Fields with operators satisfying compatibility conditions (joint work with Omar Leon Sanchez)

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Derivations

Definition

Let R, S be rings (we assume rings are commutative and with 1). A function $\partial : R \to S$ is called a *derivation*, if it satisfies

- (additivity) $\partial(a+b) = \partial(a) + \partial(b)$ for all $a, b \in R$.
- (Leibniz rule) $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in R$.

If K is a field equipped with several commuting derivations $\partial_1, \ldots, \partial_n$, we will sometimes call ∂_i *partial derivations*.

How to determine whether a system of partial differential equations over K is consistent (i.e. whether it has a solution in an extension $L \supseteq K$ in which the derivations still commute)?

Example

Consider the system

$$\begin{cases} \partial_1(z) = z \\ \partial_2(z) = 1 \end{cases} \tag{1}$$

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Consistency of systems of differential equations

$$\begin{cases} \partial_1(z) = z \\ \partial_2(z) = 1 \end{cases} \tag{2}$$

This system is inconsistent, as if z were a solution, then we would have $\partial_1\partial_2(z) = \partial_1(1) = 0$ but $\partial_2\partial_1(z) = \partial_2(z) = 1 \neq 0 = \partial_1\partial_2(z)$.
Note that the system above is algebraically consistent, that is, if we replace $\partial_1(z)$ and $\partial_2(z)$ with new variables z_1 and z_2, then we obtain a system of polynomial (even linear) equations in variables z, z_1, z_2

$$\begin{cases} z_1 = z \\ z_2 = 1 \end{cases} \tag{3}$$

which of course is consistent. However, applying ∂_2 to the equation $\partial_1(z) = z$, we obtain $\partial_2(\partial_1(z)) = \partial_2(z)$ and applying ∂_1 to $\partial_2(z) = 1$ we obtain $\partial_1(\partial_2(z)) = 0$, and replacing both $\partial_1(\partial_2(z))$ and $\partial_2(\partial_1(z))$ with a new variable z_{12} and $\partial_i(z)$ with z_i, we get

$$\begin{cases} z_{12} = z_2 \\ z_{12} = 0 \end{cases} \tag{4}$$

Together with the equation $z_2 = 1$ we had before, this gives an inconsistent system of polynomial equations. So, differentiating the given system once revealed inconsistency.

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Revealing inconsistency of a system

It is not hard to see that a given system of partial differential equations is consistent if and only if, for every $k < \omega$, the system of equations obtained by differentiating the system $\leq k$ many times is algebraically consistent.

Fact (Pierce 2007; Leon Sanchez and Gustavson 2017)

There exists a number $k < \omega$ depending only on the complexity of a given system of polynomial partial differential equations such that if the system is inconsistent, then the system of polynomial equations obtained by differentiating the given system $\leq k$ many times is algebraically inconsistent.

This yields an algorithm deciding consistency of a system of polynomial partial differential equations, and also allows to deduce the existence of a model companion of the theory of fields with n commuting derivations for any n.
The methods used both by Pierce and by Leon Sanchez and Gustavson are based on the notion of a differential kernel, which was introduced by Lando in 1970. Below, for simplicity, we restrict ourselves to the case of two commuting derivations and characteristic 0.

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Differential kernels (after Lando, Cohn, Pierce,...)

Suppose k is a field (of characteristic 0) with two commuting derivations ∂_1 and ∂_2.

When we start from an element $a = a_{0,0}$, we consider elements a_{k_1,k_2} that are "prototypes" of $\partial_1^{k_1}\partial_2^{k_2}(a)$.

A differential kernel of height r over k is a field $k(a_{k_1,k_2})_{k_1,k_2\in\omega\wedge k_1+k_2\leq r}$ such that whenever $f((a_{k_1,k_2})_{k_1+k_2\leq r-1}) = 0$ for some polynomial f over k, then the polynomial obtained by differentiating f with respect to ∂_1 and ∂_2 both vanish on $(a_{k_1,k_2})_{k_1+k_2\leq r}$.

Equivalently, $a_{k_1,k_2} \mapsto a_{k_1+1,k_2}$ defines a derivation on $k(a_{k_1,k_2})_{k_1,k_2\in\omega\wedge k_1+k_2\leq r-1}$ extending ∂_1, and $a_{k_1,k_2} \mapsto a_{k_1,k_2+1}$ defines a derivation on $k(a_{k_1,k_2})_{k_1,k_2\in\omega\wedge k_1+k_2\leq r-1}$ extending ∂_2.

Write $(\ell_1,\ell_2) \prec (k_1,k_2)$ for $(\ell_1+\ell_2,\ell_1,\ell_2) <_{lex} (k_1+k_2,k_1,k_2)$. We say (k_1,k_2) is a *leader*, if a_{k_1,k_2} is algebraic over $k(a_{\ell_1,\ell_2})_{(\ell_1,\ell_2)\prec(k_1,k_2)}$.

A leader is minimal, if there is no leader $(\ell_1,\ell_2) \neq (k_1,k_2)$ with $\ell_1 \leq k_1$ and $\ell_2 \leq k_2$ (we will write $(\ell_1,\ell_2) \leq (k_1,k_2)$ for $\ell_1 \leq k_1 \wedge \ell_2 \leq k_2$).

The companionability of the theory of fields with commuting derivations was deduced by Pierce from his following result:

Fact(Pierce)

If a kernel L of height $2r$ has no minimal leaders at levels higher than r, then L has a differential kernel extension of arbitrary height $s \geq 2r$.

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Fields with operators

Let R be a ring.
Example 1 Consider the algebra $D = R[x]/(x^2)$. Then a function $e : R \to D$ of the form $e(s) = s + f(s)x + (x^2)$ is a ring homomorphism iff f is additive and $e(st) = e(s)e(t)$ for all $s, t \in R$.
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More generally, if k is a field and D a finite-dimensional algebra over k equipped with a basis $(1, \epsilon_1, \ldots, \epsilon_m)$, then a D-operator on a ring $R \supseteq k$ is a homomorphism $e : R \to D \otimes_k R$ of the form $s \mapsto s + \partial_1(s)\epsilon_1 + \cdots + \partial_m(s)\epsilon_n$ with $\partial_i : R \to R$. We will denote $D \otimes_k R$ by $D(R)$.

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This means $st + f(st) + (x^2) = (s + f(s))(t + f(t)) + (x^2) = st + (sf(t) + f(s)t)x + (x^2)$ i.e. $f(st) = sf(t) + f(s)t$, i.e. f is a derivation.
Example 2 If $D = R \times R$, then $e : R \to D$ given by $e(s) = (s, f(s))$ is a ring homomorphism iff $f : R \to R$ is a ring endomorphism.
Example 3 If $D = R[x]/(x^3)$, then $e : R \to D$ given by $e(s) = s + f(s)x + g(s)x^2 + (x^3)$ is a ring homomorphism iff f is a derivation, and g is an order two derivation w.r.t. f, i.e. g is additive and $g(st) = g(s)t + 2f(s)f(t) + sg(t)$.

Definition (Moosa-Scanlon)

More generally, if k is a field and D a finite-dimensional algebra over k equipped with a basis $(1, \epsilon_1, \ldots, \epsilon_m)$, then a D-operator on a ring $R \supseteq k$ is a homomorphism $e : R \to D \otimes_k R$ of the form $s \mapsto s + \partial_1(s)\epsilon_1 + \cdots + \partial_m(s)\epsilon_n$ with $\partial_i : R \to R$. We will denote $D \otimes_k R$ by $D(R)$.

Fields with operators

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Example 1 Consider the algebra $D = R[x]/(x^2)$. Then a function $e : R \to D$ of the form $e(s) = s + f(s)x + (x^2)$ is a ring homomorphism iff f is additive and $e(st) = e(s)e(t)$ for all $s, t \in R$.

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More generally, if k is a field and $\mathcal{D}$ a finite-dimensional algebra over k equipped with a basis $(1, \epsilon_1, \ldots, \epsilon_m)$, then a $\mathcal{D}$-operator on a ring $R \supseteq k$ is a homomorphism $e : R \to \mathcal{D} \otimes_k R$ of the form $s \mapsto s + \partial_1(s)\epsilon_1 + \cdots + \partial_m(s)\epsilon_n$ with $\partial_i : R \to R$. We will denote $\mathcal{D} \otimes_k R$ by $\mathcal{D}(R)$.

Applications of model theory of fields with operators usually rely on the existence of a *model companion*. Recall a model M of a theory T is *existentially closed* if every quantifier-free formula $\phi(x)$ over M that has realisation in some $M \subseteq N \models T$, has a realisation in M.
So, informally speaking, a field with operators is existentially closed when it is closed under adding solutions of systems of equations.

A *model companion* of a theory T is a theory axiomatising the class of existentially closed models of T. We say T is *companionable* if T has a model companion.
Robinson has proved that the theory of fields of characteristic zero with a derivation has a model companion called DCF_0, and Macintyre has proved that the theory of fields with an automorphism has a model companion called *ACFA*.
More generally, Moosa and Scanlon have proved that, in characteristic zero, the theory of fields with $\mathcal{D}$-operators has a model companion for every finite dimensional algebra $\mathcal{D}$ (with a distinguished basis) assuming $res(B_i) = k$ for a local decomposition $\mathcal{D} = B_1 \times \cdots \times B_k$ of $\mathcal{D}$.
In positive characteristic, Beyarslan, Hoffmann, Kamensky and Kowalski have proved that a model companion exists iff the nilradical of $\mathcal{D}$ coincides with the kernel of the Frobenius morphism $\mathcal{D} \to \mathcal{D}$.

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Compatibility conditions

The results of Moosa-Scanlon and Beyarslan-Hoffmann-Kamensky-Kowalski deal with the *free* case, that is, ∂_i are not required to satisfy any compatibility with each other.

The most straightforward compatibility condition on the operators ∂_i is that they commute with each other: $\partial_i \partial_j = \partial_j \partial_i$ for all $1 \leq i, j \leq m$.

In case of $\mathcal{D} = k \times k \times k$, this yields the theory of fields with two commuting automorphisms, which is **not** companionable by a result of Hrushovski.

However, the theory of fields with n commuting derivations is known to be companionable for any n by a result of Pierce.

In our work, we have proved that for a local $\mathcal{D}$, and a general compatibility notion that we call Γ-commutativity, the theory of fields with Γ-commuting $\mathcal{D}$-operators is always companionable in characteristic zero, and in positive characteristic is companionable if the maximal ideal of $\mathcal{D}$ coincides with the kernel of the Frobenius morphism $Fr : \mathcal{D} \to \mathcal{D}$. The examples falling into this framework include:

- fields with Lie-commuting derivations (studied by Yaffe),
- fields with iterative Hasse-Schmidt derivations (studied by Ziegler),
- fields with $\mathfrak{g}$-iterative Hasse-Schmidt derivations for a finite group scheme $\mathfrak{g}$ (studied by Hoffmann-Kowalski), and
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In our work, we have proved that for a local $\mathcal{D}$, and a general compatibility notion that we call Γ-commutativity, the theory of fields with Γ-commuting $\mathcal{D}$-operators is always companionable in characteristic zero, and in positive characteristic is companionable if the maximal ideal of $\mathcal{D}$ coincides with the kernel of the Frobenius morphism $Fr : \mathcal{D} \to \mathcal{D}$. The examples falling into this framework include:

- fields with Lie-commuting derivations (studied by Yaffe),
- fields with iterative Hasse-Schmidt derivations (studied by Ziegler),
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The results of Moosa-Scanlon and Beyarslan-Hoffmann-Kamensky-Kowalski deal with the *free* case, that is, ∂_i are not required to satisfy any compatibility with each other.

The most straightforward compatibility condition on the operators ∂_i is that they commute with each other: $\partial_i \partial_j = \partial_j \partial_i$ for all $1 \leq i, j \leq m$.
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Compatibility conditions (a simplified version)

We fix a field k for the rest of the talk. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be local algebras over k with $res(\mathcal{D}_i) = k$ (we assume all algebras and rings are commutative and have 1). Recall that by $\mathcal{D}_i(R)$ we denote the base change of $\mathcal{D}_i$ from k to R, i.e. $\mathcal{D}_i(R) = \mathcal{D}_i \otimes_k R$.
Fix a k-algebra homomorphism $r : \mathcal{D}_2 \to \mathcal{D}_1 \otimes_k \mathcal{D}_2$. Let $R \supseteq k$. We we have two lifts of r:

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where the lift r^{ι} is with respect to the standard R-algebra structure on $\mathcal{D}_2(R)$ and on $\mathcal{D}_1(\mathcal{D}_2(R))$, while the lift r^{e_1} is with respect to the R-linear structure on $\mathcal{D}_1(\mathcal{D}_2(R))$ given by

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Example: usual commutativity

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\mathcal{D}_1(e_2) \circ e_1(a) &= \mathcal{D}_1(e_2)\left(a + \epsilon_{1,1}\partial_{1,1}(a) + \cdots + \epsilon_{1,m_1}\partial_{1,m_1}(a)\right) \\
&= 1 \otimes 1 \otimes e_2(a) + \epsilon_{1,1} \otimes 1 \otimes e_2(\partial_{1,1}(a)) + \cdots + \epsilon_{1,m_1} \otimes 1 \otimes e_2(\partial_{1,m_1}(a)) \\
&= 1 \otimes 1 \otimes a + 1 \otimes \epsilon_{2,1} \otimes \partial_{2,1}(a) + \cdots + 1 \otimes \epsilon_{2,m_2} \otimes \partial_{2,m_2}(a) + \\
&\quad \epsilon_{1,1} \otimes 1 \otimes \partial_{1,1}(a) + \epsilon_{1,1} \otimes \epsilon_{2,1} \otimes \partial_{2,1}\partial_{1,1}(a) + \cdots + \epsilon_{1,1} \otimes \epsilon_{2,m_2} \otimes \partial_{2,m_2}\partial_{1,1}(a) \; + \ldots \\
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\end{aligned}
$$

As $r^{e_1} = \mathcal{D}_2(e_1)$, for $r^{e_1} \circ e_2(a)$ we get the same expression but with $\partial_{1,i}\partial_{2,j}$ in place of $\partial_{2,j}\partial_{1,i}$. As $(\epsilon_{1,i} \otimes \epsilon_{2,j} : 0 \leq i \leq m_1, 0 \leq j \leq m_2)$ is an R-linear basis of $\mathcal{D}_1(\mathcal{D}_2(R))$, it follows that $\mathcal{D}_1(e_2) \circ e_1(a) = r^{e_1} \circ e_2(a)$ if and only if
$\partial_{1,i}\partial_{2,j}(a) = \partial_{2,j}\partial_{1,i}(a) \quad$ for all $1 \leq i \leq m_1, 1 \leq j \leq m_2$.

Example: Lie-commuting derivations

Let $m \in \mathbb{N}$ and

$$\mathcal{D} = k[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^2$$

with $\pi(\epsilon_i) = 0$ and ranked basis $(1, \epsilon_1, \ldots, \epsilon_m)$. This recovers differential rings with m-many derivations. Let $(c_\ell^{ij})_{i,j,\ell=1}^m$ be a tuple from k such that for each ℓ the $m \times m$-matrix $(c_\ell^{ij})_{i,j=1}^m$ is skew-symmetric. Consider the k-algebra homomorphism $r : \mathcal{D} \to \mathcal{D}(\mathcal{D}(k))$ determined by

$$r(\epsilon_\ell) = 1 \otimes \epsilon_\ell + \sum_{i,j=1}^m \epsilon_i \otimes \epsilon_j \otimes c_\ell^{ji}$$

for $\ell = 1, \ldots, m$. Then, on any $\mathcal{D}$-ring (R, e), e commutes on R with respect to r^e if and only if

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Example: Iterative truncated Hasse-Schmidt derivations

Assume $char(k) = p > 0$. Let $n \in \mathbb{N}$ and $\mathcal{D} = k[\epsilon]/(\epsilon)^{p^n}$ with $\pi(\epsilon) = 0$ and ranked basis $(1, \epsilon, \dots, \epsilon^{p^n-1})$. This recovers rings equipped with a $(p^n - 1)$-truncated Hasse-Schmidt derivation. Consider the k-algebra homomorphism $r : \mathcal{D} \to \mathcal{D} \otimes_k \mathcal{D}$ determined by

$$r(\epsilon) = \epsilon \otimes 1 + 1 \otimes \epsilon$$

(the fact that $char(k) = p$ yields that r is indeed a homomorphism). Then, on any $\mathcal{D}$-ring (R, e), e commutes on R with respect to r^ι if and only if for $1 \leq i, j \leq n$ we have

$$\partial_j \partial_i = \begin{cases} \binom{i+j}{i} \partial_{i+j} & i + j \leq p^n - 1 \\ 0 & i + j \geq p^n \end{cases}$$

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Lie commutativity

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{D}$ and let $d < \omega$ be minimal such that $\mathfrak{m}^{d+1} = 0$.

We say that a homomorphism $r : \mathcal{D} \to \mathcal{D} \otimes_k \mathcal{D}$ is of Lie-commutation type if there is a tuple $(c_\ell^{ij})_{i,j,\ell=1}^m$ from k such that

$$r(\epsilon_\ell) = \epsilon_\ell \otimes 1 + \sum_{i,j=1}^{m} \epsilon_i \otimes \epsilon_j \otimes c_\ell^{ji}$$

and $c_\ell^{ji} = 0$ unless $\epsilon_i, \epsilon_j \in \mathfrak{m}^d$. We call the tuple (c_ℓ^{ji}) the *Lie-coefficients* of r.

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HS-commutativity

We say $r : \mathcal{D} \to \mathcal{D} \otimes_k \mathcal{D}$ is of HS-iteration if there is a tuple $(c_\ell^{ij})_{i,j,\ell=1}^m$ from k such that

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Associativity condition

Let r be of HS-iteration type. Suppose there exists a $\mathcal{D}^r$-ring (R, e) where the operators $(\partial_1, \dots, \partial_m)$ are k-linearly independent (as functions $R \to R$).

Then $\partial_i \partial_j \partial_k = \sum_\ell c_\ell^{ij} \partial_\ell \partial_k = \sum_r \left(\sum_\ell c_\ell^{ij} c_r^{\ell k} \right) \partial_r$. On the other hand,

$$\partial_i \partial_j \partial_k = \sum_\ell \partial_i (c_\ell^{jk} \partial_\ell) = \sum_\ell c_\ell^{jk} \partial_i \partial_\ell = \sum_\ell \sum_r c_\ell^{jk} c_r^{i\ell} \partial_r$$

As $(\partial_1, \dots, \partial_m)$ are k-linearly independent, comparing coefficients on both sides yields that $\sum_\ell c_\ell^{ij} c_r^{\ell k} = \sum_\ell c_\ell^{jk} c_r^{i\ell}$ for all $1 \leq i, j, k, r \leq m$.

Definition

Let r be of HS-type. We say that r is *associative* if for all $1 \leq i, j, k, r \leq m$ we have

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Jacobi condition

For similar reasons, we introduce the following definition.

Definition

Let r be of Lie type. We say r is Jacobi, if for each ℓ, the $m \times m$ matrix $(c_\ell^{ij})_{i,j=1}^m$ is skew-symmetric, and for each $1 \leq i, j, k, r \leq m$ we have

$$\sum_{p=1}^{m} \left(c_p^{ij} c_r^{pk} + c_p^{ki} c_r^{pj} + c_p^{jk} c_r^{pi} \right) = 0$$

(this is a form of the Jacobi identity)

Remark

Both the iterativity and the Jacobi conditions become more complicated when we do not assume that c_ℓ^{ij} are in k (hence constant for all ∂_r).

Let $r_1 : \mathcal{D}_1 \to \mathcal{D}_1 \otimes_k \mathcal{D}_1$ be of Lie type and Jacobi, and let $r_2 : \mathcal{D}_2 \to \mathcal{D}_2 \otimes_k \mathcal{D}_2$ be of HS-iteration type and associative. Put $\Gamma = (r_1, r_2)$. We say that Γ is Jacobi-associative if e_1 is Jacobi and e_2 is associative.

For operators $e_1 : R \to \mathcal{D}_1(R)$ and $e_2 : R \to \mathcal{D}_2(R)$, we say that (e_1, e_2) Γ-commutes if

- e_1 commutes with respect to $r_1^{e_1}$,
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1. Let Γ be Jacobi-associative. If $char(k) = 0$, then the theory of fields with Γ-commuting $(\mathcal{D}_1, \mathcal{D}_2)$-operators is companionable. In positive characteristic the same is true if the maximal ideal of $\mathcal{D}_i$ coincides with the kernel of the Frobenius homomorphism $Fr : \mathcal{D}_i \to \mathcal{D}_i$ for $i = 1, 2$.
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If a kernel L of height $2r$ has no minimal separable leader of length bigger than r, then L has a differential kernel extension of arbitrary height $s \geq 2r$.

The proof of this statement is by inductive construction of partial kernels $k(a_{(\ell_1,\ell_2)})_{(\ell_1,\ell_2)\prec(k_1,k_2)}$ satisfying the differential condition:
If a polynomial f over k vanishes on $(a_{(\ell_1,\ell_2)})_{(\ell_1,\ell_2)\prec(k_1-1,k_2)}$, then the polynomial $\partial_1(f)$ vanishes on $k(a_{(\ell_1,\ell_2)})_{(\ell_1,\ell_2)\prec(k_1,k_2)}$, and likewise for ∂_2. (*)
While the notion of a differential kernel has a natural analogue in our context of Γ-commuting operators, the above condition (*) for "partial" kernel $(a_{(\ell_1,\ell_2)})_{(\ell_1,\ell_2)\prec(k_1-1,k_2)}$ does not seem to have any reasonable analogue in our context.
Hence, we do the kernel construction differently - given a kernel of length s, we first we construct some "approximation" of a kernel of length $r+1$ that induces **non-Γ-commuting** operators, on which we then perform suitable a sequence of specialisations that eventually yield Γ-commutativity.

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Some more details

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Let $\mathbb{N}_0^{\mathfrak{d}}(r) = \{\beta \in \mathbb{N}_0^{\mathfrak{d}} : \ |\beta| \leq r\}$. Write $\beta \in \mathbb{N}_0^{\mathfrak{d}}(r)$ as

$$\beta = ((\beta_{1,1}, \ldots, \beta_{1,m_1}), (\beta_{2,1}, \ldots, \beta_{2,m_2}))$$

and consider the map $\psi : \mathbb{N}_0^{\mathfrak{d}}(r) \to \mathfrak{d}^{\leq r}$ given by

$$\psi(\beta) = ((1, m_1), \ldots, (1, m_1), \ldots, (1, 1), \ldots, (1, 1), (2, m_2), \ldots, (2, m_2), \ldots, (2, 1), \ldots, (2, 1))$$

where (u, i) appears $\beta_{u,i}$-times. Clearly ψ is injective, thus we identify $\mathbb{N}_0^{\mathfrak{d}}(r)$ with its image in $\mathfrak{d}^{\leq r}$.

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We prove by induction on $(\mu, t) \in \mathbb{N}_0^{\mathfrak{d}}(s+1) \times \mathfrak{n}$ with respect to $\trianglelefteq$ that there is a specialisation $(a_t^\xi)_{(\xi,j)\in\mathfrak{d}^{s+1}\times\mathfrak{n}}$ of the tuple $({a'}_t^\xi)_{(\xi,t)\in\mathfrak{d}^{s+1}\times\mathfrak{n}}$ over L_s such that, considering the $\underline{\mathcal{D}}$-structure on L_s given by the kernel $K(a_t^\xi)_{(\xi,t)\in\mathfrak{d}^{s+1}\times\mathfrak{n}}$ satisfying:

For all $(\tau, t') \in \mathbb{N}_0^{\mathfrak{d}} \times \mathfrak{n}$ and $i, j \in \mathfrak{d}$ with $(\rho(i,j,\tau), t') \trianglelefteq (\mu, t)$ we have

$$\partial_i \partial_j (a_{t'}^\tau) = \chi_{ij} \partial_j \partial_i (a_{t'}^\tau) + \sum_\ell c_\ell^{ij} \partial_\ell (a_{t'}^\tau).$$

i.e. $\partial_i \partial_j (a_{t'}^\tau)$ is what it is expected to be,
together with a number of auxiliary conditions letting us carry out the construction.