solution and c_α is not yet satisfied, show that $\{x \in \mathbb{R} \mid x \text{ satisfies } c_\alpha \text{ and } A \cup \{x\} \text{ is a partial solution}\}$ is cofinal in $\leq_T$.
3. Pick such x_α, and define $A = \{x_\alpha \mid \alpha < \omega_1\}$.

Example

$(V{=}L)$ There is a $\underset{\sim}{\mathbf{\Pi}}^1_1$ decomposition of $\mathbb{R}^3$ into disjoint circles.

Theorem (R.)

(*V*=*L*) *There exists a* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E \subseteq \mathbb{R}^2$ *for which* $\dim_H(E) = 1$ *yet for every* θ *we have* $\dim_H(\text{proj}_\theta(E)) = 0$.

Theorem (R.)

(*V*=*L*) *For every* $\epsilon \in (0, 1)$ *there exists a* $\underset{\sim}{\mathbf{\Pi}}^1_1$ *set* $E \subseteq \mathbb{R}^2$ *for which* $\dim_H(E) = 1 + \epsilon$ *yet for every* θ *we have* $\dim_H(\text{proj}_\theta(E)) = \epsilon$.

This is optimal by classical facts of geometric measure theory (e.g. Hausdorff dimension cannot drop by more than 1 under projection).

Takeaway

The complexity of the set is determined by the properties of real numbers—both globally, and locally!

Part II: From Reals to Elementary Substructures

Set-theoretical Structures in Topology

Two set-theoretical structures have found interesting relationships with topology.

Roitman's Model Hypothesis is an axiom due to J. Roitman (2011) to settle variants of the box product problem (is $\mathbb{R}^\omega$ under the box topology normal?).

Paul. E. Cohen's Pathways (1979) are a sequence of sets of reals, whose existence implies the existence of P-points (a special type of ultrafilter, whose existence in the random model is still open).

Recently, Barriga-Acosta, Brian, and Dow related these two.

Definition (P. E. Cohen's Pathways PE)

There exists a cardinal κ and an increasing sequence of sets $(A_\alpha)_{\alpha<\kappa}$ such that:

- $A_\alpha \subset \omega^\omega$
- $\bigcup_{\alpha<\kappa} A_\alpha = \omega^\omega$
- for every α, there exists $f \in A_{\alpha+1}$ such that if $g \in A_\alpha$ then $f \not<^* g$
- A_α is a Turing ideal

Call the sequence $(f_{\alpha+1})_{\alpha<\kappa}$ the fundamental sequence. The fundamental sequence traces the structure ω^ω.

Definition (Roitman's Model Hypothesis MH)

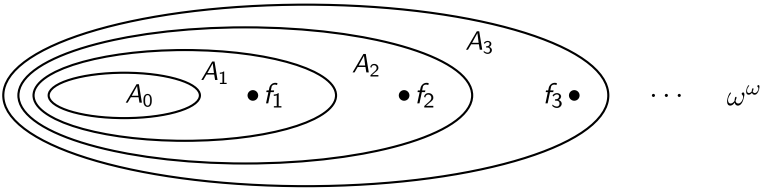
There exists a cardinal κ and an increasing sequence of sets $(M_\alpha)_{\alpha<\kappa}$ such that:

- $M_\alpha \subset H(\omega_1)$
- $\bigcup_{\alpha<\kappa} M_\alpha = H(\omega_1)$
- for every α, there exists $f \in M_{\alpha+1} \cap \omega^\omega$ such that if $g \in M_\alpha \cap \omega^\omega$ then $f \not<^* g$
- $M_\alpha \prec H(\omega_1)$

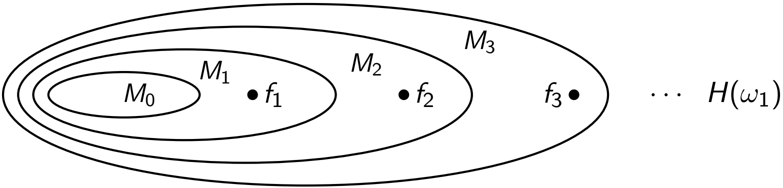
Call the sequence $(f_{\alpha+1})_{\alpha<\kappa}$ the fundamental sequence. The fundamental sequence traces the structure $H(\omega_1)$.

For the sake of emphasis...

Paul E. Cohen's Pathways PE:



Roitman's Model Hypothesis MH:



MH vs PE

There exists a cardinal κ and an increasing sequence of sets

$$(A_\alpha)_{\alpha<\kappa} \qquad\qquad (M_\alpha)_{\alpha<\kappa}$$

such that

$$A_\alpha \subset \omega^\omega \qquad\qquad M_\alpha \subset H(\omega_1)$$

$$\bigcup_{\alpha<\kappa} A_\alpha = \omega^\omega \qquad\qquad \bigcup_{\alpha<\kappa} M_\alpha = H(\omega_1)$$

and for every α, there exists

$$f \in A_{\alpha+1} \qquad\qquad f \in M_{\alpha+1} \cap \omega^\omega$$

such that if

$$g \in A_\alpha \qquad\qquad g \in M_\alpha \cap \omega^\omega$$

then $f \not<^* g$.

AND:

A_α is a Turing ideal $\qquad\qquad$ $M_\alpha \prec H(\omega_1)$

From Models to Reals

Theorem (Barriga-Acosta, Brian, Dow)

MH *implies* PE

Proof.

Use the fact that each M_α is an elementary substructure of $H(\omega_1)$—and hence closed under first-order definable truths—to "pull out" the sets of reals. □

Can we go the other way? Can one construct a sequence of elementary substructures of $H(\omega_1)$ from certain sets of reals alone?

With stronger hypotheses, here is one way to do this.

Going the Other Way?

Let $(A_\alpha)_{\alpha<\kappa}$ with fundamental sequence $(f_{\alpha+1})_{\alpha<\kappa}$ be given.

One approach by recursion:

1. Take some "minimal" structure induced by A_α.
2. Find witnesses to satisfy a countable sequence of Tarski-Vaught-conditions to build an elementary substructure M_α with
   - $A_\alpha \subset M_\alpha$
   - $f_{\alpha+1} \notin M_\alpha$

Question

What is a natural choice for the "induced" structure?
How do we find "nice" witnesses?

Computability theory helps!

Structures Induced by Sets of Reals

For a set $A \subseteq \omega^\omega$, consider

$$L^A := \bigcup_{x \in A} L_{\omega_1^x}[x].$$

These sets code a version of computational reduction, called hyperarithmetic reduction $\leq_h$:

Theorem (Kleene)

$$y \in L_{\omega_1^x}[x] \cap \omega^\omega \iff y \leq_h x$$

This is our "minimal" structure, since:

$$L^A \subset H(\omega_1)$$

Note: This resembles the Turing ideal structure of the A_α's, but our version is quite a bit stronger.

Coding Elements and Sets

Suppose we're at stage α. We are looking at

$$L^{A_\alpha} \quad \text{and} \quad f_{\alpha+1} \in A_{\alpha+1}.$$

We build

$$M_{\alpha+1}.$$

Instead of witnesses (elements), we choose codes (reals).

Lemma

Every set $a \in H(\omega_1)$ can be coded by a real $x \in 2^\omega$.

Given a formula φ true in $H(\omega_1)$, look at the set of codes of witnesses, $W(\varphi) \subset \omega^\omega$. This set is always projective:

Lemma (Folklore)

$A \subseteq \omega^\omega$ is Σ^1_{n+1} if and only if it is Σ_n over $(H(\omega_1), \in)$.

To complete the proof, we assume the following:

1. A_α is not only a Turing ideal, but a HYP-ideal.
2. The fundamental sequence $(f_{\alpha+1})_{\alpha<\kappa}$ satisfies that if $y \in \Delta^1_n(x)$ for any $x \in A_\alpha$ then

$$f_{\alpha+1} \not<^* y.$$

Call this a $(*)$-pathway.

Using Projective Determinacy and a Basis Lemma due to Moschovakis, we get:

Lemma

If $H(\omega_1) \vDash \varphi$, then the set of codes for witnesses $W(\varphi)$ contains an element that does not dominate $f_{\alpha+1}$.

Theorem (R.)

(PD) *If there is a $(*)$-pathway, then* MH *holds.*

Conclusions

Definable properties of real numbers determine interesting properties of sets:

- set theory $\longleftrightarrow$ regularity properties
- to characterise them—and other objects in classical mathematics—use computability theory
  - locally: point-to-set principle for Hausdorff dimension, $\utilde{\mathbf{\Pi}}^1_1$-recursion, all countable sets
  - globally: placement of objects in hierarchies, e.g. Borel/projective hierarchy, arithmetic hierarchy, to prove provability
- many other examples beyond descriptive set theory: e.g. reverse mathematics, computable structure theory

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Thank you