The model theory of large fields

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Conventions/Background

All rings are commutative with unit.

K is a field and K^{alg} is the algebraic closure of K.

V, W, V', W' are K-varieties. $V \to W$ is a K-variety morphism.

$V(K)$ is the set of K-points of V.

$\mathbb{A}^n$ is n-dimensional affine space over K, so $\mathbb{A}^n(K) = K^n$.

Roughly speaking:

V is defined by a finite system of polynomial equations and inequations with coefficients from K. $V \to W$ is a polynomial map.

$V(K)$ is the solution set of the system in K.

We are primarily interested in $V(K)$ – basically quantifier free definable sets in K.

Even more roughly:

V "is" $V(K^{\mathrm{alg}})$, K-varieties are sets definable in K^{alg} with parameters from K.

Suppose W is smooth.

$f\colon V \to W$ is **étale** if V is smooth and $T_pV \to T_{f(p)}W$ is an iso. for any $p \in V$.

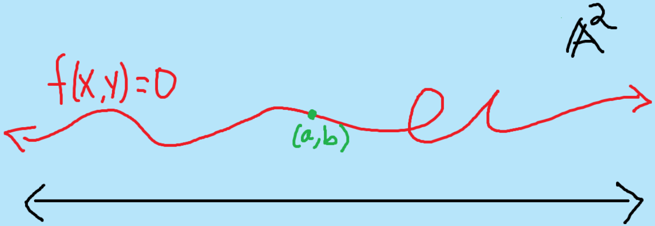
Largeness

K is **large** if it satisfies one of the following equivalent conditions:

- A smooth 1-dim K-variety with a K-point has infinitely many K-points.
- If $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

  then f has infinitely many zeros in K. (Large fields form a $\forall\exists$ elem. class.)



Pop ('96) introduced largeness, proved inverse Galois theorem over $K(t)$, K large.

"the 'right class' of fields over which one can do a lot of interesting mathematics."
–Pop, "Little survery on large fields"

Non-large fields

K is **large** if whenever $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

then f has infinitely many zeros in K.

Finite fields are not large.

$f(x, y) = x^4 + y^4 - 1$.

$f(0, 1) = 0 \neq 4 = \partial f / \partial y(0, 1)$

Fermat: Only zeros of f in $\mathbb{Q}^2$ are $(\pm 1, 0)$, $(0, \pm 1)$.

Faltings: f has only finitely many zeros in any number field.

Similar (hard) results show that function fields are not large.

function field = f.g. extension of a field F that is not algebraic over F, e.g. $F(t)$.

Most other fields you have heard of are large.

All logically tame infinite fields known before 2022 are large.

Large fields

K is **large** if whenever $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

then f has infinitely many zeros in K.

K algebraically closed $\Longrightarrow$ non-constant $f \in K[x, y]$ has infinitely many zeros.

More generally, separably closed fields are large.

Implicit function theorem $\Longrightarrow$ $\mathbb{R}$ is large $\Longrightarrow$ real closed fields are large.

Polynomial IFT $\Longrightarrow$ henselian valued fields are large.

$\mathbb{Q}_p$, $K((t))$ are henselian valued. Local fields are large, global fields are not.

K is **pseudofinite** if K infinite and satisfies Th(finite fields).

Weil conj. for curves $\Longrightarrow$ Pseudofinite is PAC $\Longrightarrow$ Pseudofinite is large.

Infinite algebraic extensions of finite fields are also PAC.

(PAC means **p**seudo **a**lgebraically **c**losed.)

More large fields

Hasse Principle: A $\mathbb{Q}$-variety with a point in $\mathbb{R}$ and in each $\mathbb{Q}_p$ has a $\mathbb{Q}$-point.

Often fails.

K satisfies a **local-global principle** if it satisfies a form of the Hasse principle.

(For more precision see "Little survey on large fields" by Pop.)

K satisfies a local-global principle $\implies$ K is large.

$\alpha \in \mathbb{Q}^{\mathrm{alg}}$ is **totally real** if all roots of its min. poly. are in $\mathbb{R}$.

$\mathbb{Q}^{\mathrm{tr}} =$ field of totally real algebraic numbers.

An abs. irred. $\mathbb{Q}^{\mathrm{tr}}$-variety with a point in any completion of $\mathbb{Q}^{\mathrm{tr}}$ has a $\mathbb{Q}^{\mathrm{tr}}$-point.

Equivalently: A absolutely irreducible $\mathbb{Q}^{\mathrm{tr}}$-variety with a point in every real closure of $\mathbb{Q}^{\mathrm{tr}}$ with respect to a field order has a $\mathbb{Q}^{\mathrm{tr}}$-point.

Hence $\mathbb{Q}^{\mathrm{tr}}$ is large.

The field of totally p-adic algebraic numbers is also large.

More generally pseudo real closed and psuedo p-adically closed field are large.

Possibly large fields

$\mathbb{Q}_{\mathrm{ab}}$ = max. abelian extension of $\mathbb{Q}$ = extension of $\mathbb{Q}$ by all roots of unity.

Open Question: Is $\mathbb{Q}_{\mathrm{ab}}$ large?

Large fields are closed under algebraic extensions.

$\mathbb{Q}_{\mathrm{solv}}$ = max. solvable extension of $\mathbb{Q}$ = smallest root-closed subfield of $\mathbb{Q}^{\mathrm{alg}}$.

Famous Conjecture: $\mathbb{Q}_{\mathrm{solv}}$ is PAC, hence large.

Conjecture (Koenigsmann):

K has $< \infty$ separable extensions of any given degree $\implies$ K is large.

Colliot-Thélène, Jarden: $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ is pro-p $\implies$ K is large.

The étale-open topology

An **étale-image** is a subset of $W(K)$ of the form $f(V(K))$ for étale $f\colon V \to W$.

Étale-images are closed under finite intersections and unions.

Étale-images form a basis for the **étale-open topology** on $W(K)$.

Also call it the $\mathcal{E}_K$-topology.

Some Properties:

- Refines the Zariski topology.
- $V(K) \to W(K)$ is continuous for any $V \to W$.
- $V(K) \to W(K)$ is an open map for $V \to W$ étale.
- Topology on K^n refines, but may not agree, with the product topology.
- Topology on K^n = product topology $\iff$ $\mathcal{E}_K$ induced by a field top. on K.

Theorem (Johnson, Tran, W, Ye):
The $\mathcal{E}_K$-topology on K is discrete $\iff$ K is not large.

K is **large** if whenever $f \in K[x,y]$ and $(a,b) \in K^2$ satisfy

$$f(a,b) = 0 \neq \frac{\partial f}{\partial y}(a,b)$$

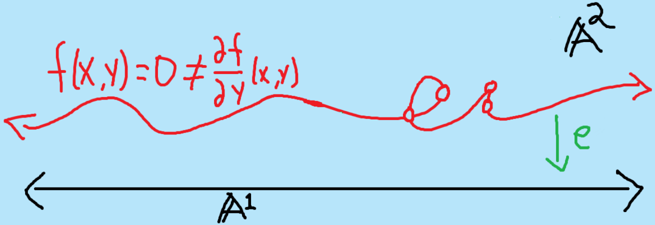
then f has infinitely many zeros in K.

An **étale-image** is a subset of $W(K)$ of the form $f(V(K))$ for étale $f \colon V \to W$.

Étale-images form a basis for the **étale-open ($\mathcal{E}_K$-) topology** on $W(K)$.

Theorem (JTWY): The $\mathcal{E}_K$-topology on K is discrete $\iff$ K is not large.

Proof: Any étale $V \to \mathbb{A}^1$ is (Zariski) locally isomorphic to a morphism e:



Sets of the form $\{a \in K : (\exists b \in K)\ \ f(a,b) = 0 \neq \frac{\partial f}{\partial y}(a,b)\}$ form a basis on K.

étale vs étale

The étale-open topology is **not** the étale topology!!!!!!!

étale-open topology	**étale topology**
a topology	not a topology
defined on $V(K)$	defined on V
not like the analytic topology	like the analytic topology

Examples

Theorem (JTWY):

K separably closed	$\implies$	$\mathcal{E}_K$ is the Zariski topology
K real closed	$\implies$	$\mathcal{E}_K$ is the order topology
K henselian valued & not sep. closed	$\implies$	$\mathcal{E}_K$ is the valuation topology
K is PAC and not sep. closed	$\implies$	$\mathcal{E}_K$ is something new

K is PAC, $\mathrm{Char}(K) \neq 2 \quad \implies \quad U - U$ cofinite for any nonempty open $U \subseteq K$.

With respect to the étale-open topology:

K is HD	$\iff$	K is not sep. closed
K is zero-dimensional	$\iff$	K is not sep. closed or $\mathbb{R}$.
K is loc. compact HD	$\iff$	K is a local field other than $\mathbb{C}$.
$\mathcal{E}_K$ given by abs. value/valuation on K	$\iff$	K is not sep. closed and we have $K \equiv K^*$ for henselian K^*.

Our old theorem

The stable fields conjecture: An infinite field is stable iff separably closed.

True when "stable" is replaced by stronger stability-theoretic conditions.

$\mathbb{C}(t)$ might be stable?????????

Theorem (JTWY): A large field is stable iff separably closed.

Uses that the topology is HD, non-discrete, and has a definable basis.

Our new theorem

R is a local (i.e. unique maximal ideal $\mathfrak{m}$) integral domain that is not a field.
$\mathrm{Frac}(R)$ is its fraction field.
R is **henselian** if any simple root of $f \in R[x]$ in $R/\mathfrak{m}$ lifts to a root of f in R.
simple root: $f(\alpha) = 0 \neq f'(\alpha)$.
Examples: Val. ring of henselian valuation, $K[[x_1, \ldots, x_n]]$, complete local rings

> it was generally believed that the above fields $K = k((x, y))$, and more general $K = \mathrm{Quot}(R)$ with R complete Noetherian local and $\mathrm{Krull.dim}(R) > 1$, were not large fields. Note that these fields are definitely *not* Henselian valued fields!

–Pop, "Henselian implies large"

$K((x, y)) = \mathrm{Frac}(K[[x, y]])$.

Theorem (Pop '07): $\mathrm{Frac}(R)$ is large when R is henselian.

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Theorem (JTWY): K is large $\iff$ $K \equiv \mathrm{Frac}(R)$ for R henselian.

$\mathbb{R}$ is not the fraction field of a henselian local domain.

Problem: Find logically tame henselian R with $\mathrm{Frac}(R)$ pseudofinite.

Can't be noetherian.

Other perspective: Def. of largeness axiomatizes the theory of fraction fields of henselian local domains.

Proof of our new theorem

Theorem (JTWY): K is large $\iff$ $K \equiv \mathrm{Frac}(R)$ for R henselian.

$\Longleftarrow$ Pop + large fields are an elementary class.

Separably closed case of $\Longrightarrow$ follows by standard valuation theory.

Main tool in non-separably closed case: Polynomial inverse function theorem.

Theorem (JTWY): If K is not separably closed then $V(K) \to W(K)$ is a local homeomorphism for étale $V \to W$.

Corollary (not used for new theorem):
V smooth irreducible $\Longrightarrow$ $V(K)$ locally homeomorphic to $K^{\dim V}$.

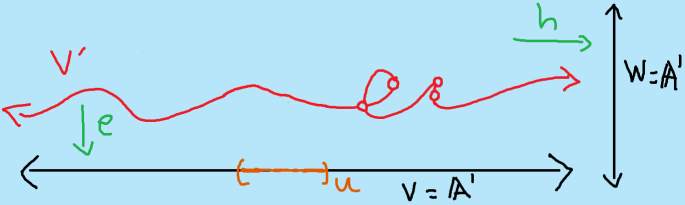
$V(K)$ is a "manifold".

Proof of our new theorem: Nash functions

$U \subseteq \mathbb{R}^n$ open and definable, $s\colon U \to \mathbb{R}$.
s is a **Nash function** if s is C^∞ and definable. (Equiv: analytic and algebraic.)
Artin-Mazur: s is Nash $\iff$ s is the composition of the inverse of an étale map with a polynomial map.

Theorem (JTWY): If K is not separably closed then $V(K) \to W(K)$ is a local homeomorphism for étale $V \to W$.

Suppose K is large and not separably closed. $U \subseteq V(K)$ étale-open.
$s\colon U \to W(K)$ is **Nash** if $s = h \circ (e|_P)^{-1}$ for $e\colon V' \to V$ étale,
$h\colon V' \to W$ a morphism, $P \subseteq V'(K)$ open, e gives a homeo. $P \to U$.



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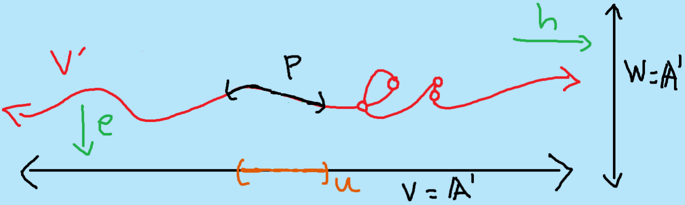

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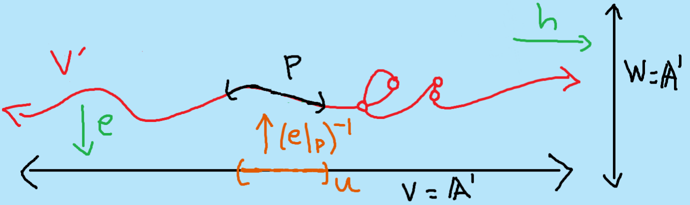

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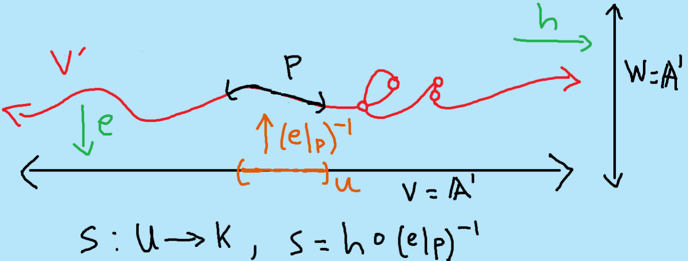

Suppose K is large and not separably closed. $O \subseteq V(K)$ étale-open.

$f\colon O \to W(K)$ is **Nash** if $f = h \circ (e|_P)^{-1}$ for
$e\colon V' \to V$ étale, $h\colon V' \to W$ a morphism, e gives a homeo. $P \to O$.

Example: $\sqrt[n]{x}$ on a nbhd of 1 when $\mathrm{Char}(K) \nmid n$.

Nash maps are closed under compositions.

Nash maps $O \to K$ form a ring.

Theorem (JTWY): The ring of germs of Nash maps to K on K^n at the origin is isomorphic to the ring of algebraic formal power series over K in n variables. (Real closed case well-known.)

$f \in K[[t_1, \ldots, t_n]]$ is **algebraic** if algebraic over $K[t_1, \ldots, t_n]$.

$K[[t_1, \ldots, t_n]]^{\mathrm{alg}}$ is a henselian local domain.

We can evaluate $f \in K[[t_1, \ldots, t_n]]^{\mathrm{alg}}$ on sufficiently small elements of K^n.

$\mathbb{K}$ is a $|K|^+$-saturated elementary extension of K.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{K}^n$ be K-**infinitesimal**, i.e. in every nbhd of 0 def. over K.

$\mathrm{Eval}_\varepsilon\colon K[[t_1, \ldots, t_n]]^{\mathrm{alg}} \to \mathbb{K}$, $\mathrm{Eval}_\varepsilon(f) = f(\varepsilon_1, \ldots, \varepsilon_n)$. A ring morphism.

Image of $\mathrm{Eval}_\varepsilon$ is a henselian local domain (they are closed under quotients).

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$K[[t_i]]^{\mathrm{alg}}_{i<\kappa}$ is the ring of algebraic power series in κ variables.

Still a henselian local domain.

$\varepsilon = (\varepsilon_i)_{i<\kappa} \in \mathbb{K}^\kappa$ is K-**infinitesimal** if every finite subtuple is.

Can still define $\mathrm{Eval}_\varepsilon \colon K[[t_i]]^{\mathrm{alg}}_{i<\kappa} \to \mathbb{K}$, each series depends on only finitely many t_i.

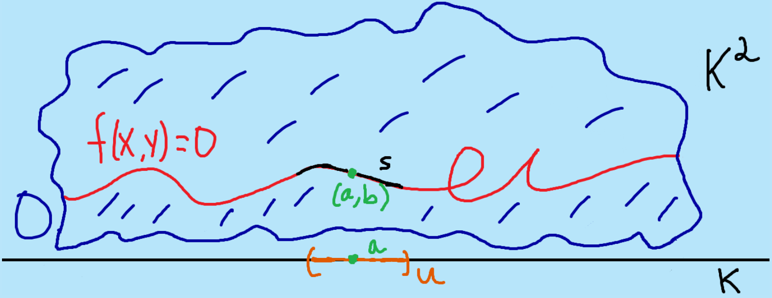
Final Lemma: $\exists$ K-infinitesimal $\varepsilon = (\varepsilon_i)_{i<\kappa}$ s.t. the fraction field of the image of $\mathrm{Eval}_\varepsilon$ is an elementary extension of K.

IFT for Nash maps

Suppose K is large and not separably closed.

Theorem (W): Nash maps over K satisfy the inverse and implicit function theorems with respect to the étale-open topology.

Ex: $O \subseteq K^2$ open, $f: O \to K$ Nash, $(a,b) \in O$ s.t. $f(a,b) = 0 \neq \partial f / \partial y(a,b)$.



$\exists$ nbhd $a \in U$, unique Nash $s: U \to K$ s.t. $s(a) = b$ and $f(a^*, s(a^*)) = 0 \ \forall a^* \in U$.

Roughly:

Large $\iff$ inverse/implicit func. thm holds $\iff$ $\equiv \mathrm{Frac}(R)$ for R henselian

Thank you.

“Large implies Henselian”, on arxiv soon.