Higher Structural Reflection and Very Large Cardinals

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Incompleteness and Gödel's Program

ZFC, the standard axioms of mathematics, is incomplete. Numerous interesting questions from various areas of mathematics have been shown to be unresolved by ZFC.

Gödel's program seeks *well-justified* additional axioms to *resolve important questions independent of ZFC*.

Strong axioms of infinity, or *large cardinal axioms* (LCAs), assert the existence of extraordinarily large infinities.

LCAs are the central additional axioms, and they do resolve numerous independent questions.

Are they also well-justified?

Reflection Principles

The idea that the universe of sets is too rich and complicated to be simply definable is often seen to be *intrinsically plausible*.

The *Levy-Montague* reflection principle, asserting that any statement true in V is also true in some V_α, is seen as an expression of this idea. The reflection principle is provable from ZFC.

Reflection Principles

An approach to justify large cardinals is to strengthen the reflection principle, and derive large cardinal axioms from them. This approach was endorsed by Gödel[1]:

> *Generally I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is undefinable, where definability is to be taken in [a] more and more generalized and idealized sense.*

However, traditional proposals along this line faced many difficulties.[2]

[1] Hao Wang, *A Logical Journey*, 1997

[2] For a discussion see Peter Koellner, "On Reflection Principles", *APAL*, 2009.

The Structural Reflection Approach

A class $\mathcal{C}$ of first order structures in the same language is definable by a formula φ if $\mathcal{C} = \{A = \langle A, \langle R_i \rangle_{i \in I} \rangle : \varphi(A)\}$

Joan Bagaria proposed the following principle schema[3]:

SR (Structural Reflection): For every definable class $\mathcal{C}$ of structures in the same language there exists an ordinal α that *reflects* C, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\alpha$ and an elementary embedding from B into A.

[3]First appeared in J.Bagaria "$C^{(n)}$-cardinals," *Archive for Mathematical Logic*, 2012

The Structural Reflection Approach

The idea: instead of reflecting the *theory* of V, we reflect the *structural content of V*.

Instead of requiring any definable property of V to already hold in some V_α, we require any definable class of structures to be already *almost contained* in some V_α, in the sense that for any $A \in \mathcal{C}$ there is some $B \in \mathcal{C} \cap V_\alpha$ that *structurally resembles A*.

B structurally resembles A: B is isomorphihc to some elementary substructure of A, i.e., B is elementary embeddable into A.

The Structural Reflection Approach

Γ-SR is SR restricted to Γ definable (without parameters) class of structures.

Theorem (Bagaria)

For every natural number $n \geq 1$:

1. Π_1-SR *holds if and only if there exists a supercompact cardinal.*
2. Π_2-SR *holds if and only if there exists an extendible cardinal.*
3. Π_{n+1}-SR *holds if and only if there exists a $C^{(n)}$-extendible cardinal.*

Theorem (Bagaria)

The following schema are equivalent:

1. *SR.*
2. VP*.*
3. *There exists a $C^{(n)}$-extendible cardinal for all natural numbers n.*

The Structural Reflection Approach

Over the years, Bagaria and his collaborators have characterized large cardinals of varying strengths through variants of SR.

The ultimate goal of Bagaria's SR Program: Identifying a single, general SR principle that is intrinsically plausible, such that every LCA is equivalent to some instance of this principle.

The limits of the SR program

However, the variants of SR proposed are very different principles, such as principles of *product* SR, *generic SR*, and *exact SR*, rather than instances of a single principle. This is in tension with the ideal of uniformity.

Among these variants, the strongest are the principles of exact structural reflection (ESR), proposed by Bagaria and Lücke, and possess consistency strength at the level of (almost) huge cardinals, and beyond. However, ESR does not seem to admit the same kind of justification as SR. This is in tension with the ideal of intrinsic justification.

The limit of the SR program

For cardinals $\kappa < \lambda$, Γ-ESR(κ, λ) holds if for any Γ definable class C of structures in the same language and any structure $A \in C$ of rank λ, there is some $B \in C$ of rank κ and an elementary embedding from B to A.

Problem: ESR is not about reflecting the structures in V to structures in some V_α.

The lack of justification for ESR was acknowledged by Bagaria himself:[4]

> *The only SRPs which would, thus, be prone to the objection are those yielding large cardinals stronger than VP; hence, by this argument's lights, it would only be ESR and the corresponding large-cardinal notions, that would be lacking intrinsic evidence.*

[4] Bagaria and Ternullo, "Intrinsic justification for large cardinals and structural reflection", *Philosophia Mathematica*, 2025

Motivating Question

SR is a natural principle, though of course the question of whether it is *intrinsically plausible*—and what such plausibility would entail—is hard to resolve conclusively. However, the above problems show that even one assumes the plausibility of SR, it does not extend to the stronger principle of ESR.

A key question concerning the prospects of the SR program: Is there a general form of SR that

1. is stronger than SR,
2. has SR as one of its instances, and
3. can be given a justification similar to that of SR?

A positive answer would fulfill both the ideal of uniformity and the ideal of justification (assuming SR is justified).

Structural Resemblance

In the formulation of SR, the notion of *structural resemblance* is explicated by *elementary embeddability*[5]:

> *Since, in general, A may be much larger than any B in V_α, the closest resemblance of B to A is attained in the case B is isomorphic to an elementary substructure of A, i.e., B can be elementarily embedded into A.*

The key to our result is exactly to find an even closer resemblance between structures.

[5] J. Bagaria, "Large cardinals as principles of structural reflection," Bulletin of Symbolic Logic, 2023

Elementary Covering

The idea is simple: instead of requiring a single elementary embedding, we require many.

Definition (Elementary Covering)

For any structures A, B in the same language, we say B **elementarily covers** A if for any $a \in A$, there is some elementary embedding $j : B \to A$ with $a \in ran(j)$.

When the context is clear we simply say that B *covers* A.

Elementary covering demands a stronger resemblance between B and A than mere elementary embeddability.

Higher Structural Reflection

CSR (Covering Structural Reflection): For every definable class $\mathcal{C}$ of structures in the same language there exists an ordinal α that *cover-reflects* $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\alpha$ such that B covers A.

As mentioned, CSR is based on the same rationale as SR. We simply strengthen the notion of structural resemblance and change nothing else. Accordingly, the same justification for SR arguably extends to CSR.

The *Covering Vopěnka's Principle*, CVP, says that for any definable proper class $\mathcal{C}$ of structures, there are $A \neq B \in \mathcal{C}$ s.t. A covers B.

Related Work

In a related joint project with Hamkins, Lietz, and Schlutzenberg, we investigate the principle assertaing the existence of some cardinal κ such that any structure A in a countable language is covered by a structure B of size $< \kappa$ in the same language.

Very Large Cardinals

Let us introduce some large cardinal notions.

Definition (m-supercompactness)

For a natural number $m \geq 1$, a cardinal κ is m-**supercompact** if for any $\lambda > \kappa$ there is some $\bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ such that $j^m(crit(j)) = \kappa$.

1-supercompactness is equivalent to supercompactness (Magidor).

Very Large Cardinals

Definition (*m*-fold extendibility, Sato[6])

Given a natural number $m \geq 1$, a cardinal κ is ***m*-fold extendible** if for any λ there is some elementary embedding $j : V_{j^{m-1}(\lambda)} \to V_\delta$ for some δ, with $\mathrm{crit}(j) = \kappa$ and $\lambda < j(\kappa)$.

κ is said to be $C^{(n)}$-m-fold extendible, for a natural number n, if we require $V_{j(\kappa)} \prec_{\Sigma_n} V$ in the above definition.

1-fold extendibility is extendibility.

[6] "Double helix in large large cardinals and iteration of elementary embeddings", *APAL*, 2007

Normal Measures

Definition (m-normal measure[7])

Given a natural number $m \geq 1$, a cardinal κ, and a sequence $\kappa_0 \leq \lambda_0 < \kappa_1 \leq \lambda_1 \cdots < \kappa_{m-1} = \kappa \leq \lambda_{m-1} = \lambda$.

A set $\mathcal{U}$ is an m-**normal measure** for (κ, λ) if $\mathcal{U}$ is a κ_0-complete normal fine ultrafilter over $\mathcal{P}_\kappa \lambda$ such that

$$\{x \in \mathcal{P}_\kappa \lambda : ot(x \cap \kappa_{i+1}) = \kappa_i \ \ \& \ \ ot(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}$$

for any $0 \leq i \leq (m-2)$.

Additionally, for a natural number $n \geq 1$, an m-normal measure $\mathcal{U}$ for (κ, λ) is n-**reflecting** if $\mathcal{T}^n_\lambda = \{x \in \mathcal{P}_\kappa \lambda : V_{ot(x)} \prec_{\Sigma_n} V\} \in \mathcal{U}$.

[7] The notion of n-reflecting measure, in the context without the m-fold requirement, is due to Bagaria and Goldberg, "Reflecting Measures", *Advances in Mathematics*, 2024

Combinatorial Characterization

Theorem (Combinatorial Characterization)

For $m, n \geq 1$ and $\kappa \leq \lambda$ the following holds:

1. *κ is measurable if and only if there is an 1-normal measure for (κ, κ).*
2. *κ is m-supercompact if and only if for any $\lambda \geq \kappa$ there is an m-normal measure for (κ, λ).*
3. *κ is $C^{(n)}$-m-fold extendible if and only if for every $\lambda \in C^{(n+1)}$ greater than or equal to κ, there is an $(n+1)$-reflecting m-normal measure for (κ, λ).*
4. *κ is the mth target of an m-huge cardinal if and only if there is an $(m+1)$-normal measure for (κ, κ).*

Strength

These large cardinals are very strong:

Theorem

For all natural numbers $m, n \geq 1$, the following lists the consistency strength of the statements in strictly decreasing order:

1. *There exists an almost $(m+1)$-huge cardinal.*
2. *There exists a $C^{(n)}$-$(m+1)$-fold extendible cardinals.*
3. *There exists an $(m+1)$-supercompact cardinal*
4. *There exists a super m-huge cardinal.*

In particular, we have superhuge $<$ 2-supercompact $<$ $C^{(n)}$-2-fold extendible $<$ almost 2-huge, in terms of consistency strength.

A Question of Sato

The following is a question of Sato[8]:

Question

Are the statements "there is an $(m+1)$-fold-supercompact cardinal" and "there is an $(m+1)$-fold extendible cardinal" equiconsistent, for $m \geq 1$?

We answer the question positively:

Theorem

For natural numbers $m \geq 1$, a cardinal κ is $(m+1)$-fold supercompact if and only if κ is $(m+1)$-fold extendible.

[8] "Double helix in large large cardinals and iteration of elementary embeddings", *APAL*, 2007

Correspondence

The following are analogous to Bagaria's results:

Theorem

For any natural number $n \geq 1$:

1. Π_1-CSR *holds if and only if there exists a 2-supercompact cardinal.*
2. Π_2-CSR *holds if and only if there exists a 2-fold extendible cardinal.*
3. Π_{n+1}-CSR *holds if and only if there exists a $C^{(n)}$-2-fold extendible cardinal.*

Theorem

The following schema are equivalent:

1. CSR.
2. CVP.
3. *There exists a $C^{(n)}$-2-fold extendible cardinal, for all n.*

Generalizations

Given two structures A and B and an ordinal δ, we define the *δ-elementary covering game of length δ on A and B*, $Cov_\delta(A, B)$. Two players: P_A and P_B. At stage α, where $\alpha < \delta$, P_A chooses some $a_\alpha \in A$, and then P_B chooses some $b_\alpha \in B$. After δ steps, we have two sequences $a = (a_\alpha : \alpha < \delta)$ and $b = (b_\alpha : \alpha < \delta)$. P_B wins if there is some elementary embedding $e : B \to A$ with $e(b_\alpha) = a_\alpha$ for all $\alpha < \delta$. P_A wins otherwise.

Definition (δ-elementary covering)

Given structures A and B in the same language and some ordinal δ, B *δ-elementarily covers* A if P_B has a winning strategy in the game $Cov_\delta(A, B)$.

B 0-covers A if B is elementarily embeddable into A, and B 1-covers A if B covers A.

δ-CSR

As δ increases, δ-covering gives stronger notions of structural resemblance. Thus we may formulate stronger principles of SR with the same conceptual motivation, and merely strengthen the technical notion of resemblance.

δ-CSR (δ-Covering Structural Reflection): For every definable class $\mathcal{C}$ of structures in the same language there exists an ordinal α that *δ-cover-reflects* $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\alpha$ such that B δ-covers A.

The *δ-Covering Vopěnka's Principle*, δ-CVP, says that for any definable proper class $\mathcal{C}$ of structures, there are $A \neq B \in \mathcal{C}$ s.t. A δ-covers B.

0-CSR is the same as SR, and 1-CSR is CSR. Similarly, 0-CVP is VP, and 1-CVP is CVP.

Generalization

Now we see that the pattern of correspondence between LCAs and CSR further generalizes:

Theorem

For natural numbers $n \geq 1$ and m:

1. Π_1-m-CSR *holds if and only if there exists an* $(m+1)$*-supercompact cardinal.*
2. Π_2-m-CSR *holds if and only if there exists an* $(m+1)$*-fold extendible cardinal.*
3. Π_{n+1}-m-CSR *holds if and only if there exists a* $C^{(n)}$-$(m+1)$*-fold extendible cardinal.*

Generalization

Theorem

The following schema are equivalent for every natural number m:

1. m-CSR.
2. m-CVP.
3. *There exists a* $C^{(n)}$-$(m+1)$*-fold extendible cardinal for all natural numbers* n.

These results generalize Bagaria's results, which is the case $m = 0$, and our previous results, which is the case $m = 1$.

Positive Results

In summary, we have shown that the schema of m-CSR for all m has the following property:

1. It has SR as one of its instances, namely 0-CSR.
2. It is much stronger than SR, as it gives m-huge cardinals for all m.
3. It can be given similar justification as that of SR, since we have only strengthened the notion of structural resemblance.

Moreover, m-CSR corresponds to LCAs in exactly the same way SR corresponds to LCAs. This also makes m-CSR natural extensions of SR.

Thus these results arguably provide a positive answer to our motivating question.

On the other hand, δ-CSR generalizes to inconsistency once we reach the first infinite ordinal:

Theorem (Inconsistency)

Σ_0-ω-CSR does not hold.

It seems that ω-CSR is also based on the same motivation, with a strengthened notion of resemblance, which however leads to falsity. One might interpret this as a problem of *extendibility to inconsistency*: the justifiability of SR itself could be questioned, given that similar justifications extend to false principles.

Applications to ESR

We have suggested that the large cardinals and reflection principles introduced in the previous section serve as alternatives to ESR, more aligned with the original motivation for SR. Surprisingly, they can also be applied to give interesting answers to open questions concerning ESR.

Definition (Exact Structural Reflection[9])

Given a definability class Γ and cardinals $\kappa < \lambda$, Γ-ESR(κ, λ) holds if for any Γ definable class $\mathcal{C}$ of structures of the same type and any structure $A \in \mathcal{C}$ of rank λ, there is some $B \in \mathcal{C}$ of rank κ and an elementary embedding from B to A.

Γ-ESR(κ) holds if Γ-ESR(κ, λ) holds for some λ, and Γ-UESR(κ) holds if Γ-ESR(κ, λ) holds for a proper class of λ.

[9] Bagaria and Lücke, "Huge Reflection", *APAL*, 2022

Strength of ESR

Bagaria and Lücke showed that the strength of Γ-ESR(κ) holds for some κ is bounded below by a proper class of almost huge cardinals, and bounded above by an I3 cardinal. They asked the following question:

Question[10] Does Con(ZFC+"there is a huge cardinal") imply Con(ZFC+"Σ_2-ESR(κ) holds for some κ")?

A negative answer was considered more likely.

[10] Bagaria and Lücke, "Huge Reflection", *APAL*, 2022

Strength

The answer turns out to be positive in the strongest way.

Theorem

If κ is a huge cardinal, then for any natural number n, V_κ is a model of ZFC in which there is a proper class of cardinals μ such that Σ_n-UESR(μ) holds.

A Conjecture

The SR program aims not only to show that the *existence* of large cardinals are equivalent to some SR principles, but also to show that a cardinal κ *being* (the least instance of) certain large cardinals is equivalent to κ witnessing some SR principles.

Conjecture[11] A cardinal κ is the least cardinal satisfying some large cardinal notion iff κ is the least cardinal satisfying some Structural Reflection Principle that implies (in some inner model) κ is (weakly) inaccessible.

[11] Bagaria and Ternullo, "Intrinsic justification for large cardinals and structural reflection", *Philosophia Mathematica*, 2025

A Conjecture

The conjecture has been confirmed in all the cases so far. It was shown that the least cardinal that witnesses Σ_2-SR is the least supercompact cardinal, and that the least cardinal that witnesses Σ_3-SR is the least extendible cardinal.

For some other examples, the least cardinal that witnesses Σ_2-PSR (product structural reflection) is the least strong cardinal, and that the least cardinal that witnesses Σ_2-WPSR (weak product structural reflection) is the least strongly unfoldable cardinal.

ESR

The conjecture was also confirmed in the case of ESR:

Theorem (Bagaria and Lücke)

The following are equivalent for cardinals κ and natural numbers $n \geq 1$:

1. *κ is the least such that Π_n-ESR(κ) holds.*
2. *κ is the least cardinal that is weakly parametrically n-exact for some cardinal $\lambda > \kappa$.*

Theorem (Bagaria and Lücke)

The following are equivalent for cardinals κ and natural numbers $n \geq 1$:

1. *κ is the least such that Σ_{n+1}-ESR(κ) holds.*
2. *κ is the least cardinal that is parametrically n-exact for some cardinal $\lambda > \kappa$.*

Questions

However, there are several remaining cases. Bagaria and Lücke asked:

Question A
Are the following statements equivalent for every cardinal κ and every natural number $n \geq 1$?

1. κ is the least cardinal such that Π_n-UESR(κ) holds.
2. κ is the least super weakly parametrically n-exact cardinal.

Question B
Are the following statements equivalent for every cardinal κ and every natural number $n \geq 1$?

1. κ is the least cardinal such that Σ_{n+1}-UESR(κ) holds.
2. κ is the least super parametrically n-exact cardinal.

Answers

Interestingly, we can show the following.

Theorem

The following statements are equivalent:

1. *The answer to Question A is positve in the case* $n = 1$*.*
2. *The answer to Question B is positive in the case* $n = 1$*.*
3. *There are no* 2*-supercompact cardinals.*

Theorem

The following statements are equivalent:

1. *The answer to Question B is positive in the case* $n \geq 2$
2. *There are no* $C^{(n-1)}$*-*2*-fold extendible cardinals.*

A Challenge

In the paper of Bagaria and Lücke, the *sequential* versions of ESR are also considered. The results above have corresponding generalizations to the case of sequential ESR of length m, where m is a natural number.

Thus the conjecture of the SR program, in the case of UESR, turns out to be an *anti-large cardinal hypothesis*. This seems to be a problem for the SR program, which aims to *justify* large cardinals.

Summary

The incompleteness phenomenon motivates Gödel Program. An approach to justifying large cardinals, favored by Gödel himself, is to strengthen the reflection principles to derive large cardinals. This motivates Bagaria's SR program.

Problems for the SR program: the non-uniformity of the variants of SR, and the fact that the justification given for SR does not extend to ESR, the strongest variant of SR.

We have shown that m-CSR, which includes SR as one of its instances and is subject to a similar justification as SR, provides much greater strength and corresponds to LCAs in the same way that SR corresponds to LCAs. These results seem to support the SR program.

Summary

On the other hand, the failure of ω-CSR might be taken to constitute the problem of extendibility to inconsistency.

We also apply our results to resolve open questions concerning ESR. However, in doing so we have uncovered a potential counterexample of a conjecture of the SR program. This suggests another problem for the SR program.

This concludes our attempt to provide rational evaluation of the SR program. Although it is unclear whether the question of justification will ever be settled conclusively, what is clear is the power and coherence of the notion of reflection, which has taken us from finitude to infinity, and far, far beyond.

Thanks!