Automorphism groups and random dynamics

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Joint work with Daniel Hoffmann and Krzysztof Krupiński

Outline

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Outline of the talk:

1. Additional background/motivation
2. A new semigroups of types [over arb. theories]
3. A new convolution product [over arb. theories]
4. Idempotents and the classification of subgroups of automorphism group

Notation + Preliminaries

Remarks:

1. T will always be a complete first order theory.
2. $G(x)$ is an $\emptyset$-definable group w.r.t. T.
3. $\mathcal{U}, M \models T$; $\mathcal{U}$ will be a monster model; M a small elementary submodel.
4. Stable and NIP are properties of first order theories; they are combinatorial dividing lines.
5. Stable theories are very tame (e.g., Abelian or definable in $(\mathbb{C}; +, \times, 0, 1)$).
6. NIP theories are relatively tame (e.g., definable in $(\mathbb{R}; +, \times, 0, 1)$ or p-adics).
7. All stable theories are NIP.

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   2. Ellis semigroup, Newelski conjecture (Chernikov-Simon), WAP/tame flows
4. Convolution dynamics over definable groups
   1. Randomized variants of above connection
   2. (good) idempotent measures $\leftrightarrow$ (good) type-definable subgroups

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2. Suppose that M is any first order structure. Then $\mathrm{Aut}(M)$ acts on $S_x(M)$ via
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Solution: Encode a variant of the system into a *model theoretic semigroup* [a semigroup of types]; Use model theory to study the semigroup.

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Also, does not depend on choice of $\mathcal{U}$.

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We define an operation $* : S^{\mathrm{fs}}_G(\mathcal{U}, M) \times S^{\mathrm{fs}}_G(\mathcal{U}, M) \to S^{\mathrm{fs}}_G(\mathcal{U}, M)$ via

$$\begin{aligned} \theta(x, c) \in p * q &\iff \theta(x \cdot y, c) \in p \otimes q \\ &\iff \models \theta(a \cdot b, c) \end{aligned}$$

where $b \models q|_{Mc}$ and $a \models p|_{Mcb}$.

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Take away: The dynamical system can be encoded in a type space semigroup with a natural model theoretic product.

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1. Replace x with an infinite tuple corresponding to an enumeration of our model M. Then one could identify σ with the type $\mathrm{tp}(\sigma(\bar{m})/M)$.
2. Still need to work in the global finitely satisfiable* setting so that we can construct an analogue of the Newelski product.

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Let $q \in S^{\mathrm{fs}}_{\bar{m}}(\mathcal{U}, M)$. Then suppose that $\mathcal{U} \prec \mathcal{U}'$ and $\bar{\alpha} \models q$. Then there exists an automorphism $\sigma \in \mathrm{Aut}(\mathcal{U}')$ such that $\sigma(\bar{m}) = \bar{\alpha}$.

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We can now define the analogue product.

A product for types

Let $p, q \in S^{\mathrm{fs}}_{\bar{m}}(\mathcal{U}, M)$. Then

$$p * q := (\sigma \cdot \hat{p})|_{\mathcal{U}},$$

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Notice that since $\hat{p}$ is M-invariant, actually whether or not $\theta(x, \sigma^{-1}(\bar{b})) \in \hat{p}$ just depends on the type of $\sigma^{-1}(\bar{b})$ over M. Since $\mathcal{U}$ is saturated, the type of $\sigma^{-1}(\bar{b})$ over M is realized in $\mathcal{U}$.

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Hence,

$$\theta(\bar{x}, \bar{b}) \in (p * q) \iff \theta(\bar{x}, \bar{c}) \in p$$

where $\bar{b}\bar{\alpha} \equiv \bar{c}\bar{m}$.

An important map (twisting)

Fix a tuple $\bar{b} = b_1, ..., b_n$ from $\mathcal{U}$. Then we have a map $h_{\bar{b}} : S_{\bar{m}}(\mathcal{U}) \to S_{\bar{y}}(M)$ via

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Take Away: $(S^{\mathrm{fs}}_{\bar{m}}(\mathcal{U}, M), *)$ is the* appropriate semigroup of types in the automorphism group context.

Part II

Convolution for random automorphisms

Classical setting

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If $\mu, \nu \in \mathcal{P}(G)$, then the convolution product of μ and ν, denoted $\mu * \nu$, is the unique element of $\mathcal{P}(G)$ such that for any bounded continuous function $f : G \to \mathbb{R}$,

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Keep in mind: This operation naturally extends the product.

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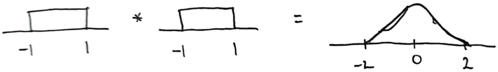
3. If L is the Lesbegue measure restricted to the interval $[-1, 1]$, then



A space of Keisler Measure

Much of our work is in the context of the following spaces:

Definition

Let $\pi(\bar{x};\bar{m})$ be the partial type over M which states "$\mathrm{tp}(\bar{m}/\emptyset) = \mathrm{tp}(\bar{m}'/\emptyset)$". Then

$$\mathfrak{M}^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M) := \{\mu \in \mathfrak{M}_{\bar{m}}(\mathcal{U}, M) : \mu([\pi(\bar{x};\bar{m})]) = 1, \mu \text{ is } M\text{-invariant}\}.$$

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By f.s. in M, we mean that if $\mu(\varphi(x,c)) > 0$, then there exists some $d \in M^x$ such that $\mathcal{U} \models \varphi(d,c)$.

By M-invariant, we mean that if $a, b \in \mathcal{U}^z$ and $a \equiv_M b$, then

$$\mu(\varphi(x,a)) = \mu(\varphi(x,b)).$$

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$$\mathfrak{M}^{\mathrm{fs}}_{\bar{m}}(\mathcal{U}, M) := \{\mu \in \mathfrak{M}_{\bar{m}}(\mathcal{U}, M) : \mu([\pi(\bar{x};\bar{m})]) = 1, \mu \text{ is f.s. in } M\}.$$

By f.s. in M, we mean that if $\mu(\varphi(x,c)) > 0$, then there exists some $d \in M^x$ such that $\mathcal{U} \models \varphi(d,c)$.

By M-invariant, we mean that if $a, b \in \mathcal{U}^z$ and $a \equiv_M b$, then

$$\mu(\varphi(x,a)) = \mu(\varphi(x,b)).$$

When T is NIP, these spaces admit a convolution operation.

Twisted Morley product

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Then for $\mu, \nu \in \mathfrak{M}^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M)$, we define the convolution product as follows:

$$\begin{aligned}(\mu * \nu)(\varphi(x_{i_1}, ..., x_{i_n}, b_1, ..., b_k)) &= \int_{S^{fs}_{\bar{m}}(\mathcal{U},M)} \left(F^{\varphi}_{\mu} \circ h_{\bar{b}}\right) d\nu \\ &= \int_{S_{\bar{y}}(M)} F^{\varphi}_{\mu} d\left(h_{\bar{b}}\right)_* (\nu) \\ &= (\mu \otimes (h_{\bar{b}})_*(\nu))(\varphi(x_{i_1}, ..., x_{i_n}, y_1, ..., y_n)).\end{aligned}$$

Convolution in automorphism setting

Theorem (G., Hoffmann, Krupiński (2025))

Suppose that T is NIP

1. If $\mu, \nu \in \mathfrak{M}^{\dagger}_{\bar{m}}(\mathcal{U}, M)$, then $\mu * \nu \in \mathfrak{M}^{\dagger}_{\bar{m}}(\mathcal{U}, M)$ for $\dagger \in \{\mathrm{inf}, \mathrm{fs}\}$.
2. Definable convolution extends the product on types, i.e. If $p, q \in S^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M)$, then $\delta_{p*q} = \delta_p * \delta_q$.
3. The convolution operation is left continuous, i.e. for any $\mu \in \mathfrak{M}^{\mathrm{inv}}_{G}(\mathcal{U}, M)$, the map $- * \mu : \mathfrak{M}^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M) \to \mathfrak{M}^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M)$ is continuous.
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4. The definable convolution operation is associative on fs.
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Open question: Is the convolution product associative on $\mathfrak{M}^{\text{inv}}_{\bar{m}}(\mathcal{U}, M)$?

Encoding

This new convolution product *encodes* the standard convolution operation.

Theorem (G., Hoffmann, Krupiński (2025))

Suppose that T is an NIP structure and G be a $\emptyset$-definable group. If $M \models T$ we let $M_S = (M, S, \cdot)$ be the expansion of M by a new sort S with a regular action $\cdot$ of $G(M)$ on S and no other structure. Then there exists a type-definable set π_G such that

$$(\mathfrak{M}^{\mathrm{inv}}_{\pi_G}(\mathcal{U}_S, M_S), *) \cong (\mathfrak{M}^{\mathrm{inv}}_{G}(\mathcal{U}, M), *)$$

As consequence, (counter)examples from the definable group setting transfer the to automorphism group setting.

An application

Classifying subgroups of the Automorphism group

Relatively definable subgroups

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What else?

Idempotent measures

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Proposition (G., Hoffmann, Krupiński (2025))

Suppose that $\mu \in \mathfrak{M}^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M)$ and μ is definable. Then $\mathrm{stab}(\mu)$ is a relatively $\bar{m}$-type definable subgroup of $\mathrm{Aut}(\mathcal{U})$ (over M).

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Consequence: If T is stable, then all measures are definable and so any M-invariant idempotent Keisler measure implies the existence of a relatively $\bar{m}$-type definable subgroup of $\mathrm{Aut}(\mathcal{U})$.

Example I: An idempotent

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Enumerate M with $\bar{m} = m_1, m_2, m_3, \ldots$ For each $i < \omega$ consider the type $p_i(x_i) \in S^{\mathrm{inv}}_{x_i}(\mathcal{U}, M)$ where $m_i E x_i \in p_i$ and $p_i \vdash x \neq c$ for any $c \in \mathcal{U}$. Consider the type given by

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$$\text{stab}(p) = \left\{ \sigma \in \text{Aut}(\mathcal{U}) : \mathcal{U} \models \bigwedge_{i \in \omega} m_i E \sigma(m_i) \right\}.$$

Example II: An idempotent

Let T be the theory of the random graph. Let $\bar{m}$ be an enumeration of M. Let $\Phi(\bar{x})$ be a formula without parameters. Then there is a unique measure μ in $\mathfrak{M}^{\mathrm{inv}}_{\bar{m}}(\mathcal{U}, M)$ which satisfies the following: For any finite sets of parameters $B_1, \ldots, B_n$, possibly pairwise indistinct, and for any $\epsilon \colon \mathbb{N} \times \bigcup_{i=1}^{n} B_i \to \{0,1\}$ we have that

$$\mu\left(\Phi(\bar{x}) \wedge \bigwedge_{i=1}^{n} \bigwedge_{b \in B_i} R^{\epsilon(i,b)}(x_i, b)\right) = \begin{cases} \frac{1}{2^{|B_1|+\cdots+|B_n|}} & \models \Phi(\bar{m}), \\ 0 & \text{otherwise,} \end{cases}$$

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Results in stable theories

In the stable case, idempotent measures completely classify relatively type-definable subgroups of the automorphism group.

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Remark: Similar theorem from the **definable group setting**; G is a stable group, then idempotent Keisler measures $\leftrightarrow$ type-definable subgroups.

The proof relies on an automorphism variant of Newelski's variant of Hrushovski's group chunk theorem. To do this, we needed to develop some stable group theory for relatively type-definable subgroups of $\mathrm{Aut}(\mathcal{U})$.

Conjecture (CGK + GHK)

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Conjecture (CGK + GHK)

Suppose that T is an arbitrary theory. There there is a one-to-one correspondence between fim measures and fim relatively $\bar{m}$-type definable subgroups of $\mathrm{Aut}(\mathcal{U})$ (over M) via $\mu \to \mathrm{stab}(\mu)$.

Thank you

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