An Infinite Lone Wolf Theorem

Robert M. Anderson (Harbin Institute of Technology and UC Berkeley)
Haosui Duanmu (Harbin Institute of Technology)
Ming Xiao (Wuhan University of Science and Technology
Yahui Zhang (Harbin Institute of Technology)

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Our Model

In this talk, we work with the simple one-to-one "marriage" matching market, as in Gale and Shapley(1962).

Definition

A one-to-one matching market is a list $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ such that:

- M and W are (possibly infinite) sets of men and women, respectively;
- $\mathcal{P}_M = \{(W_m, \succ_m) : m \in M\}$ is a set of preferences for the men over the women. For each $m \in M$, $W_m \subset W$ is the set of acceptable women for m and $\succ_m$ is a strict total order on W_m. Any woman in W_m is preferred by m over being unmatched;
- The set $\mathcal{P}_W = \{(M_w, \succ_w) : w \in W\}$ of preferences for the women over the men is defined similarly.

Matching

Definition

A one-to-one matching between M and W is a collection of pairwise disjoint set of man-woman pairs. Formally, a matching is a partial function $\mu : M \bigcup W \rightarrow M \bigcup W$ such that :

- If f is not in the domain of μ, we denote $\mu(f) = \emptyset$. This means that f is unmatched in μ;
- If m is matched under μ, then $\mu(m) \in W$, for all $m \in M$;
- If w is matched under μ, then $\mu(w) \in M$, for all $w \in W$;
- $\mu(m) = w$ if and only if $\mu(w) = m$, for all $m \in M$ and $w \in W$.

Remark:

- Each candidate in $M \cup W$ has at most 1 partner.
- Let μ_f denote the partner of f for any $f \in M \cup W$.

Blocking Pair and Stable Matching

Definition

Given a matching μ, a blocking pair is a pair (m, w) such that

- If $m \in M$ is matched, then $w \succ_m \mu_m$, i.e., m prefers w to his partner in μ. If $m \in M$ is unmatched, then $w \in W_m$, i.e., m prefers w to being unmatched;
- If $w \in W$ is matched, then $m \succ_w \mu_w$. If w is unmatched, then $m \in M_w$.

Definition

A matching μ is stable if

- no candidate is matched to someone he or she finds unacceptable;
- there is no blocking pair.

Background

- A classical result of Gale and Shapley shows that stable matching always exists if M and W are both finite.

Theorem (Gale-Shapley, 1962)

Suppose M and W are finite. Then every one-to-one market $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ has a stable matching.

The Lone Wolf Theorem

- The Lone Wolf Theorem suggests that the set of stable matchings does not depend on the choice of stable matchings.

Theorem (McVitie-Wilson, 1970; Gale-Sotomayor, 1985)

Suppose that M and W are finite. If a candidate is matched in one stable matching, then the candidate is matched in all stable matchings.

Generalizations of the Lone Wolf Theorem imply the Rural Hospitals Lemma.

An Example in Infinite Setting

In this talk, we consider the Lone Wolf Theorem in infinite setting. Jagadeesan(2018) considers the following example:

Example (Jagadeesan, 2018)

Let $M = \{m_1, m_2, ...\}$ and let $W = \{w_1, w_2, ...\}$. Let the mens' preferences be given by

$$m_i : w_{i+1} \succ_{m_i} w_i \succ_{m_i} \emptyset.$$

For $i > 1$, let woman w_i's preferences be given by

$$w_i : m_i \succ_{w_i} m_{i-1} \succ_{w_i} \emptyset,$$

and let w_1's preferences be given by

$$w_1 : m_1 \succ_{w_1} \emptyset.$$

An Example in Infinite Setting (Continued)

Consider the matchings

$$A \equiv \{(m_i, w_i) | i \in \mathbb{N}\};$$

$$B \equiv \{(m_i, w_{i+1}) | i \in \mathbb{N}\}.$$

Conclusion

- Both A and B are stable.
- w_1 is matched in A but not B. Moreover, the set of unmatched candidates is strictly larger in B than in A.
- Taking a countable disjoint union of markets of the above form, there are infinitely many candidates that are matched in one stable matching but not in another.

Unstable Candidate

Definition

A person $x \in M \cup W$ is called an unstable candidate if x is matched in a stable matching, while is unmatched in another stable matching.

Our aim is to identify an upper bound on unstable candidates for infinite one-to-one matching markets, hence establishing a generalization of the classic Lone Wolf Theorem.

Main Result

Based on the previous counterexample, we focus on infinite one-to-one matching markets where $M = W$ is $\mathbb{N}$ or $\mathbb{Z}$. However, the next example shows that there may be no upper bound on unstable candidates even if there is a uniform bound on the size of acceptable sets of men or women.

Another Counterexample

Example

Let $M = \{m_1, m_2, ...\}$ and let $W = \{w_1, w_2, ...\}$. For $i \geq 1$, let men's preferences be given by

$$m_{2i+1} : w_{2i+1} \succ_{m_{2i+1}} \emptyset; \quad m_{2i} : w_{2i} \succ_{m_{2i}} w_i \succ_{m_{2i}} \emptyset;$$

let woman w_i's preferences be given by

$$w_i : m_{2i} \succ_{w_i} m_i \succ_{w_i} \emptyset.$$

Another Counterexample (Continued)

Consider the matchings

$$A \equiv \{(w_i, m_{2i}) | i \in \mathbb{N}\}; \quad B \equiv \{(w_i, m_i) | i \in \mathbb{N}\}.$$

Conclusion

- Both A and B are stable. Indeed, every man is matched to his most preferred woman in B, while every woman is matched to her most preferred man in A. Thus, there are no blocking pairs in A or B.
- For any $i \in \mathbb{N}$, m_{2i+1} is matched in B but not in A. Hence, "half" of the men are unstable candidates.
- We can modify this example by enlarging the gap between elements in acceptable sets, hence obtaining more unstable candidates.

Uniformly Bounded Preferences

To obtain an infinite Lone Wolf Theorem, we make the following definition:

Definition

For an integer $d \in \mathbb{N}$, we say that $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ has uniformly bounded preferences by d if:

- For every $m \in M$ or $w \in W$, the distance between any pairs in W_m or M_w is no greater than d.

Remark: If we interpret candidates' labelling as candidates as characteristics, few people is willing to accept candidates with very different characteristics.

Main Theorem Statement

Theorem

Let $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ be a one-to-one matching market with uniformly bounded preference d. If $M = W = \mathbb{N}$, then there are at most d unstable candidates. If at least one of M, W equals $\mathbb{Z}$, then there are at most $2d$ unstable candidates.

Proof Sketch

The proof of the above theorem is divided into three main steps:

- Given stable matchings π and π', there are at most d candidates who have different status between π and π'.
- Let $\bar{\Pi}$ be the stable matching generated from the usual delayed acceptance algorithm proposed by women. If $w \in W$ is an unstable candidate, then w must be matched in $\bar{\Pi}$. If $m \in M$ is an unstable candidate, then m must be unmatched in $\bar{\Pi}$;
- Let $\bar{\Pi}'$ be the stable matching generated from the usual delayed acceptance algorithm proposed by men. If $m \in M$ is an unstable candidate, then m must be matched in $\bar{\Pi}'$. If $w \in W$ is an unstable candidate, then w must be unmatched in $\bar{\Pi}'$;
- Hence, we conclude that there are at most d unstable candidates.

The Chain of Forward

- To prove the above theorem, we will need the following concept:

Definition 7

Let $x \in M \cup W$ be a person. Given two matchings π_0 and π_1, the chain of forward inference induced by x from π_0 to π_1 is a sequence $F(x, \pi_0, \pi_1) = \langle x_0, x_1, \ldots, x_n \ldots \rangle$ such that:

- $x_0 = x$.
- For every $n \in \mathbb{N}$, $(x_{2n}, x_{2n+1}) \in \pi_1$.
- For every $n \in \mathbb{N}$, $(x_{2n+1}, x_{2n+2}) \in \pi_0$.
- If there exists $n \in \mathbb{N}$ such that x_n is unmatched in $\pi_{(n \bmod 2)+1}$, we define $F(x, \pi_0, \pi_1) = \langle x_0, x_1, \ldots, x_n \rangle$.

F: $x_0 \xrightarrow{\pi_1} x_1 \xrightarrow{\pi_0} x_2 \xrightarrow{\pi_1} x_3 \xrightarrow{\pi_0} x_4 \ldots$

The Chain of Forward (Countinued)

Remark

- We start with $x = x_0$, look for his(or her) match x_1 in π_1, then come back to π_0 to see who is x_1 matched, denote this person by x_2, then go back to π_1...this progress generates the chain of forward inference.
- In particular, at step n, we look for x_{n-1}'s match x_n in $\pi_{(n \bmod 2)}$. If x_n is unmatched in $\pi_{(n \bmod 2)+1}$, then terminate the progress. In this case, $F(x, \pi_0, \pi_1) = \langle x_0, x_1, \ldots, x_n \ldots \rangle$ is finite.

An Important Lemma

- By the chain of forward inference, we obtain an upper bound of candidates that change status in two given stable matchings.

Lemma 1

Suppose $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ is a one-to-one matching market that has uniformly bounded preferences by d. Let π_0 and π_1 are two stable matchings, then there are at most d candidates have different status (i.e. matched in one matching and unmatched in the other) between π_0 and π_1.

Delayed Acceptance Algorithm

Let $\bar{\Pi}$ be obtained with the usual delayed acceptance algorithm proposed by women and $\bar{\Pi}'$ be the stable matching produced by the same algorithm, but proposed by males. Formally, $\bar{\Pi}$ is:

- Initializing steps:
    - For each $w \in W$, initialize $l_{w,-1} = \emptyset$. These will be increasing lists $l_{w,n}$ of initial segments of M_w. The list $l_{w,n}$ contains all persons who have been proposed by w but rejected w at step $k \leq n$ and the element in $l_{w,n} \setminus l_{w,n-1}$ is the person who has been proposed by w but rejected w at step n.
    - Initialize $\bar{\Pi}_{-1} = \emptyset$ being an empty matching.

Delayed Acceptance Algorithm (Continued)

- At n'th step, suppose that we have already constructed a partial match $\bar{\Pi}_{n-1}$ and lists $l_{w,n-1}$ for every $w \in W$:
  - $\bar{\Pi}_{n,0} = \bar{\Pi}_{n-1} \cup \{(m,w) : w$ is unmatched in $\bar{\Pi}_{n-1}$ and m is the $\succ_w$ maximum element in $M_w \setminus l_{w,n-1}\}$.
  - $\bar{\Pi}_n = \{(m,w) \in \bar{\Pi}_{n,0} : w$ is the best choice for m among all w so that $(m,w) \in \bar{\Pi}_{n,0}\}$. For every pair $(m,w) \in \bar{\Pi}_{n,0} \setminus \bar{\Pi}_n$, let $l_{w,n} = l_{w,n-1} \cup \{m\}$.

Remark: At step $n > 1$: Each woman who was rejected at Step $n-1$ proposes to her most-preferred man who has not rejected her as yet. If no proposals are made, then terminate the algorithm. Otherwise, each man is tentatively matched with his most-preferred woman from the set of new proposals and his previous tentative match. Each man then rejects all proposals from women other than his tentative match. Let $\bar{\Pi}_n$ denote all tentatively matched at step n.

Delayed Acceptance Algorithm (Final Slide)

- Define $\bar{\Pi} = \bigcup_{n \in \mathbb{N}} \bigcap_{k > n} \bar{\Pi}_k$ to be the delayed acceptance algorithm proposed by women.
- $\bar{\Pi}$ is a stable matching: let (m, w) be an unmatched pair. Either w has not proposed to m, in which case w is matched with a higher preference, or w has proposed to m but is rejected, in which case m is matched with a higher preference.
- The above two bullet points remain true for $\bar{\Pi}'$, the delayed acceptance algorithm proposed by men.

Properties of Unstable Candidates

Lemma 2

Let $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ be a one-to-one matching market. Let $w \in W$ and $m \in M$ be two unstable candidates. Then:

- w must be matched in $\bar{\Pi}$ and unmatched in $\bar{\Pi}'$.
- m must be matched in $\bar{\Pi}'$ and unmatched in $\bar{\Pi}$.

Proof of the Main Theorem

Theorem

Suppose $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ is a one-to-one matching market that has a uniformly bounded preference d. Then there are at most d unstable candidates.

Proof.

By lemma 2, all unstable candidates change status between $\bar{\Pi}$ and $\bar{\Pi}'$. By lemma 1, there can be at most d candidates change status between $\bar{\Pi}$ and $\bar{\Pi}'$. So there are at most d unstable candidates. □

Future Works

- Consider the Lone Wolf Theorem in the measure-theoretic setting.
- Generalize our infinite Lone Wolf Theorem into the many-to-one matching markets.
- Incorporate externality into our matching model.