EXERCISES

STRONG MINIMALITY AND GEOMETRIC STABILITY

The following are exercises accompanying the lecture notes for the course *Geometric Stability Theory and Strong Minimality*. Most of the exercises appear as natural lemmas and examples in the text, and are copied from where they appear. Each section also has some added exercises at the end (many of which are more challenging) under the heading **Additional Exercises for ...**.

2. Uncountable Categoricity and Strong Minimality

(1) Give an example of an infinite structure M such that every unary definable set in M is finite or cofinite, but M is not strongly minimal.
(2) Let T be a complete theory in a countable language with infinite models.
  (a) Suppose that for every $M \models T$, and every finite subset $A \subset M$, the automorphism group $\mathrm{Aut}(M/A)$ (automorphisms of M fixing A pointwise) has a co-countable orbit. Show that T is strongly minimal.
  (b) Use (a) to show that each of pure sets, vector spaces, and algebraically closed fields are strongly minimal.
(3) Show that the theory of pairs of infinite sets with a bijection is uncountably categorical, while the theory of pairs of infinite sets (without a bijection) is not.
(4) Show that the theory of a fibered family of affine vector spaces (Example 2.9(4)) is uncountably categorical.

(5) Additional Exercises for Section 2

(6) Show that $T := \mathrm{Th}((\mathbb{Z}/4\mathbb{Z})^\omega, +)$ is uncountably categorical.
(7) Let M be an infinite set, and E an equivalence relation on M, so that $T := \mathrm{Th}(M, E)$ (in the language with one binary relation) is uncountably categorical. Show that one of the following holds:
  (a) There is a cofinite E-class.
  (b) There is a positive integer n such that cofinitely many elements of M belong to classes of size n.

  In each of the above cases, conclude that T is strongly minimal.

3. Definable Sets in Strongly Minimal Theories

Throughout, assume M is a strongly minimal structure.

(1) Show that uniform finiteness holds in M^n: That is, let $X \subset M^m \times M^n$ be definable. Then there is N so that for all $y \in M^m$, either X_y is infinite or $|X_y| \leq N$.
(2) Show the following:
  (a) $\dim(M^n) = n$ for each n.

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   (b) If $X \subset M^n$ is definable, we have $\dim(X) = -\infty$ if and only if X is empty, and $\dim(X) = 0$ if and only if X is non-empty and finite.
(3) Suppose $M \models ACF_p$ for some p. Let $P(x, y)$ be a non-constant binary polynomial with coefficients in M. Show that $\dim(\{(x, y) : P(x, y) = 0\}) = 1$.
(4) Show the following:
   (a) $\dim(X \times Y) = \dim(X) + \dim(Y)$ for any definable X and Y.
   (b) If $f : X \to Y$ is a definable surjection, then $\dim(X) \geq \dim(Y)$.
   (c) If $f : X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$.
   (d) More generally, say that the definable sets X and Y are in *definable finite-to-finite correspondence* if there is a definable $Z \subset X \times Y$ so that both projections $Z \to X$ and $Z \to Y$ are surjective with finite fibers. So that in this case we have $\dim(X) = \dim(Y)$.
(5) Show the following:
   (a) A finite set is stationary if and only if it is a single point.
   (b) Each M^n is stationary.
   (c) If $\dim(X) = 1$, then X is stationary if and only if X is infinite and every definable subset of X is finite or cofinite. In this case we say X is *strongly minimal* (as a definable set).
   (d) Suppose $T = \mathrm{ACF}_p$, and $P(X, Y)$ is a non-constant binary polynomial with coefficients in M. Then $\{(x, y) : P(x, y) = 0\}$ is stationary if and only if $P = Q^k$ for some irreducible Q and some k.
(6) Let X be A-definable and stationary of dimension d. Show that the d-dimensional A-definable subsets of X are closed under finite intersections, and thus they determine a complete consistent type over A.
(7) Show that stationarity is preserved under elementary extensions. That is, let N be an elementary extension of M, and let $\phi(x, a)$ be a formula in n variables with parameters in M. Show that $\phi(x, a)$ defines a stationary subset of M^n if and only if it defines a stationary subset of N^n.
(8) (a) Show that if X and Y are almost equal, then $\dim(X) = \dim(Y)$.
   (b) Show that almost equality is an equivalence relation, but almost containment is *not* a partial order.
   (c) If X and Y are stationary of dimension d, show that X and Y are almost equal if and only if $\dim(X \cap Y) = d$.
(9) Let $X \subset M^n$ be A-definable of dimension d. Show that there is an A-definable function $X \to M^d$ with all fibers finite. Hint: first show that if $X \subset M^n$ and $\dim(X) \neq n$, there is a finite-to-one A-definable function $X \to M^{n-1}$. Now use induction.
(10) (a) Show that if X and Y are definable and almost equal, then X is stationary if and only if Y is. Thus the term *stationary class* is well-defined.
   (b) Show that if $[X]$ and $[Y]$ are d-dimensional almost equality classes, then the class $[X] \cup [Y] = [X \cup Y]$ is well-defined.
   (c) Conclude that every d-dimensional almost equality class decomposes uniquely into a union of finitely many stationary classes.
(11) Let $a \in M^n$. Then there is an A-definable set X containing a such that for every other A-definable Y containing a, X is almost contained in Y.

Additional Exercises for Section 3

(12) Let $X \subset M^n$ be non-empty and definable.
  (a) Show that $\dim(X) \geq d$ if and only if there is a projection $X \to M^d$ whose image is almost equal to M^d. Thus, show that $\dim(X)$ is the largest d such that there is a projection $X \to M^d$ whose image is almost equal to M^d.
  (b) Similarly, show that $\dim(X) \leq d$ if and only if there is a definable finite-to-one function $X \to M^d$. Thus, show that $\dim(X)$ is also the smallest d such that there is a finite-to-one definable function $X \to M^d$.
  (c) Show that $\dim(X)$ is the unique d such that there is a finite-to-one definable function $X \to M^d$ whose image is almost equal to M^d.

(13) Suppose M is a vector space over the field F, and $X \subset M^n$ is a definable vector subspace. Show that $\dim_F(X) = \dim_F(M) \times \dim(X)$, where $\dim_F$ denotes the usual dimension of F-vector spaces in linear algebra.

(14) Suppose F is an infinite field, M is an F-vector space, and $\emptyset \neq X \subset M^n$ is definable. Show that X is stationary if and only if it is almost equal to a translate of a vector subspace of M^n. What happens if F is finite?

(15) (For those comfortable with some algebraic geometry). Suppose $M \models ACF_p$.
  (a) Suppose V is an affine variety over M. Show that the set $V(M)$ is definable, and $\dim(V(M))$ is the same as $\dim(V)$ (the dimension as a variety).
  (b) Suppose $X \subset M^n$ is definable. Show that M is stationary if and only if it is almost equal to $V(M)$ for some irreducible affine variety V over M.

(16) (Hard) Show that unlike dimension, stationarity is *not* always definable in families: in some strongly minimal structure, there is a definable family $X \subset M^n \times T$ so that $\{t : X_t \text{ is stationary}\}$ is not definable. Do this as follows:
  (a) Let $M = \mathbb{R} \times \{1, 2\}$, and let $\pi : M \to \mathbb{R}$ be the projection. Equip V with the following:
    - The ternary relation $\pi(x) + \pi(y) = \pi(z)$.
    - The unary function $(x, i) \mapsto (x + 1, i)$.

  Show that M with this structure is strongly minimal.
  (b) Consider the definable family $\{X_t : t \in M\}$ where X_t is the set of $(x, y) \in M^2$ with $\pi(y) = \pi(x) + \pi(t)$. Show that X_t is both stationary for infinitely many t and non-stationary for infinitely many t. Conclude that $\{t : X_t \text{ is stationary}\}$ is not definable.

(17) On the other hand, show that stationarity *is* definable in the theory of F-vector spaces for a fixed field F. (In fact, stationarity is definable in all natural examples – in particular also in algebraically closed fields, but this is much harder to prove).

4. Interpretable Sets

For the first two exercises, let M be any structure.

(1) (a) $\mathrm{Th}(M^{eq})$ only depends on $\mathrm{Th}(M)$.

(b) If $S = M^n/E$ is a sort in M^{eq}, a set $X \subset S$ is $\emptyset$-definable if and only if its preimage in M^n is $\emptyset$-definable in M.

(c) In particular, the $\emptyset$-definable sets in each M^n are unaffected by passing to M^{eq}.

(2) Let $X \subset M^n$ be A-definable, and let E be an A-definable equivalence relation on X. Show that the quotient X/E is naturally identified with an A-definable set in M^{eq}.

Hint: Let X and E be definable over a finite tuple $t \in M^m$ from A. Construct a $\emptyset$-definable equivalence relation on $M^m \times M^n$.

For the remaining exercises, assume M is strongly minimal.

(3) Check that dimension is well-defined on interpretable sets.

(4) Show that all items in Theorem 3.31 remain true in M^{eq}. Hint: many of (1)-(7) are interdependent, so you don't have to prove them all directly. For those you do have to prove, try to use weak elimination of imaginaries to reduce to a property of dimension in definable sets. This should work everywhere *except* definability of dimension (because there you need to control parameters, and weak elimination of imaginaries requires uncontrollable extra parameters). Instead, prove definability of dimension using function additivity.

Additional Exercises for Section 4

(5) A structure N is said to have *elimination of imaginaries* if each sort in N^{eq} is in $\emptyset$-definable bijection with a subset of some N^k. This says, roughly, that N^{eq} is no different that N itself. A very useful fact in the model theory of fields is that every algebraically closed field has elimination of imaginaries. Let us sketch a proof. So, suppose $M \models ACF$.

(a) Show that $\mathrm{acl}(\emptyset) \cap M$ is infinite.

(b) By rerunning the proof and using (a), show that we do not need added parameters in the statement of weak elimination of imaginaries for M: that is, every $\emptyset$-definable set in M^{eq} is the image of a $\emptyset$-definable set in some M^n under a finite-to-one $\emptyset$-definable function.

(c) Conclude that to prove M has elimination of imaginaries, it suffices to show that each symmetric power $(M^n)^{(m)}$ is in $\emptyset$-definable bijection with a subset of some M^k (here $(M^n)^{(m)}$ is the set of all m-element subsets of M^n, which is in particular a sort in M^{eq}).

(d) Use symmetric functions to prove elimination of imaginaries for symmetric powers as in (3), and conclude that M has elimination of imaginaries. (Precisely, for $M^{(m)}$, symmetric functions suffice. For $(M^n)^{(m)}$, something similar but more complicated is required).

5. Dimension of Types

For the first two exercises, M and N denote any structures.

(1) Suppose M is saturated. Then so is M^{eq}.

(2) Let M and N be saturated models of the same complete theory. If $|M| = |N|$ then M and N are isomorphic.

For the remaining exerices, work in a fixed uncountable saturated strongly minimal structure M.

(3) Show the following basic properties of the notation $\dim(a/A)$:

   (a) $\dim(a/A)$ only depends on $\text{tp}(a/A)$.
   (b) If $a \in M^n$ then $\dim(a/A) \leq n$.
   (c) If σ is a permutation of $a_1, ..., a_n$ then $\dim(\sigma(a_1)...\sigma(a_n)/A) = \dim(a_1...a_n/A)$.
   (d) If $A \subset B$ then $\dim(a/B) \leq \dim(a/A)$.

(4) Let X and Y be A-definable.
   (a) Show that X is almost contained in Y if and only if every generic point of X over A belongs to Y.
   (b) Show that X and Y are almost equal if and only if they have the same generic points over A.

(5) (Model-Theoretic Galois Theory)
   (a) Show that if A is small, $a, b \in M^{eq}$, and $\text{tp}(a/A) = \text{tp}(b/A)$, there is an automorphism of M fixing A point-wise and sending a to b. Hint: use the uniqueness of saturated models.
   (b) Conclude that for small A, the assertion that $a \in \text{acl}(A)$ is equivalent to the assertion that a has finite orbit under the action of $\text{Aut}(M/A)$ (automorphisms fixing A point-wise).
   (c) Similarly, say that a is *definable over* A (denoted $a \in \text{dcl}(A)$) if the set $\{a\}$ is A-definable. Show that for small A, $a \in \text{dcl}(A)$ is equivalent to the assertion that a is fixed by all of $\text{Aut}(M/A)$.
   (d) Give examples to show that (2) and (3) fail if A is not small (i.e. if $|A| = |M|$).
   (e) (Hard) On the other hand, show that (1) *does* hold even if A is not small (this is a unique feature of strongly minimal theories).

(6) Characterize the acl operator in natural examples, as follows:
   (a) Suppose $M \models \text{ACF}_p$ and $A \subset M$. Show that $\text{acl}(A) \cap M$ is the (field-theoretic) algebraic closure of the field generated by A. (It may help to use the previous exercise).
   (b) Find similar characterizations of algebraic closure (inside the sort M only) in the theories of the pure set, the integers with successor, and vector spaces.

(7) Use additivity to prove the following:
   (a) $\dim(ab/A) \leq \dim(a/A) + \dim(b/A)$.
   (b) If $b \in \text{acl}(A)$ then $\dim(a/Ab) = \dim(a/A)$.
   (c) If a and b are interalgebraic over A then $\dim(a/A) = \dim(b/A)$.

(8) Show the following properties of independence:
   (a) If $b \in \text{acl}(A)$ then every a is independent from b over A.
   (b) a is independent from itself over A if and only if $a \in \text{acl}(A)$.
   (c) If a is independent from bc over A if and only if it is independent from both b over A and c over Ab.

Additional Exercises for Section 5

(9) Let $A \subset M$. Show that A is an elementary submodel of M if and only if A is infinite and algebraically closed (i.e. $\text{acl}(A) \cap M = A$).

(10) Let us prove Zilber's non-finite axiomatizability theorem in the strongly minimal case. So, suppose M is moreover $\aleph_0$-categorical. Let $a_1, a_2, ...$ be an infinite sequence of independent generics of M (i.e. $\dim(a_{n+1}/a_1...a_n) = 1$ for all n). Let $A_n = \text{acl}(a_1...a_n) \cap M$ (a substructure of M).

   (a) Fix any non-principal ultraproduct $\mathcal{U}$ on $\mathbb{Z}^+$. Let $A^* = \Pi_{n\to\mathcal{U}} A_n$, and $M^* = \Pi_{n\to\mathcal{U}} M$. Show that A^* is algebraically closed as a subset of M^* (i.e. $\mathrm{acl}(A^*) \cap M^* = A^*$).
   (b) Conclude from the previous exercise that $A^* \models \mathrm{Th}(M)$.
   (c) Show that each A_n is finite, and thereby conclude that $\mathrm{Th}(M)$ is not finitely axiomatizable.

6. Canonical Bases

Continue to assume M is a fixed uncountable, saturated strongly minimal structure.

(1) Suppose c and d are both canonical parameters for the definable set X. Show that c and d are interdefinable (each can be defined using the other).
(2) Show that any two canonical bases of the same stationary almost equality class are interdefinable.
(3) Prove Theorem 6.10 using Theorem 6.7 and compactness.
(4) Let X be A-definable of dimension $d \geq 0$.
   (a) If X is stationary, then $\mathrm{Cb}([X])$ is definable over A.
   (b) In general, if $[Y]$ is any stationary component of $[X]$, then $\mathrm{Cb}([Y]) \in \mathrm{acl}(A)$.
   (c) X can be decomposed over $\mathrm{acl}(A)$: that is, there are finitely many disjoint stationary d-dimensional $\mathrm{acl}(A)$-definable sets whose union is X.
(5) (a) Prove that $\mathrm{Loc}(a/A)$ (and thus $\mathrm{Cb}(a/A)$) are well-defined.
   (b) Prove that $\dim(a/\mathrm{Cb}(\mathrm{a/A})) = \dim(a/A)$ for all a and A.
(6) Prove that a stationary plane curve is *trivial* (= not non-trivial) if and only if it is almost equal to a horizontal or vertical line.
(7) Characterize the possible sizes of families non-trivial plane curves in the common examples:
   (a) In a pure set, every stationary non-trivial plane curve X is almost equal to the diagonal $y = x$, and thus $\dim(\mathrm{Cb}(X)) = 0$.
   (b) In an F-vector spacee, every stationary non-trivial plane curve X is almost equal to the graph of an affine linear map $y = cx + v$. In this case, v is a canonical base, so $\dim(\mathrm{Cb}(X)) \leq 1$.
   (c) In ACF, $\dim(\mathrm{Cb}(X))$ can be arbitrarily large (consider the graph of a generic polynomial function of degree d).
(8) Show that if $(M, \cdot, ...)$ is an expansion of a group, then M is non-trivial.

Additional Exercises for Section 6

(9) Let X be stationary, and $c \in M^{eq}$. Let $\mathrm{Aut}(M/c)$ be the automorphisms of M fixing c, and let $\mathrm{Aut}(M/[X])$ be the automorphisms of M that preserve $[X]$ as a class (i.e. $\sigma(X) \in [X]$). Show that c is a canonical base of $[X]$ if and only if $\mathrm{Aut}(M/c) = \mathrm{Aut}(M/[X])$. (In general stability, this is really the definition of canonical bases).
(10) Show that M is trivial if and only if whenever $A \subset M$, $b \in M$, and $b \in \mathrm{acl}(A)$, there is $a \in A$ with $b \in \mathrm{acl}(a)$.
(11) Let $f : X \to Y$ be a finite-to-one A-definable function, and let $W \subset X$ be definable and stationary.

(a) Show that $f(W)$ is stationary, and $\mathrm{Cb}([f(W)])$ is definable over A together with $\mathrm{Cb}([W])$.
(b) Show that if a and b are interalgebraic over A, then so are $\mathrm{Cb}(a/A)$ and $\mathrm{Cb}(b/A)$.

(12) Let X be a strongly minimal set in M^{eq}.
(a) Show that there is a definable finite-to-finite correspondence between X and M.
(b) Show that X is trivial (viewed with its induced structure from M) if and only if M is, and X is locally modular if and only if M is.

(13) Let X be stationary and A-definable. A *Morley sequence* in X over A is an infinite sequence $a_1, a_2, ...$ of elements of X so that each a_n is generic in X over $Aa_1...a_{n-1}$.
(a) Show that any permutation of a Morley sequence is a Morley sequence.
(b) Show that a Morley sequence is indiscernible over A: for all $i_1 < ... < i_n$ and $j_1 < ... < j_n$, we have $\mathrm{tp}(a_{i_1}...a_{i_n}/A) = \mathrm{tp}(a_{j_1}...a_{j_n}/A)$.
(c) (Hard) Conversely, show that every indiscernible sequence arises this way: let $a_1, a_2, ...$ be an A-indiscernible sequence of tuples in M^{eq}. Show that there are $B \supset A$, and a stationary B-definable set X, so that $a_1, a_2, ...$ is a Morley sequence in X over B.
(d) (Hard) Show that if $a_1, a_2, ...$ is a Morley sequence in X over A, then $\mathrm{Cb}([X]) \in \mathrm{acl}(a_1 a_2 ...)$. (This says that $[X]$ is determined by a sufficiently big finite sample of points).

7. The Locally Modular Case

Throughout, assume M is uncountable, saturated, strongly minimal, non-trivial, and locally modular.

(1) Show that the germ groupoid is a well-defined groupoid.
(2) Let C, D be objects in the germ groupoid, and F a definable collection of morphisms $C \to D$. Then $\dim(F) \leq 1$. Thus, if $f : C \to D$ is a morphism, then $\dim(f/CD) \leq 1$.
(3) Suppose F, G, H are A-definable collections of morphisms from C to D, D to E, and C to E, respectively. Then $\{(f, g, h) \in F \times G \times H : g \circ f = h\}$ is A-definable.
(4) Suppose $[X]$ is an almost equality class of stationary non-trivial plane curves, with canonical base t satisfying $\dim(t) = 1$. So $[X]$ contains some t-definable member – without loss of generality X itself. Now let $(a, b) \in X$ be generic over t. Show that (a, b, t) is a non-trivial configuration. Conversely, if (a, b, t) is any non-trivial configuration, then there is a stationary t-definable non-trivial plane curve X so that $(a, b) \in X$ is generic over t.
(5) Show that if (a, b, t) is a non-trivial configuration, then so is (b, a, t).

Hint: this just says that if X is a stationary non-trivial plane curve with canonical base t, then t is also a canonical base for $X^{-1} = \{(y, x) : (x, y) \in X\}$.
(6) Let $[X]$ and $[Y]$ be almost equality classes of stationary non-trivial plane curves, with canonical bases t_{12} and t_{23} satisfying $\dim(t_{12}) = \dim(t_{23}) = 1$ and $\dim(t_{12}t_{23}) = 2$. Without loss of generality assume X is t_{12}-definable and Y is t_{23}-definable. Let $a_1 \in X$ be generic over $t_{12}t_{23}$. Show that there

are a_2, a_3 with $(a_1, a_2) \in X$ and $(a_2, a_3) \in Y$, and that for any such a_2 and a_3, $s = (a_1, a_2, a_3, t_{12}, t_{23}, t_{13})$ is a composition configuration.

Additional Exercises for Section 7

(7) (Hard) Prove our more general characterization of local modularity, namely that $\mathrm{Cb}(a/A) \in \mathrm{acl}(a)$ for all $a \in M^{eq}$ and $A \subset M^{eq}$. More generally, for each $a \in M^{eq}$, $b \in M^{eq}$, and $A \subset M^{eq}$, consider the following statements:

I. $\mathrm{Cb}(a/Ab) \in \mathrm{acl}(Aa)$.
II. $\dim(\mathrm{Cb}(a/Ab)/A) \leq \dim(a/A) - \dim(a/Ab)$.
III. There is $c \in M^{eq}$ with $c \in \mathrm{acl}(Aa) \cap \mathrm{acl}(Ab)$ and $\dim(c/A) = \dim(a/A) + \dim(b/A) - \dim(ab/A)$.

(a) Show that for all a, b, A, the statements I, II, and III are equivalent.
(b) Now show that I, II, and III hold for all a, b, A. First, using weak elimination of imaginaries, show that we may assume $a \in M^n$ for some n, and that $\dim(a/A) = n$.
(c) Now work by induction on n. First show the case $n \leq 1$ directly. Then show that the case $n = 2$ is a restatement of local modularity. Finally, for the inductive step when $n \geq 3$, replace A by $A' = Aa_n$, and replace a by $a' = a_1 ... a_{n-1}$.
(d) Lastly, show that I, II, and III imply the desired statement ($\mathrm{Cb}(a/A) \in \mathrm{acl}(a)$ for all a and A).

(8) Let us sketch the proof of the weak trichotomy theorem. Suppose that instead of knowing M is locally modular, we know M is k-linear for some $k \geq 1$: namely, k is the maximum value of $\dim(\mathrm{Cb}(X))$ over all stationary plane curves X.
(a) By mimicking the entirety of our Section 7, show that there is a k-dimensional definable stationary group of automorphisms of some object in the germ groupoid of M (this is very long but ultimately it adapts quite directly.)
(b) Show that in fact, there are a definable strongly minimal set X, and a k-dimensional stationary definable group of permutations of X, acting transitively on X.
(c) A theorem of Hrushovski (coming from the theory of groups of finite Morley rank) says that if G is a k-dimensional stationary definable transitive group of permutations of a strongly minimal set X, and $k \geq 2$, then some cofinite subset of X can be definably endowed with the structure of an algebraically closed field (precisely, X is either a field itself, or is the projective line over such a field – and then the group is the algebro-geometric automorphism group of X). Use this to show that M is in finite-to-finite definable correspondence with an algebraically closed field, and that using the field structure we can contradict the maximality of k.

(This is how model theorists construct fields: we really just contruct higher-dimemsional group actions, and use that such actions always arise from fields).

8. The Structure of the Group

Throughout, assume M is saturated, uncountable, strongly minimal, and locally modular, and G is a $\emptyset$-definable strongly minimal group in M^{eq}.

(1) (a) Let $H \leq G$ be a definable subgroup. Then either H is finite, or $H = G$.
(b) Let $f : G \to G$ be a definable endomorphism. Then either f is trivial, or f is surjective with finite kernel.

(2) Suppose X and Y are both cosets of definable subgroups of G^n. If X and Y are almost equal, then X and Y are equal.

(3) Suppose $\{f_t : t \in T\}$ is a definable family of endomorphisms of G. We show there are only finitely many distinct endomorphisms amont the f_t. Toward a contradiction, assume there are infinitely many distinct maps.
(a) By modding T by an equivalence relation, show that there is such a family which is *faithful*: whenever $s \neq t$ the maps f_s and f_t agree in only finitely many points. Conclude in this case that $\dim(T) \geq 1$.
(b) Now assume $\{f_t\}$ is faithful, and consider the family $\{g_{t,a} : (t, a) \in T \times G\}$, where $g_{t,a}(x) = a \cdot f_t(x)$. Use the previous exercise to show that $\{g_{t,a}\}$ is still faithful.
(c) Conclude that for generic $(t, a) \in T \times G$, the canonical base of the graph of $g_{t,a}$ has dimension at least 2, and thus we contradict local modularity.

(4) Prove that every stationary definable subgroup of every G^n is $\operatorname{acl}(\emptyset)$-definable, as follows:
Suppose $H \leq G^n$ is a stationary definable subgroup of dimension d. Let c be a canonical parameter of H, so H is c-definable. Let $a \in G^n$ be generic over c, and let c_a be a canonical parameter of the coset $a \cdot H$.
(a) Show that c_a is also a canonical base of $[a \cdot H]$.
(b) Show that $\dim(a/cc_a) = d$, by computing $\dim(ac_a/c)$ in two ways. Conclude that a is generic in $a \cdot H$ over c_a.
(c) Conclude that $\operatorname{Loc}(a/c_a) = [a \cdot H]$ and $\operatorname{Cb}(a/c_a) = c_a$, and thus by local modularity, $c_a \in \operatorname{acl}(a)$.
(d) Show that c is definable over (c_a, a) (hint: $H = a^{-1} \cdot (a \cdot H)$). Conclude that $c \in \operatorname{acl}(a)$.
(e) Finally, show that $c \in \operatorname{acl}(\emptyset)$ by computing $\dim(ac)$ in two ways. Conclude that H is $\operatorname{acl}(\emptyset)$-definable.

Additional Exercises for Section 8

(5) (Very Hard) Prove that *every* strongly minimal group is abelian (regardless of local modularity). A bit easier: just do it assuming there is an element of infinite order.

Bonus Section: Totally Categorical Implies Locally Modular

Here we sketch the proof of our black box, that totally categorical theories are locally modular. This will be quite involved. Throughout, work in a saturated uncountable strongly minimal structure M.

(1) Suppose $X \subset M^m$ and $Y \in M^n$ are definable, $\dim(X) = \dim(Y) = d$, Y is stationary, and $f : X \to Y$ is definable and finite-to-one. Show that there

is a positive integer k so that $f^{-1}(y)$ has size k for almost all $y \in Y$. We will say that f is *generically k-to-1*.

Definition 0.1. Say that M is *unimodular* if for all X, Y, and f as above, the value k only depends on X and Y, and not on f.

(2) Show that vector spaces are unimodular, and algebraically closed fields are not.

(3) (Hard) Show that if M is $\aleph_0$-categorical, then M is unimodular.

Hint: Given X, Y, f, and k as above, assume X, Y, and f are definable over a finite tuple t. Let $b \in Y$ be generic over t. Let $Y(b)$ be the (finite) set of points $y \in Y$ with $\operatorname{acl}(yt) = \operatorname{acl}(bt)$, and let $X(b)$ be the (finite) set of points $x \in X$ with $\operatorname{acl}(xt) = \operatorname{acl}(bt)$. By restricting the whole problem to $X(b)$ and $Y(b)$, show that $k = \frac{|X(b)|}{|Y(b)|}$, and that $|X(b)|$ and $|Y(b)|$ only depend on X and Y.

From now on, assume M is unimodular.

(4) Show that we can assign a value $\operatorname{Z.deg}(X) \in \mathbb{Q}$ (called the *Zilber degree*) to every definable set X in M^{eq} with the following properties:

- $\operatorname{Z.deg}(M^n) = 1$ for all n.
- If $\dim(X) = \dim(Y)$ and X and Y are disjoint then $\operatorname{Z.deg}(X \cup Y) = \operatorname{Z.deg}(X) + \operatorname{Z.deg}(Y)$. In particular, if X is finite then $\operatorname{Z.deg}(X) = |X|$.
- $\operatorname{Z.deg}(X)$ only depends on the almost equality class $[X]$.
- If $\dim(X) = \dim(Y)$, Y is stationary, and $f : X \to Y$ is definable and generically k-to-1, then $k = \frac{\operatorname{Z.deg}(X)}{\operatorname{Z.deg}(Y)}$.

Definition 0.2. Let $f : X \to Y$ be A-definable in M^{eq}, where Y is stationary. Say that f is *generically dominant* if for any generic $a \in X$ over A, $f(a)$ is generic in Y over A.

(5) Show that the definition of generically dominant is well-defined (it doesn't depend on A).

(6) Suppose $f : X \to Y$ is generically dominant (so Y is stationary). Show that for all generic $y \in Y$ we have $\operatorname{Z.deg}(f^{-1}(y)) = \frac{\operatorname{Z.deg}(X)}{\operatorname{Z.deg}(Y)}$.

Recall that if M is not locally modular, we can find faithful families of plane curves $X \subset M^2 \times T$ with $\dim(T)$ arbitrarily large. Moreover, we may assume that T is stationary and that for generic $t \in T$, X_t is stationary.

Definition 0.3. Let $X \subset M^2 \times T$ be a faithful definable family of plane curves, where T is stationary and $\dim(T) \geq 2$. A *common point* of X is a point $a \in M^2$ belonging to almost all X_t.

(7) Let $X \subset M^2 \times T$ be a faithful definable family of plane curves, where T is stationary and $\dim(T) \geq 2$, and X_t is stationary for generic $t \in T$.

(a) Show that X has only finitely many common points, and thus X is almost equal to a family without any common points.

(b) Show that the projections $X \to T$ and $X \to M^2$ are generically dominant.

(c) Let $I \subset M^2 \times T^2$ be the 'intersection family' of X: $(x, t, u) \in I$ if $(x, t), (x, u) \in X$. Show that $\dim(I) = 2 \cdot \dim(T)$, and if X has no common points, then the projections $I \to X$ and $I \to T^2$ are generically dominant.

(8) Let $X \subset M^2 \times T$ be a $\emptyset$-definable faithful family of plane curves as above: T is stationary of dimension at least 2, X_t is stationary for generic t, and X has no common points. By analyzing various generically dominant projections, show that for generic $(t, u) \in T^2$ we have

$$|X_t \cap X_u| = \mathrm{Z.deg}(X_t) \times \mathrm{Z.deg}(X_u).$$

In particular, the number of intersection points of X_t and X_u only depends on their Zilber degrees, which in turn only depends on the almost equality classes $[X_t]$ and $[X_u]$.

(9) Now find a contradiction by deleting an intersection point off of two curves: their Zilber degrees don't change, but their intersection size does. Precisely, take $X \subset M^2 \times T$ as above, where $\dim(T) \geq 3$. Fix $p \in M^2$ generic, and form a new family $Y \subset M^2 \times U$ as follows:
- U is any stationary component of the set of $t \in T$ with $p \in X_t$.
- Let X_U be the subfamily of X indexed by U. Then let Y be X_U with all common points removed: so $(x, t) \in Y$ if $t \in U$, $x \in X_t$, and x is not a common point of X_U.

(a) Show that $\dim(U) \geq 2$.

(b) Let $(t, u) \in U^2$ be generic over p. Show that (t, u) is also generic in T^2 over $\emptyset$. Conclude that $|X_t| \cap |X_u| = |Y_t| \cap |Y_u|$, as both equal $\mathrm{Z.deg}(X_t) \times \mathrm{Z.deg}(X_u)$. This is absurd, since p is removed from both X_t and X_u when passing to Y_t and Y_u.