STRONG MINIMALITY AND GEOMETRIC STABILITY

BEN CASTLE

1. Introduction

This course serves as an introduction to geometric stability theory, focusing on the special case of *strongly minimal* structures. Geometric stability is a beautiful and powerful part of modern model theory aimed at recovering familiar algebraic objects from abstract logical data, and has been at the center of celebrated applications of model theory in other areas. Unfortunately, it is also quite abstract and technical. On the other hand, the strongly minimal case (where the subject originated) is both accessible and still quite relevant to modern research. Thus, our aim is to give a sensible account of some of the main notions and theorems while staying entirely in the strongly minimal setting. Our main result will be the 'locally modular group configuration theorem' (essentially [2]) – an important special case of the celebrated (general) group configuration theorem of Hrushovski. This theorem roughly says that strongly minimal structures of 'linear' complexity always arise from definable abelian groups. We will also emphasize the ensuing classification theorems one obtains for totally categorical strongly minimal theories.

Note that our general approach is to be as elementary and 'hands-on' as possible, focusing on developing useful intuition for any further studies in stability theory. As a consequence, we will develop the material in a rather unique way (going in a different order and with different emphases from other treatments). Nevertheless, some useful references may include [6] and [4].

Throughout the text, there are various exercises appearing as natural lemmas and examples that we will use. Such things are relegated to the exercises largely due to the time constraints of this particular course. In a separate document, we also include a fully compiled exercise list which contains all in-text exercises plus a few supplementary challenging ones.

2. Uncountable Categoricity and Strong Minimality

For now, we give an imprecise summary of the earliest developments of geometric stability theory. Our starting point is Morley's Theorem. Let T denote a complete theory in a countable language with infinite models. By Lowenheim Skolem, T has at least one model in every infinite cardinality. Recall:

Definition 2.1. For κ an infinite cardinal, T is *κ-categorical* if T has only one model of cardinality κ up to isomorphism. T is *totally cateogircal* if it is κ-categorical for all infinite κ.

Fact 2.2 (Morley 1965, [5]). *If T is κ-categorical for some uncountable κ, then T is κ-categorical for all uncountable κ.*

Date: Fudan Logic Summer School 2025.

(Thus we just say *uncountably categorical.*)

This theorem is most notable because of the ideas introduced in the proof – it directly led to many ideas in modern *stability theory* and its generalizations. Modern treatments of Morley's theorem typically follow a later version due to Baldwin-Lachlan (1971, [1]). One of the advantages of the Baldwin-Lachlan proof is that it characterizes which theories are uncountably categorical. Recall that countably categorical theories already have a satisfying characterization:

Fact 2.3. *T is $\aleph_0$-categorical if and only if for all n, there are only finitely many complete n-types over $\emptyset$ (equivalently, there are only finitely many formulas in n variables up to equivalence).*

Baldwin-Lachlan gave a characterization of uncountable categoricity in terms of a new notion called *strong minimality*:

Definition 2.4. T is *strongly minimal* if in all models $M \models T$, every unary definable subset $X \subset M$ (over any parameters) is finite or cofinite. In general, a structure (in any language) is strongly minimal if its complete theory is.

It is important that X is unary above; otherwise the diagonal $\{(x, x)\} \subset M$ is always infinite and coinfinite.

It is also important that one considers all models of T instead of just one:

Exercise 2.5. Give an example of an infinite structure M such that every unary definable set in M is finite or cofinite, but M is not strongly minimal.

Here are some positive examples:

Example 2.6. Each of the following theories is strongly minimal, by quantifier elimination:

(1) The theory of pure infinite sets.
(2) The theory of the integers with the successor function.
(3) The theory of infinite vector spaces over a field. To clarify, this means we have a fixed field F (countable for now), and we take the theory of infinite F-vector spaces in the language $(+, -, 0)$ augmented by unary function symbols for each $c \in F$ (interpreted as scaling by c).
(4) The theory ACF_p of algebraically closed fields of a fixed characteristic p (which could be zero).

Let us check that ACF_p is strongly minimal; the others are easier. Let $M \models \mathrm{ACF}_p$, and let $X \subset M$ be unary and definable. By quantifier elimination, X is a finite Boolean combination of solution sets of polynomials. The solution set of a single polynomial $p(x) = 0$ is either all of M (if $p = 0$), or of cardinality at most $\deg(p)$ – hence is finite or cofinite. So X is a finite Boolean combination of finite and cofinite sets, so is itself finite or cofinite.

Exercise 2.7. We give a more general test for strong minimality. Let T be a complete theory in a countable language.

(1) Suppose that for every $M \models T$, and every finite subset $A \subset M$, the automorphism group $\mathrm{Aut}(M/A)$ (automorphisms of M fixing A point-wise) has a co-countable orbit. Show that T is strongly minimal. (This is actually an exact characterization but the converse requires further material).
(2) Use (a) to show that examples (1)-(4) above are strongly minimal.

Note that each of (1)-(4) above is uncountably categorical. Consider, for example, F-vector spaces where F is countable. In this case, a model is determined by its dimension as an F-vector space. If M is uncountable, then $\dim(M)$ is easily seen to be $|M|$ – thus two uncountable models of the same cardinality have the same dimension, so are isomorphic.

Similar arguments can made in each of the examples using a different numerical invariant in place of vector space dimension:

(1) For pure infinite sets, a model is determined by its *cardinality*.
(2) For $\mathrm{Th}(\mathbb{Z}, s)$, a model is determined by the number of copies of $\mathbb{Z}$ it contains.
(3) For ACF_p, a model is determined by its *transcendence degree* over the prime model – that is, the largest size of an algebraically independent subset of the field.

Baldwin and Lachlan show that this is the typical situation:

Fact 2.8. (1) *If T is strongly minimal, there is a cardinal-valued map $M \mapsto \dim(M)$ on models $M \models T$ so that two models $M, N \models T$ are isomorphic if and only $\dim(M) = \dim(N)$. If M is uncontable, then $\dim(M) = |M|$ – thus T is uncountably categorical.*
(2) *In general, T is uncountably categorical if and only if there is a strongly minimal theory T' such that T is 'prime over T''.*

We won't go into details, but clause (2) means that arbitrary uncountably categorical theories are built out of strongly minimal ones, so one can understand uncountably categorical theories well by just focusing on the strongly minimal case. The idea of 'prime' is that any two strongly minimal pieces of T should be 'linked' by some definable relation to make sure they have the same cardinality.

Example 2.9. (1) Let M be strongly minimal. Consider M^2 with its induced structure from M. Then $\mathrm{Th}(\mathrm{M}^2)$ is uncountably categorical, and is prime over $\mathrm{Th}(\mathrm{M})$. One can decompose M^2 into two strongly minimal pieces (the two M factors), which are 'linked' by the diagonal.
(2) Let T be the theory of pairs of infinite sets with a bijection: a model is the disjoint union of two infinite sets X and Y, and the language consists of a bijection $X \to Y$. Then T is prime over the theory of pure infinite sets.
(3) On the other hand, let T be the theory of pairs of infinite sets (with no bijection). Then T is not prime over the theory of infinite sets.

Exercise 2.10. Show explicitly that (2) (the theory of pairs of infinite sets with a bijection) is uncountably categorical, while (3) (pairs of infinite sets without a bijection) is not.

(4) (More complicated example): We give an example of an uncountably categorical theory that can't be decomposed cleanly into finitely many strongly minimal ones (again, without going into details about what exactly that means; the relevant property of the theory is that it is not *almost strongly minimal*). Let T be the theory of a fibered family of affine vector spaces, in the following sense: Fix a countable field F, and an infinite F-vector space V. Let M be the disjoint union of V and V^2, with the following language:
   - The vector space structure on V.
   - The left projection $V^2 \to V$ $((x, y) \mapsto x)$.

- The fiber-wise difference map $(V^2)^2 \to V$, given by $(x, y)(x, z) \mapsto y - z$ and $(x, y), (w, z) \mapsto 0$ if $x \neq w$.

Let T be the theory of M in this language. In general, T describes a vector space V, in addition to a V-indexed family of affine copies of V (affine means that each copy of V has no clearly labeled identity element, and thus only becomes a vector space after fixing one point). Using this idea, one can show that any two copies of V are definably isomorphic, but not in a uniform way. Nevertheless:

Exercise 2.11. Show that T is uncountably categorical. (In this case T is prime over the theory of F-vector spaces).

Note: a similar example to (4) above is $T := \text{Th}((\mathbb{Z}/4\mathbb{Z})^\omega, +)$, the (theory of the) product of infinitely many cyclic groups of order 4. This theory T is the 'canonical' example of an uncountably categorical theory that isn't essentially a product of strongly minimal factors (i.e. is not almost strongly minimal). While T is easier to define than (4) above, it is less obvious why it has the relevant properties. Nevertheless, the reason is really the same: T is prime over the theory of $\mathbb{F}_2$-vector spaces, and decomposes into an $\mathbb{F}_2$-vector space V together with a family of affine copies of V. We leave this as an additional exercise.

Another advantage of the Baldwin-Lachlan treatment is it makes clear which uncountably categorical theories are also *totally categorical*:

Fact 2.12. (1) *If T is strongly minimal, then T is totally categorical if and only if every model of T has infinite dimension.*
(2) *If T is prime over the strongly minimal theory T', then T is totally categorical if and only if T' is.*

Thus pure sets are totally categorical, as are F-vector spaces when F is finite; while our other strongly minimal examples are not totally categorical. Notice that in our totally categorical examples, it's not really that there are *no* finite-dimensional models – it's just that the finite-dimensional models are finite. For example, for the theory of F-vector spaces (for a finite field F), one has a unique 'n-dimensional model' for all n (namely an n-dimensional F-vector space) – it's just not actually a model of the complete theory of infinite F-vector spaces under consideration. The following was formerly a major open problem aimed at capturing this phenomenon:

Conjecture 2.13 (Now Zilber's Theorem)**.** *Suppose T is totally categorical. Then T is not finitely axiomatizable. More precisely, there is a collection of finite structures $\{M_n\}$ such that any non-principal ultraproduct of the M_n is a model of T. Thus T cannot be finitely axiomatizable because any sentence of T also holds of some M_n.*

Here the M_n are thought of as the 'finite-dimensional subspaces' of models of T.

Zilber gave a simple proof of this conjecture in the strongly minimal case, and an extremely complicated proof in general. If T is the theory of pure infinite sets, his construction produces for M_n a finite set of size n. If T is the theory of F-vector spaces for F a finite field, then M_n is the unique n-dimensional F-vector space.

Let's imagine the general case by considering our fibered family of affine spaces $V \cup V^2$ above. One should take for M_n a fibered family of n-dimensional spaces. To start, fix an n-dimensional $V_n \leq V$. This gives us a distinguished V_n-indexed family of affine copies of V (the fibers above V_n in V^2). Call these $X_1, ..., X_m$. We would like to shrink each X_i into an affine copy of V_n. This basically amounts to choosing a point $a_i \in X_i$ and then taking the V_n-translate of a_i (which is uniformly a_i-definable once a_i is chosen). So for the construction to work, we have to introduce some arbitrary choices. Now as it turns out, this is not a problem if our original strongly minimal piece (i.e. the base 'V' in this example) is a pure set or vector space. On the other hand, the abstract proof runs into a problem when we choose arbitrary points (speficially the proof that ultraproducts of the finite structure are models of T). So one has a major issue (and this is why the proof got complicated).

As it turned out, Zilber got around this issue in a totally unexpected and remarkable way – by showing that the two cases he could handle were exhaustive:

Fact 2.14 (Zilber)**.** *Every totally categorical theory T is prime over either the theory of pure infinite sets or the theory of infinite F-vector spaces for some finite field F. More precisely, in any model $M \models T$, there is an infinite interpretable set X such that the induced structure on X is precisely a pure set or a vector space over a finite field.*

By 'the induced structure is a vector space' above, we mean one can endow X with the structure of an F-vector space so that the definable subsets of each X^n are exactly those definable in the vector space language (and similarly for pure sets).

This is an amazing fact: totally categorical theories are by definition the most model-theoretically well-behaved, and out of nowhere, they all arise from pure sets and linear algebra. This started a trend of results in model theory of a similar form – from purely model-theoretic assumptions on a theory, one recognizes a very concrete algebraic structure. Results in this style are collectively known as 'geometric stability theory'. One of the goals of this course is to cover as much as possible of the proof of Fact 2.14 (essentially giving the whole proof modulo a couple black boxes).

Note that Fact 2.14 can be proven by combining model theory with some classification theorems for abstract 'combinatorial geometries' (this is really why the subject is called 'geometric' stability theory). The fact that this works is specific to $\aleph_0$-categoricity. We will instead use a purely model-theoretic approach (roughly developed by Hrushovski in his 'group configuration theorem'), which generalizes to prove similar facts in the non-$\aleph_0$-categorical setting, and thus has proven more useful over time.

3. Definable Sets in Strongly Minimal Theories

Until otherwise stated, fix a strongly minimal structure M**.** We now allow the case that the language of M is unountable: then $\text{Th}(M)$ might not be uncountably categorical, but it is still κ-categorical for uncountable κ larger than the language.

The main tool associated to strong minimality is a dimension theory for definable sets in M^n. This means every definable set $X \subset M^n$ is assigned a number $\dim(X)$, and this assignment has nice properties. Actually, we will develop two dimension theories: one for definable sets, and one for tuples (equivalently, for types) – these two theories are in a sense 'dual' to each other. The first is more concrete and hands-on; the second is smoother and more generalizable. In the end, having both perspectives will be advantageous.

3.1. **Definability of Finiteness.** The most fundamental building block of the dimension theory is referred to as *uniform finiteness*:

Lemma 3.1 (Uniform Finiteness)**.** *Let $X \subset M^m \times M$ be definable. Then there is $N \in \mathbb{Z}^+$ so that for all $y \in M^m$, either $|X_y| \leq N$ or $|M - X_y| \leq N$.*

Proof. Otherwise, by compactness, we could build an elementary extension N of M and a tuple $a \in N^m$ so that for all k, $|X_a| > k$ and $|N - X_a| > k$. Thus X_a is infinite and coinfinite in N, contradicting the strong minimality of T. □

Exercise 3.2. Show that uniform finiteness holds in M^n: That is, let $X \subset M^m \times M^n$ be definable. Then there is N so that for all $y \in M^m$, either X_y is infinite or $|X_y| \leq N$.

Uniform finiteness shows that finiteness is 'definable in families':

Corollary 3.3 (Finiteness is Definable)**.** *Let $X \subset M^m \times M$ be A-definable. Then $\{y \in M^n : X_y$ is finite$\}$ is A-definable.*

Proof. By uniform finiteness, this set is defined by 'X_y has at most N distinct elements' for some sufficiently large N. □

3.2. **The Definition of Dimension.** We will now assign every definable set $X \subset M^n$ a dimension. Let us sketch how this is done. For $X \subset M$, we set $\dim(X) = 0$ if $X \neq \emptyset$ is finite and $\dim(X) = 1$ if X is cofinite (and $\dim(\emptyset) = -\infty$)) − the idea is that X is a 'line'. Then we extend inductively. Uniform finiteness (or rather definability of finiteness) is the key to making this work: for example, suppose $\emptyset \neq X \subset M^2$ is definable. Consider the left projection $X \to M$. By definability of finiteness, the set $X_{cof} = \{y \in M : X_y \text{ is cofinite}\}$ is definable − so this set is itself either finite or cofinite. If X_{cof} is cofinite, we set $\dim(X) = 2$. If X_{cof} is finite, we set $\dim(X)$ to be 0 if X is finite and 1 if X is infinite. Then we do something similar in M^3, considering the leftmost projection $X \to M^2$.

Let us make the sketch above precise. Throughout, let us view M^0 as a single point (this is useful for inductive arguments).

Definition 3.4. For each $n \geq 0$ and each definable set $X \subset M^n$, we inductively associate the *dimension* of X, $\dim(X) \in \{-\infty, 0, 1, ..., n\}$. We do this as follows:

- For $n = 0$, we set $\dim(\emptyset) = -\infty$ and $\dim(M^0) = 0$.
- For $n = 1$, we set $\dim(X) = -\infty$ if X is empty, $\dim(X) = 1$ if X is cofinite, and $\dim(X) = 0$ otherwise.
- Suppose $n \geq 1$ and we have defined dimension for subsets of M^n. Write M^{n+1} as $M^n \times M$. If $X \subset M^n \times M$ is definable, then for $e \in \{-\infty, 0, 1\}$ we set
$$X(e) = \{y \in M^n : \dim(X_y) = e\}.$$
By definability of finiteness, each $X(e)$ is definable, and thus has a dimension. Now set
$$\dim(X) = \max\{\dim(X(e)) + e : e \in \{-\infty, 0, 1\}\}.$$

Example 3.5. Suppose M is a pure infinite set, and fix distinct elements $a, b, c \in M$. Let $X \subset M^2$ be defined by '$x = a$ or $y = b$, and $x \neq c$'. X is the union of a horizontal line and a vertical line, with one point removed from the horizontal line. Basic intuition should suggest that $\dim(X) = 1$. Let us check:

- $X(-\infty) = \{c\}$, since c is the only point not extendable to an element of X. So $\dim(X(-\infty)) = 0$.
- $X(1) = \{a\}$, since a is the unique point extendable to infinitely many points of X. So $\dim(X(1)) = 0$.
- $X(0) = M - \{a, c\}$ (every other point). So $\dim(X(0)) = 1$.

Thus $\dim(X) = \max\{0 + (-\infty), 0 + 1, 1 + 0\} = 1$.

A key feature of dimension is its *definability* (we will use this essentially constantly):

Exercise 3.6 (Dimension is Definable)**.** Show by induction that dimension is definable in families: if $X \subset M^m \times M^n$ is A-definable, then for each $d \in \{-\infty, 0, ..., n\}$ the set $\{y \in M^m : \dim(X_y) = d\}$ is A-definable.

Here are some properties that fall straight out of the definition:

Exercise 3.7. Show the following:

(1) $\dim(M^n) = n$ for each n.
(2) If $X \subset M^n$ is definable, we have $\dim(X) = -\infty$ if and only if X is empty, and $\dim(X) = 0$ if and only if X is non-empty and finite.

Exercise 3.8. Suppose $M \models ACF_p$ for some p. Let $P(x, y)$ be a non-constant binary polynomial with coefficients in M. Show that $\dim(\{(x, y) : P(x, y) = 0\}) = 1$.

3.3. **Basic Properties.** The definition of dimension above is natural but clunky and hard to work with. Most notably, the definition changes if we permute the coordinates, which is undesirable. We now want to develop some basic properties, ultimately showing that dimension is actually very well-behaved (in particular invariant under coordinate permutations and much more). When working with dimension later on, it will be much more useful to have this basic set of properties than the original definition.

First, and most basically (but surprisingly tricky), is the following:

Lemma 3.9. (1) *If $X \subset Y \subset M^n$ are definable, then* $\dim(X) \leq \dim(Y)$.
(2) *If $X_1, ..., X_m \in M^n$ are definable, then*

$$\dim(X_1 \cup ... \cup X_m) = \max\{\dim(X_1), ..., \dim(X_m)\}.$$

(Put another way, if we split a d-dimensional set into finitely many definable pieces, one of them has dimension d).

Proof. We prove both statements jointly by induction on n. The cases $n = 0$ and $n = 1$ are clear (a union of finitely many finite sets is finite).

Suppose both statements are true for a fixed $n \geq 1$, and consider $n + 1$. We prove (1) and (2):

(1) Let $X \subset Y \subset M^n \times M$. Let $e \in \{-\infty, 0, 1\}$ with $\dim(X(e)) + e = \dim(X)$. By the base case, for all $y \in X(e)$ we have $\dim(Y_y) \geq e$. So $X(e)$ is contained in the union of $Y(f)$ over all $f \in \{-\infty, 0, 1\}$ with $f \geq e$. By (2) of the inductive hypothesis, there is a single $f \geq e$ with $\dim(X(e) \cap Y(f)) = \dim(X(e))$. By (1) of the inductive hypothesis, $\dim(Y(f)) \geq \dim(X(e))$. So by definition,

$$\dim(Y) \geq \dim(Y(f) + f) \geq \dim(X(e)) + e = \dim(X).$$

(2) By induction, we may assume $m = 2$. That is, we consider definable $X, Y \in M^n \times M$, and we show that $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$. The $\geq$ direction follows by (1). We show $\leq$, i.e. that one of X and Y has dimension at least $\dim(X \cup Y)$. Let $e \in \{-\infty, 0, 1\}$ with $\dim(X \cup Y) = \dim((X \cup Y)(e)) + e$. By the base case, for each $y \in (X \cup Y)(e)$ we have $y \in X(e)$ or $y \in Y(e)$. So $(X \cup Y)(e)$ is contained in $X(e) \cup Y(e)$, and by the inductive hypothesis one of $X(e), Y(e)$ has dimension at least $\dim((X \cup Y)(e))$. Without loss of generality assume $\dim(X(e)) = \dim((X \cup Y)(e))$. Then

$$\dim(X) \geq \dim(X(e)) + e = \dim((X \cup Y)(e)) + e = \dim(X \cup Y).$$

□

It is rather annoying to always 'cut' M^n into M^{n-1} and M in the inductive definition. In some cases, we may want to invoke a similar definition of dimension where we factor M^n as e.g. $M^{n-3} \times M^3$. We now want to show that doing this does not change the definition.

Lemma 3.10 ('m-cutting')**.** *Let $X \subset M^{m+n} = M^m \times M^n$ be definable. For each $i \in \{-\infty, 0, ..., n\}$ let $X_m(i) = \{y \in M^m : \dim(X_y) = i\}$. Then*

$$\dim(X) = \max\{\dim(X_m(i)) + i : i \in \{-\infty, 0, ..., n\}\}.$$

Proof. We induct on n. If $n = 0$, this just says that $\dim(X) = \max\{-\infty, \dim(X) + 0\}$, which is clear.

Suppose the statement is true for $n \geq 0$, and consider $X \subset M^m \times M^{n+1}$. If $X = \emptyset$, the statement just says that $-\infty = \max\{-\infty + i : i \in \{-\infty, 0, ..., n+1\}$, which is clear. So assume $X \neq \emptyset$, and thus $\dim(X) \geq 0$.

We prove the desired statement by establishing both inequalities:

- $\leq$: By definition of dimension, there is $e \in \{-\infty, 0, 1\}$ so that $\dim(X(e)) + e = \dim(X)$. Since $X \neq \emptyset$, $e \geq 0$, so $\dim(X(e)) = d - e$. Now $X(e) \subset M^m \times M^n$. By the inductive hypothesis, there is j so that $\dim(Y) + j = \dim(X(e)) = d - e$, where $Y \subset M^m$ is the points with j-dimensional fiber in $X(e)$. The definition of dimension gives that for $y \in Y$, we have $\dim(X_y) \geq j + e$ – thus $y \in X_m(i)$ for some $i \geq j + e$. So Y is covered by the $X_m(i)$ for $i \geq j + e$, and thus by Lemma 3.9, there is a single $i \geq j + e$ so that $\dim(X_m(i)) \geq \dim(Y)$. Then

$$\dim(X_m(i)) + f \geq \dim(Y) + j + e = \dim(X(e)) + e = \dim(X).$$

- $\geq$: Let $i \in \{-\infty, ..., n+1\}$. We want to show that $\dim(X_m(i)) + i \leq \dim(X)$. Since $X \neq \emptyset$, we may assume $i \geq 0$.

  For each $y \in X_m(i)$, there is $e \in \{-\infty, 0, 1\}$ so that $i = \dim(X_y) = \dim((X_y)(e)) + e$. By Lemma 3.9, there are a single $e \in \{-\infty, 0, 1\}$ and a definable $Y \subset X_m(i)$ so that $\dim(Y) = \dim(X_m(i))$ and $\dim((X_y)(e)) + e = i$ for all $y \in Y$. Since $i \geq 0$, we also have $e \geq 0$, so this can be rearranged as $\dim((X_y)(e)) = i - e$ for $y \in Y$. Let $Z \subset X$ be the set of points whose M^m-coordinate lies in Y. Then by the inductive hypotheses, $\dim(Z(e)) \geq \dim(Y) + (i - e)$. Thus

$$\dim(Z) \geq \dim(Z(e)) + e = \dim(Y) + i = \dim(X_m(i)) + i.$$

□

The above lemma says that dimensions in X^{m+n} can always be reduced to dimensions in X^m and X^n. Using this idea, we can now show that dimension is invariant under permutations:

Theorem 3.11. *Let $X \subset M^n$ be definable, and $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ a permutation. Then $\dim(\sigma(X)) = \dim(X)$ (where $\sigma(X)$ denotes the image of X under permuting coordinates using σ).*

Proof. We may assume $n \geq 2$ (otherwise σ is trivial).

First suppose $n = 2$, so $X \subset M^2$. We give a coordinate-free equivalent condition to '$\dim(X) = d$' for each $d \in \{-\infty, 0, 1, 2\}$, which implies the theorem in this case. The first two are easy: $\dim(X) = -\infty$ if and only if X is empty, and $\dim(X) = 0$ if and only if X is non-empty and finite (see Exercise 3.7).

We will give a coordinate-free characterization of '$\dim(X) = 2$'. This is enough, because then $\dim(X) = 1$ has the coordinate-free characterization '$\dim(X)$ is not $-\infty$, 0, or 2'. Now for dimension 2, we show:

Claim 3.12. $\dim(X) = 2$ *if and only if for every $k \in \mathbb{Z}^+$ there are $A, B \subset M$ with $|A|, |B| \geq k$ and $A \times B \subset X$.*

Proof. Let $\pi : X \to M$ be the leftmost projection. First suppose $\dim(X) = 2$, and fix k. Then cofinitely many fibers X_y under π are cofinite, so there is a set $A \subset M$ with $|A| = k$ and each X_a cofinite for $a \in A$. The intersection of all such X_a is still cofinite, so it contains a set B with $|B| = k$. Then $A \times B \subset X$.

Now assume X contains arbitrarily large boxes $A \times B$ as in the claim. By uniform finiteness, there is N so that if $|X_y| \geq N$ for some y, then X_y is cofinite. Since X contains arbitrarily large boxes $A \times B$, infinitely many fibers X_y have size at least N. Thus infinitely many fibers X_y are cofinite, and thus $\dim(X) = 2$. □

We have now proven the theorem for $n = 2$. We now show that the general case $n \geq 2$ reduces to the case $n = 2$. So let $X \subset M^n$ be definable where $n \geq 2$. Recall that the permutations of $\{1, ..., n\}$ are generated by all transpositions of the form $(i, i + 1)$. Thus, we may assume σ has this form for some i. By m-cutting for $m = i - 1$, we may assume $i = 1$, i.e. $\sigma = (1, 2)$ (that is, we know that $\dim(X)$ is determined by dimensions in M^{i-1} and $M^{n-(i-1)}$ separately; since σ doesn't move $1, ..., i - 1$, all dimensions in M^{i-1} are unchanged, so these coordinates can be disregarded). But then by a similar argument, the coordinates $i + 2, ..., n$ don't matter either: that is, by m-cutting with $m = 2$, we reduce to the case $n = 2$. This case was already done, so the theorem is proved. □

Theorem 3.11 implies a very strong consequence, that is the basis for most other basic properties of dimension:

Corollary 3.13 (Function Additivity)**.** *Let $f : X \to Y$ be a definable function, where $X \subset M^m$ and $Y \subset M^n$.*

(1) *For $y \in Y$ and $e \in \{-\infty, ..., m\}$, let $Y_f(e) = \{y \in Y : \dim(f^{-1}(y)) = e\}$. Then*
$$\dim(X) = \max\{\dim(Y_f(e)) + e : e \in \{-\infty, 0, ..., m\}\}.$$
(2) *In particular, if $\dim(f^{-1}(y)) = k$ for all $y \in Y$, then $\dim(X) = \dim(Y) + k$.*

Proof. (1) implies (2) automatically. We show (1). Let $\Gamma \subset X \times Y$ be the graph of f. By m-cutting, $\dim(\Gamma) = \dim(X)$. Now let $\Gamma' \subset Y \times X$ be the graph written backwards, $\{(y,x) : f(x) = y\}$. By Theorem 3.11, $\dim(\Gamma') = \dim(\Gamma) = \dim(X)$. But now by n-cutting, $\dim(\Gamma')$ is the max of all $\dim(Y_f(e)) + e$. □

We infer more basic properties:

Exercise 3.14. Show the following:

(1) $\dim(X \times Y) = \dim(X) + \dim(Y)$ for any definable X and Y.
(2) If $f : X \to Y$ is a definable surjection, then $\dim(X) \geq \dim(Y)$.
(3) If $f : X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$.
(4) More generally, say that the definable sets X and Y are in *definable finite-to-finite correspondence* if there is a definable $Z \subset X \times Y$ so that both projections $Z \to X$ and $Z \to Y$ are surjective with finite fibers. So that in this case we have $\dim(X) = \dim(Y)$.

3.4. **Stationarity.** So far we have developed properties that hold of many dimension theories (such as o-minimal dimension). Our goal now is to discuss a special property of strongly minimal theories (that is really the basis for a significant amount of abstract stability theory).

Note that by strong minimality, M cannot be partitioned into two infinite definable sets. That is, suppose M is the disjoint union of two definable sets X and Y. If X is infinite, then it is cofinite, and thus Y cannot also be infinite. Abstractly, we will say that M is *stationary*:

Definition 3.15. Let $\emptyset \neq X \subset M^n$ be definable. We say X is *stationary* if X cannot be partitioned into two definable subsets of the same dimension as X.

Exercise 3.16. (1) A finite set is stationary if and only if it is a single point.
(2) Each M^n is stationary.
(3) If $\dim(X) = 1$, then X is stationary if and only if X is infinite and every definable subset of X is finite or cofinite. In this case we say X is *strongly minimal* (as a definable set).
(4) Suppose $T = \mathrm{ACF}_p$, and $P(X,Y)$ is a non-constant binary polynomial with coefficients in M. Then $\{(x,y) : P(x,y) = 0\}$ is stationary if and only if $P = Q^k$ for some irreducible Q and some k.

Note that unlike dimension, stationary is *not* always definable in families. See the additional exercises.

Exercise 3.17. Let X be A-definable and stationary of dimension d. Show that the d-dimensional A-definable subsets of X are closed under finite intersections, and thus they determine a complete consistent type over A. We call this type the *generic type* of X over A.

Exercise 3.18. Show that stationarity is preserved under elementary extensions. That is, suppose N is an elementary extension of M, a is a tuple from M, and $\phi(x,a)$ is a formula in n-variables. Show that $\phi(x,a)$ defines a stationary subset of M^n if and only if it defines a stationary subset of N^n.

In general, there are non-stationary definable sets. For example, in ACF, the union of two distinct irreducible plane curves is not stationary. However, this set still decomposes into stationary pieces (namely the two curves). We will prove something similar in general. The decomposition we get is not unique, but is unique up to 'almost equality':

Definition 3.19. Let X and Y be non-empty definable sets.

(1) X is *almost contained in* Y if $\dim(X - Y) < \dim(X)$.
(2) X and Y are *almost equal* if they are almost contained in each other.

Armed with this language, we may say things like 'almost all elements of X satisfy property P' moving forward.

Exercise 3.20. (1) Show that if X and Y are almost equal, then $\dim(X) = \dim(Y)$.
(2) Show that almost equality is an equivalence relation, but almost containment is *not* a partial order.
(3) If X and Y are stationary of dimension d, show that X and Y are almost equal if and only if $\dim(X \cap Y) = d$.

Exercise 3.21. Let $X \subset M^n$ be A-definable of dimension d. Show that there is an A-definable function $X \to M^d$ with all fibers finite. Hint: first show that if $X \subset M^n$ and $\dim(X) \neq n$, there is a finite-to-one A-definable function $X \to M^{n-1}$. Now use induction.

Theorem 3.22 (Stationary Decomposition)**.** *Let $\emptyset \neq X \subset M^n$ be definable of dimension d. Then X is the disjoint union of finitely many stationary definable sets of dimension d. These sets are unique up to almost equality: if $X = X_1 \cup ... \cup X_m = Y_1 \cup ... \cup Y_k$ are two such decompositions, then each X_i is almost equal to some Y_j and vice versa.*

Proof. Let $f : X \to M^d$ be a definable function with all fibers finite. By uniform finiteness, there is N so that each fiber has size at most N. We claim now that X cannot be split into a disjoint union of $N+1$ definable sets of dimension d. Indeed, suppose X is the disjoint union of $X_1, ..., X_{n+1}$. Then each $f(X_i)$ has dimension d. Since M^d is stationary, the intersection $f(X_1) \cap ... \cap f(X_{n+1})$ has dimension d. In particular it is non-empty. If y belongs to this intersection, then $f^{-1}(y)$ has at least $N+1$ points, a contradiction.

Now it follows from the previous paragraph that X admits a *maximal* splitting into a disjoint union $X = X_1 \cup ... \cup X_m$ with each $\dim(X_i) = d$ (i.e. there is such a decomposition which can't be further refined). Then each X_i is stationary.

For uniqueness, suppose $X = X_1 \cup ... \cup X_m = Y_1 \cup ... \cup Y_k$ are two such decompositions. For each i, we have

$$X_i = \bigcup_j (X_i \cup Y_j).$$

Since $\dim(X_i) = d$, some $\dim(X_i \cap Y_j) = d$. Since X_i and Y_j are d-dimensional and stationary, this implies they are almost equal. The same argument works for each Y_j. □

Definition 3.23. Let $\emptyset \neq X \subset M^n$ be definable. The *degree* of X is the number of stationary components of X as above.

In fact, what we have is a unique factorization theorem for almost equality classes:

Exercise 3.24. (1) Show that if X and Y are definable and almost equal, then X is stationary if and only if Y is. Thus the term *stationary class* is well-defined.

(2) Show that if $[X]$ and $[Y]$ are d-dimensional almost equality classes, then the class $[X] \cup [Y] = [X \cup Y]$ is well-defined.
(3) Conclude that every d-dimensional almost equality class decomposes uniquely into a union of finitely many stationary classes.

Recall that a type is *isolated* if it is implied by a single formula – that is, $\mathrm{tp}(a/A)$ is isolated if there is an A-definable set X containing A such that for any other A-defianble Y containing A we have $X \subset Y$. It follows from the above theorem that types in strongly minimal theories are 'almost isolated' in a sense:

Exercise 3.25. Let $a \in M^n$. Then there is an A-definable set X containing a such that for every other A-definable Y containing a, X is almost contained in Y.

3.5. **Recap.** Let us summarize our work to this point:

Theorem 3.26. *[Basic Properties of Dimension] The dimension function on definable sets satisfies the following:*

(1) *Dimension is definable in families.*
(2) $\dim(X) = -\infty$ *if and only if* $X = \emptyset$ *and* $\dim(X) = 0$ *if and only if* X *is non-empty and finite. In particular, finiteness is definable in families.*
(3) $\dim(X_1 \cup ... \cup X_n) = \max\{\dim(X_1), ..., \dim(X_n)\}$. *In particular, if* $X \subset Y$ *then* $\dim(X) \leq \dim(Y)$.
(4) *If* $f : X \to Y$ *is definable then*
$$\dim(X) = \max\{\dim(Y_i) + i\}$$
where $Y_i = \{y : \dim(f^{-1}(y)) = i\}$.
(5) *In particular, if* X *and* Y *are in finite correspondence then* $\dim(X) = \dim(Y)$.
(6) *In particular,* $\dim(X \times Y) = \dim(X) + \dim(Y)$.
(7) *Every* d*-dimensional almost equality class decomposes uniquely into a union of finitely many stationary* d*-dimensional classes.*

4. Interpretable Sets

The main goal of the course is to show that many strongly minimal structures M interpret infinite groups. This will require working with *interpretable sets*: recall these are quotients of definable sets by definable equivalence relations. In particular, we need to expand the dimension theory to interpretable sets.

4.1. M^{eq}**.** To streamline the presentation, we briefly recall the construction of the expansion M^{eq}:

Definition 4.1. We define the *multisorted structure* M^{eq} as follows:

- For each $\emptyset$-definable equivalence relation on M^n, we have a sort M^n/E for the equivalence classes mod E.
- The language consists of the original language on the home sort M (viewed as M/E where E is equality), in addition to the projection map $M^n/F \to M^n/E$ whenever F refines E.
- Definable sets in M^{eq} are also called *interpretable sets* in M. Tuples in M^{eq} are also called *imaginaries* in M.

M^{eq} is no longer 'strongly minimal' in the same sense – strong minimality is specific to 1-sorted theories. Instead, one can say that the 'home' sort M is strongly minimal.

Passing from M to M^{eq} is typically harmless for many reasons, including:

Exercise 4.2.
1. $\mathrm{Th}(M^{eq})$ only depends on $\mathrm{Th}(M)$.
2. If $S = M^n/E$ is a sort in M^{eq}, a set $X \subset S$ is $\emptyset$-definable if and only if its preimage in M^n is $\emptyset$-definable in M.
3. In particular, the $\emptyset$-definable sets in each M^n are unaffected by passing to M^{eq}.

The definition of M^{eq} only allows us to quotient by $\emptyset$-definable equivalence relations (this is so the language of M^{eq} only depends on M). However, it is harmless to add parameters:

Exercise 4.3. Let $X \subset M^n$ be A-definable, and let E be an A-definable equivalence relation on X. Show that the quotient X/E is naturally identified with an A-definable set in M^{eq}.

Hint: Let X and E be definable over a finite tuple $t \in M^m$ from A. Construct a $\emptyset$-definable equivalence relation on $M^m \times M^n$.

4.2. **Dimension in M^{eq}.** As stated above, we need to extend the dimension theory from M to M^{eq}. Typically in model theory this is done by defining dimension more abstractly (e.g. with Morley rank, which is more complicated but makes sense in any sort). In our case, we use a trick specific to the strongly minimal case. Namely, we show that strongly minimal theories have an approximate definable version of the axiom of choice, and that this allows us to reduce the dimension of an interpretable set to the dimension of a definable set:

Lemma 4.4. *Let $X \subset M^n \times T$ be definable in M^{eq}, and assume that for all $t \in T$ the fiber X_t is non-empty. Then there is a definable subset $Y \subset X$ so that for all $t \in T$ the fiber Y_t is non-empty and finite (in general Y will need to be defined over extra parameters).*

Proof. By induction on n. First assume $n = 1$. By uniform finiteness, there is N so that for all t either $|X_t| < N$ or $|M - X_t| < N$. Fix any N distinct points, $a_1, ..., a_N \in M$. Now given t, we have two cases:

- If X_t is finite, let $Y_t = X_t$.
- If X_t is cofinite, let $Y_t = X_t \cap \{a_1, ..., a_N\}$.

In the second case, Y_t is non-empty because $|M - X_t| < N$.

Now for the inductive step, assume $n \geq 2$. For each X_t, we use the inductive hypothesis to choose a finite subset Z_t of images of X_t in M^{n-1}; then for each $z \in Z_t$, we use the base case to extend Z_t to finitely many points of X_t. All in all, this gives a finite subset of X_t, defined uniformly in t.

More precisely: consider any projection $\pi(X) \subset M^{n-1} \times T$. By the inductive hypothesis, there is a definable $Z \subset \pi(X)$ so that each Z_t is non-empty and finite. Let X_Z be the preimage of Z in X, and consider $X_Z \subset M \times Z$ as a family indexed by Z. By the base case, there is a definable $Y \subset X_Z$ so that all fibers Y_z are non-empty and finite. This implies that all fibers Y_t are non-empty and finite (each is a union of finitely many fibers above Z). □

Corollary 4.5. *[Weak Elimination of Imaginaries] Every interpretable set is a finite-to-one image of a definable set. That is, let $Y \subset M^n/E$ be definable in M^{eq}. Then there is a definable $X \subset M^n$ such that the map $X \to M^n/E$ is finite-to-one and has image precisely Y.*

Proof. Apply the previous lemma to definably choose a finite subset of each equivalence class in Y. ☐

Definition 4.6. Let Y be definable in M^{eq}. We set $\dim(Y) := \dim(X)$ for any definable $X \subset M^n$ admitting a definable finite-to-one surjective function $f : X \to Y$.

Exercise 4.7. Check that this definition of dimension is well-defined.

Exercise 4.8. Show that all items in Theorem 3.26 remain true in M^{eq}. Hint: many of (1)-(7) are interdependent, so you don't have to prove them all directly. For those you do have to prove, try to use weak elimination of imaginaries to reduce to a property of dimension in definable sets. This should work everywhere *except* definability of dimension (because there you need to control parameters, and weak elimination of imaginaries requires uncontrollable extra parameters). Instead, prove definability of dimension using function additivity.

5. Dimension of Types

One of the themes of our work to this point is that dimension is well-understood for definable sets and functions after breaking them into finitely many pieces and ignoring some small error. Our next goal is to develop the dimension of a *complete type*: the idea is that at the level of complete types, we won't have to break into cases, because the type will have already concentrated on one of the cases.

Precisely, we will define the dimension $\dim(a/A)$ where $a \in M^{eq}$ is a tuple and $A \subset M^{eq}$ is a parameter set – but $\dim(a/A)$ will only depend on $\mathrm{tp}(a/A)$, so we're really defining $\dim(\mathrm{tp}(a/A))$. The idea is that '$\dim(a/A) = d$' should mean 'X is generic in an A-definable set of dimension d' – so we are replacing definable sets with their generic types.

Unfortunately, expressions like $\dim(a/A)$ will only really capture everything in a big enough model (where generic types have actual realizations). So we first review saturated models.

5.1. Saturation.

Definition 5.1. An infinite structure M is *saturated* if for every $A \subset M$ with $|A| < |M|$, every consistent type over A has a realization in M.

Definition 5.2. Suppose M is saturated and uncountable. If $A \subset M$ and $|A| < |M|$, we call A a *small set*.

So saturation gives that types over small sets can be realized in M.

Exercise 5.3. Suppose M is saturated. Then so is M^{eq}.

A key fact about saturated models is uniqueness:

Exercise 5.4. Let M and N be saturated models of the same complete theory. If $|M| = |N|$ then M and N are isomorphic.

In the strongly minimal case, we have:

Lemma 5.5. *Suppose M is strongly minimal, uncountable, and strictly bigger than its language. Then M is saturated. In particular, strongly minimal theories in countable languages are uncountably categorical.*

Proof. It is enough to show that every 1-type over a small set is realized: then to realize an n-type, realize one coordinate at a time, at each stage adding all previous choices to the parameter set.

Now Let $A \subset M$ be small, and let $p(x)$ be a consistent 1-type over A. We consider two cases:

- First suppose there is a formula $\phi(x, a)$ in p with only finitely many solutions. Choose such a formula $\phi(x, a)$ with the smallest number of solutions. We claim that every solution of $\phi(x, a)$ realizes p. Indeed, for any other formula $\psi(x, b) \in p$, the minimality of $|\phi(M, a)|$ implies that $\phi(x, b) \wedge \psi(x, b)$ cannot define a smaller set than $\phi(x, a)$ – that is, $\phi(x, a)$ implies $\psi(x, b)$.
- Now suppose no such formula exists. By strong minimality, every formula in p defines a cofinite subset of M. Since M is uncountable and bigger than both A and the language, the intersection of all of these cofinite sets is non-empty.

Now over a countable language, we have shown that every uncountable model is saturated, and uncountable categoricity follows by the uniqueness of saturated models. □

From now on, we work in a saturated uncountable strongly minimal structure M. This is a harmless restriction: the theorems we prove in M will be elementary, and thus will pass down to all models.

5.2. **Dimension and Generic Points.**

Definition 5.6. Let $a \in M^{eq}$, and let $A \subset M^{eq}$ be small. We define $\dim(a/A)$ to be the smallest dimension of any A-definable set containing a.

The following are immediate:

Exercise 5.7. (1) $\dim(a/A)$ only depends on $\mathrm{tp}(a/A)$.
(2) If $a \in M^n$ then $\dim(a/A) \leq n$.
(3) If σ is a permutation of $a_1, ..., a_n$ then $\dim(\sigma(a_1)...\sigma(a_n)/A) = \dim(a_1...a_n/A)$.
(4) If $A \subset B$ then $\dim(a/B) \leq \dim(a/A)$.

Dually, we have:

Lemma 5.8. *Let X be A-definable in M^{eq}, where A is small. Then*

$$\dim(X) = \max\{\dim(a/A) : a \in X\}.$$

Proof. $\leq$: By definition of dimension. $\geq$: Let $\{Y_\alpha : a < \kappa\}$ enumerate all A-definable subsets of X which have smaller dimension than X. Let $p(x)$ be the partial type over A saying that $x \in X$, and $x \notin Y_\alpha$ for each α. Then p is finitely satisfiable: any finitely many Y_α union to a set of dimension less than $\dim(X)$, so some element of X does not belong to the union. By saturation, p is satisfiable in X. Any realization of p has $\dim(a/A) = \dim(X)$. □

Definition 5.9. Let X be A-definable. If $a \in X$ and $\dim(a/A) = \dim(X)$, we say that a is *generic in X over A*.

So the above lemma says that definable sets always have generic points (over any small set defining them).

Example 5.10. Let $M = (\mathbb{C}, +, \times)$, and let $X \subset K^2$ be defined by $y = x^2$. Then X is $\emptyset$-definable, and is also π-definable. Then:

- (π, π^2) is generic in X over $\emptyset$.
- (π, π^2) is not generic in M^2 over $\emptyset$.
- (π, π^2) is not generic in X over π.

Exercise 5.11. Let X and Y be A-definable.

(1) Show that X is almost contained in Y if and only if every generic point of X over A belongs to Y.
(2) Show that X and Y are almost equal if and only if they have the same generic points over A.

Exercise 5.11 shows that if X is stationary and A-definable, the generic type of X over A only depends on the almost equality class $[X]$ (meaning that if $Y \in [X]$ is also A-definable, Y has the same generic type over A). Thus, if $[X]$ is a stationary almost equality class with at least one A-definable member, there is a well-defined 'generic type of $[X]$ over A'. In this way, we can define *stationary types* (generic types of stationary classes), and we have a well-defined *almost equality* notion for stationary types (e.g. if $[X]$ contains an A-definable member and a B-definable member, the generic types of $[X]$ over A and B are almost equal). This is roughly how model theorists generalize stationary decompositions to arbitrary stable theories (where dimension is not defined): we give an abstract definition of stationary types, and then consider almost equality classes of such types (using instead the term 'parallelism').

5.3. **Algebraic Closure.** A key special case of dimension is the following:

Definition 5.12. Let $a \in M^{eq}$ and $A \subset M^{eq}$.

(1) We say that a is *algebraic over* A if $\dim(a/A) = 0$ – equivalently, if a belongs to some finite A-definable set.
(2) The *algebraic closure of* A, denoted $\text{acl}(A)$, is the set of all elements of M^{eq} which are algebraic over A.
(3) If b is another tuple, we say that a and b are *interalgebraic over* A if $a \in \text{acl}(Ab)$ and $b \in \text{acl}(Aa)$.

The notation $\text{acl}(A)$ is a bit confusing, because it is really two notions. If $A \subset M^{eq}$, we typically mean the version above. However, suppose $A \subset M$. Then it is natural to consider $\text{acl}(A) \cap M$ (the set of singletons algebraic over A), as this gives a closure operator on subsets of M. It is most convenient to ignore this difference and revisit it whenever things get confusing. Most of the time, though, we will try to write $\text{acl}(A) \cap M$ when we mean the second version.

Exercise 5.13. (Model-Theoretic Galois Theory)

(1) Show that if A is small, $a, b \in M^{eq}$, and $\text{tp}(a/A) = \text{tp}(b/A)$, there is an automorphism of M fixing A point-wise and sending a to b. Hint: use the uniqueness of saturated models.
(2) Conclude that for small A, the assertion that $a \in \text{acl}(A)$ is equivalent to the assertion that a has finite orbit under the action of $\text{Aut}(M/A)$ (automorphisms fixing A point-wise).

(3) Similarly, say that a is *definable over* A (denoted $a \in \mathrm{dcl}(A)$) if the set $\{a\}$ is A-definable. Show that for small A, $a \in \mathrm{dcl}(A)$ is equivalent to the assertion that a is fixed by all of $\mathrm{Aut}(M/A)$.
(4) Give examples to show that (2) and (3) fail if A is not small (i.e. if $|A| = |M|$).
(5) (Hard) On the other hand, show that (1) *does* hold even if A is not small (this is a unique feature of strongly minimal theories).

Exercise 5.14. (1) Suppose $M \models \mathrm{ACF}_p$ and $A \subset M$. Show that $\mathrm{acl}(A) \cap M$ is the (field-theoretic) algebraic closure of the field generated by A. (It may help to use the previous exercise).
(2) Find similar characterizations of algebraic closure (inside the sort M only) in the theories of the pure set, the integers with successor, and vector spaces.

Algebraic closure has the following basic properties:

Lemma 5.15. (1) $A \subset \mathrm{acl}(A)$ *for all* A.
(2) *If* $A \subset B$ *then* $\mathrm{acl}(A) \subset \mathrm{acl}(B)$.
(3) *If* $a \in \mathrm{acl}(A)$ *then there is a finite* $B \subset A$ *with* $a \in \mathrm{acl}(B)$.
(4) $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

Proof. (1), (2), and (3) are clear ((3) is because formulas are finite). We show (4). Let $a \in \mathrm{acl}(\mathrm{acl}(A))$. By (3), there is a finite $B = (b_1, ..., b_n) \subset \mathrm{acl}(A)$ with $a \in \mathrm{acl}(B)$. Let $\phi(a, b_1, ..., b_n)$ be a formula with finitely many solutions – say m – and for each i let $\psi_i(b_i, A)$ be a formula with finitely many solutions – say k_i. Then the formula 'there are $y_1, ..., y_n$ so that $M \models \psi_i(y_i, A)$ for each i, $M \models \phi(x, y_1, ..., y_n)$, and $|\phi(M, y_1, ..., y_n)| \leq m$' is true of a and has at most $m \cdot k_1 \cdot ... \cdot k_n$ solutions. $\square$

Note that $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$ is an abstract analog of the fact that the sum or product of two algebraic numbers is again algebraic (and the proof is similar).

5.4. **Additivity and Other Properties.** We want to develop some basic properties of dimension for tuples. The most important is called 'additivity'. This is the analog of 'function additivity' for definable sets. The tuple-version is much cleaner to state.

Theorem 5.16 (Additivity)**.** *Let* $a, b \in M^{eq}$ *and* $A \subset M^{eq}$. *Then*

$$\dim(ab/A) = \dim(a/Ab) + \dim(b/A).$$

Proof. Let $a \in S, b \in T$ where S, T are sorts. Fix definable sets witnessing all three dimensions. More precisely, let $X \subset S \times T$ be A-definable with $\dim(X) = \dim(ab/A)$; let $Y \subset S \times T$ be A-definable with $\dim(Y_b) = \dim(a/Ab)$; and let $Z \subset T$ be A-definable with $\dim(Z) = \dim(b/A)$. Now let $W \subset S \times T$ be the set of (x, y) with $(x, y) \in X$, $(x, y) \in Y$, and $y \in Z$. Then $W_b \subset Y_b$ is Ab-definable and contains a, so $\dim(W_b) = \dim(a/Ab) := d$, say. Let Z' be the set of $y \in Z$ with $\dim(W_y) = d$, and let W' be the set of $(x, y) \in W$ with $y \in Z'$. Then Z' (and thus W') are A-definable still.

Since all original dimensions were minimal, we have:

- $\dim(W') = \dim(ab/A)$.
- $\dim(W'_b) = \dim(W_b) = \dim(a/Ab) = d$.

- $\dim(Z') = \dim(b/A)$.

Now the projection $W' \to Z'$ is surjective with all fibers of dimension d, so by function additivity we have $\dim(W') = d + \dim(Z')$. Equivalently, $\dim(ab/A) = \dim(a/Ab) + \dim(b/A)$. □

Many other properties fall out of additivity:

Exercise 5.17. Use additivity to prove the following:

(1) $\dim(ab/A) \leq \dim(a/A) + \dim(b/A)$.
(2) If $b \in \operatorname{acl}(A)$ then $\dim(a/Ab) = \dim(a/A)$.
(3) If a and b are interalgebraic over A then $\dim(a/A) = \dim(b/A)$.

Many treatments of strongly minimal theories develop dimension using *pregeometries*. The idea is that for $a \in M^n$, we set $\dim(a/A)$ to be the length of a 'basis' of a over A: this is a minimal subtuple b of a with $a \in \operatorname{acl}(Ab)$. Additivity is still a crucial fact when defining dimension using bases: it first appears as the statement that any two bases have the same length, so dimension is well-defined. This is first proved by iterated applications of what is known as the 'exchange lemma'. (In more general settings, exchange and additivity are known to be equivalent, and are often used interchangeably). Let us now state exchange and verify it using additivity:

Lemma 5.18 (Exchange Lemma)**.** *Let $a, b \in M$, and $A \subset M^{eq}$. If $b \in \operatorname{acl}(Aa) - \operatorname{acl}(A)$, then $a \in \operatorname{acl}(Ab)$.*

Proof. The fact that $b \in \operatorname{acl}(Aa)$ implies $\dim(ab/A) \leq 1$ (by additivity). Thus, by additivity again, either $\dim(b/A) = 0$ or $\dim(a/Ab) = 0$. We are given $b \notin \operatorname{acl}(A)$, so the only option is $\dim(a/Ab) = 0$, i.e. $a \in \operatorname{acl}(Ab)$. □

Now let us use additivity to characterize dimension using bases.

Lemma 5.19 (Basis Characterization)**.** *Let $a \in M^n$ and $A \subset M^{eq}$. Then $\dim(a/A)$ is the length of any basis of a over A.*

Proof. Let $d = \dim(a/A)$, and let b be a basis. Then a and b are interalgebraic over A, so $\dim(b/A) = d$, and thus the length of b is at least d. Now write $b = (b_1, ..., b_m)$. So we have $m \geq d$, and we want that $m = d$.

Now use additivity to write

$$d = \dim(b/A) = \sum_{i=1}^{m} \dim(b_i/Ab_1...b_{i-1}).$$

Each term in the sum is 0 or 1. If $m > d$, there must be a 0. So there is i with $b_i \in \operatorname{acl}(Ab_1...b_{i-1})$. Then $b \in \operatorname{acl}(Ab^*)$, where b^* is b with b_i removed. In particular, $a \in \operatorname{acl}(Ab^*)$, contradicting the minimality of b. □

5.5. **Independence.** The dimension theory for tuples allows for a good notion of *independence*:

Lemma 5.20. *Let $a, b \in M^{eq}$ and $A \subset M^{eq}$. The following are equivalent:*

(1) $\dim(a/Ab) = \dim(a/A)$.
(2) $\dim(b/Aa) = \dim(b/A)$.
(3) $\dim(ab/A) = \dim(a/A) + \dim(b/A)$.

Proof. By additivity,

$$\dim(ab/A) = \dim(a/Ab) + \dim(b/A) \leq \dim(a/A) + \dim(b/A),$$

with equality precisely if $\dim(a/Ab) = \dim(a/A)$. Similarly with a and b reversed. □

Definition 5.21. We say that a and b are *independent over* A *if the above conditions hold.*

So independence is automatically symmetric. We will not need many other properties of independence, but we still list a couple:

Exercise 5.22.
(1) If $b \in \operatorname{acl}(A)$ then every a is independent from b over A.
(2) a is independent from itself over A if and only if $a \in \operatorname{acl}(A)$.
(3) If a is independent from bc over A if and only if it is independent from both b over A and c over Ab.

More important and less trivial is 'existence of independent extensions':

Lemma 5.23. *Let $a \in M^{eq}$ and $A \subset B$ small sets. Then there is $a' \models \operatorname{tp}(a/A)$ with a' and B independent over A.*

Proof. Let X be A-definable and containing a so that every definable set from $\operatorname{tp}(a/A)$ almost contains X. It follows that every A-generic point of X also belongs to every other definable set in $\operatorname{tp}(a/A)$. That is, every A-generic point of X realizes $\operatorname{tp}(a/A)$. Now take a' to be a generic point of X over B. □

6. Canonical Bases

One of the benefits of M^{eq} (throughout model theory) is that it allows us to *code* definable sets as tuples in the structure, thereby treating them as actual elements (see below). In the strongly minimal case, we need to do something similar but more abstract. Recall that our main goal is to give assumptions under which M interprets an infinite group. In the end, the elements of the group we build will be almost equality classes of permutations. Thus, roughly speaking, we need to 'code' almost equality classes. The tool for doing this is known as *canonical bases.* Importantly, canonical bases will give us a very clear meaning of 'how big' a family of almost equality classes is – and this will form the basis of the famous 'Zilber trichotomy'. Note that canonical bases are one of the more sophisticated ideas in abstract stability. We only give the strongly minimal version of them, and it is already quite non-trivial.

6.1. **Warm up: Canonical Parameters.**

We first preview canonical bases by coding ordinary definable sets. The term here is *canonical parameters*. Suppose X is definable by $\phi(x, a)$ for some tuple a. In general, there could be many other tuples b so that $\phi(x, b)$ also defines X: there is not a 'canonical' one. Over M^{eq}, this can be fixed:

Definition 6.1. Let X be definable in M^{eq}. A *canonical parameter* of X is a tuple c so that (1) X is defined by a formula $\phi(x, c)$, and (2) for all $c' \neq c$, $\phi(x, c')$ does not define X.

Exercise 6.2. Suppose c and d are both canonical parameters for X. Show that c and d are interdefinable (each can be defined using the other).

Lemma 6.3. *Every definable set in M^{eq} has a canonical parameter in M^{eq}.*

Proof. Let X be defined by $\phi(x, a)$ where $a \in M^{eq}$. Note that we may assume $a \in M^n$ for some n (if not, let $a \in M^n/E$, and let $b \in M^n$ project to a; then replace a with b).

Now let E be the $\emptyset$-definable equivalence relation on M^n where yEz if $\phi(x, y)$ and $\phi(x, z)$ are equivalent, and let $c \in M^n/E$ be the equivalence class of a. Then c is a canonical parameter. Precisely, X is defined by the formula $\psi(x, c)$, where $\psi(x, z)$ says that for all $y \in M^n$, if y projects to z in M^n/E, then $\phi(x, y)$ holds. □

The proof even shows more: every definable set occurs in a $\emptyset$-definable family of pairwise distinct sets:

Corollary 6.4. *Let X be definable in M^{eq}, where $X \subset S$ for some sort S. Then there is a $\emptyset$-definable family $Y \subset S \times Z$ so that (1) $X = Y_z$ for some z, and (2) for all $z_1, z_2 \in Z$, if $z_1 \neq z_2$ then $Y_{z_1} \neq Y_{z_2}$.*

6.2. **Canonical Bases.** As stated above, canonical parameters allow us to code definable sets as elements of a structure. For example, suppose $(M, +, ...)$ is a group, and $X \subset M$ is definable. Set $X + X = \{a + b : a, b \in X\}$. One might want to say that $X + X$ is 'defined uniformly from X' in some sense. One way to do it is to say that if c and d are canonical parameters for X an $X + X$, then d is definable over c.

We want to code almost equality classes, so that we can talk about them definably in a similar way. This is similar but harder.

Definition 6.5. Let $[X]$ be an almost equality class of stationary definable sets. A *canonical base* of $[X]$ is a tuple c such that:

(1) Some member $Y \in [X]$ is definable over c by a formula $\phi(x, c)$.
(2) For any $c' \neq c$, the set defined by $\phi(x, c')$ is not a member of $[X]$.

Exercise 6.6. As with canonical parameters, show that any two canonical bases of $[X]$ are interdefinable.

Because of this, we often treat the canonical base as a single object and call it $\mathrm{Cb}([X])$ (but this is abuse of notation). We may also write $\mathrm{Cb}(X)$ instead of $\mathrm{Cb}([X])$, for simplicity.

Theorem 6.7 (Canonical Bases Exist)**.** *Let $[X]$ be an almost equality class of stationary definable sets of dimension $d \geq 0$ in some sort S in M^{eq}. Then $[X]$ has a canonical base in M^{eq}.*

Proof. The set $X \in [X]$ is definable over some tuple t, say $X = Y_t$ where $Y \subset S \times T$ is $\emptyset$-definable. We may assume $T \subset M^n$ for some n; otherwise replace it with its preimage in M^n.

Let $U = M^n/E$ where uEv if Y_u and Y_v are almost equal (where by convention we say that $\emptyset$ is almost equal to itself). By definability of dimension, E is $\emptyset$-definable, so U is a sort in M^{eq}. Let $c \in U$ be the equivalence class of t. We claim that c is a canonical base of $[X]$. To show this, our main task is to find a c-definable set in $[X]$.

Let $C \subset M^n$ be the set of elements of the class c (i.e. c viewed as a definable set). Then let Z be the set of $x \in S$ such that $x \in Y_u$ for almost all $u \in C$. We claim that $Z \in [X]$, i.e. that X and Z are almost equal:

Claim 6.8. (1) *Every generic element of X over tc belongs to Z. Thus X is almost contained in Z, and* $\dim(Z) \geq d$.
(2) *Every generic element of Z over ct belongs to X. Thus Z is almost contained in X, and so X and Z are almost equal.*

Proof. (1) Suppose $a \in X$ is generic over ct. We want to show that $a \in Y_u$ for almost all $u \in C$. It suffices to show that if $u \in C$ is generic over cta, then $a \in Y_u$. But in this case, u is indepdnent from a over ct (it is generic in U over both ct and cta). So we also have

$$\dim(a/ctu) = \dim(a/ct) = \dim(X).$$

Since X is almost equal to Y_u, all generics of X over tu belong to Y_u – thus $a \in Y_u$ as desired.
(2) Suppose $a \in Z$ is generic over ct. As above, let $u \in C$ be generic over act, so that a and c are independent over ct. Since $a \in Z$, we have $a \in Y_u$, and thus

$$d = \dim(Z) = \dim(a/ct) = \dim(a/ctu)$$

$$\leq \dim(a/u) \leq \dim(Y_u) = d.$$

Everything above must be an equality, so $\dim(a/ctu) = d$, and thus a is generic in Y_u over ctu. Since Y_u is almost equal to X, all generics of Y_u over ctu belong to X. Thus $a \in X$.

□

By the claim, $Z \in [X]$. Let Z be defined by $\phi(x, c)$. Then M^{eq} knows that $\phi(x, c)$ is almost equal to Y_v for every v projecting to c. Let $\psi(x, w)$ be '$\phi(x, w)$ and $\phi(S, w)$ is almost equal to Y_v for every v projecting to w'. Then $\psi(x, c)$ defines Z – while if $c' \neq c$, then by definition $\psi(x, c')$ belongs to a different almost equality class. This means c is a canonical base. □

Canonical bases let us represent stationary sets as generic members of *faithful families*:

Definition 6.9. Let $Y \subset S \times T$ be an M^{eq}-definable family of subsets of a sort S, and assume $\dim(Y_t) = d \geq 0$ for all t. We say X is *faithful* if $\dim(Y_t \cap Y_{t'}) < d$ whenever $t \neq t'$.

Theorem 6.10 (Faithful Families Exist)**.** *Let X be a stationary definable set of dimension d in the sort S in M^{eq}. Let c be a canonical base of $[X]$. Then there is a faithful $\emptyset$-definable family $Y \subset S \times T$ of d-dimensional subsets of S, such that c is generic in T and $Y_c \in [X]$.*

Exercise 6.11. Prove Theorem 6.10 using Theorem 6.7 and compactness.

Theorem 6.10 gives a convenient tool for measuring the 'true size' of a family of definable sets (by 'true size', we mean we view sets up to almost equality). Namely, the following are now equivalent for a stationary definable set X:

- X has a canonical base of dimension k.
- X is almost equal to a generic member of a k-dimensional faithful definable family.

Since any two canonical bases are interdefinable, the notation $\dim(\mathrm{Cb}(X))$ is well-defined (it refers to $\dim(c)$ for any canonical base c). So we have a shorthand for a fairly intricate notion: rather than say 'we have a k-dimensional faithful family of d-dimensional sets containing a generic member almost equal to X', we can now just say '$\dim(\mathrm{Cb}(X)) = k$'. One should always retain the geometric intuition: the dimension of a canonical base is the size of a faithful family and vice versa.

Our definition of canonical bases only applies to stationary classes. However, with a little work, we can define the notation '$\mathrm{Cb}(a/A)$' for any a and A:

Exercise 6.12. Let X be A-definable of dimension $d \geq 0$.

(1) If X is stationary, then $\mathrm{Cb}([X])$ is definable over A.
(2) In general, if $[Y]$ is any stationary component of $[X]$, then $\mathrm{Cb}([Y]) \in \mathrm{acl}(A)$.
(3) X can be decomposed over $\mathrm{acl}(A)$: that is, there are finitely many disjoint stationary d-dimensional $\mathrm{acl}(A)$-definable sets whose union is X.

Definition 6.13. Let $a \in M^{eq}$ and $A \subset M^{eq}$.

(1) Set $\mathrm{Loc}(a/A)$ (the locus of a over A) to be the class $[X]$, where X is any $\mathrm{acl}(A)$-definable d-dimensional stationary set containing A.
(2) Set $\mathrm{Cb}(a/A)$ to be the canonical base of $\mathrm{Loc}(a/A)$.

Exercise 6.14. (1) Prove that $\mathrm{Loc}(a/A)$ (and thus $\mathrm{Cb}(a/A)$) are well-defined.
(2) Prove that $\dim(a/\mathrm{Cb}(a/A)) = \dim(a/A)$ for all a and A.

6.3. **Families of Plane Curves.** We now introduce Zilber's Trichotomy, one of the most influential and fruitful ideas in modern model theory. In his early investigation of strongly minimal structures (particularly in the totally categorical case), Zilber isolated two crucial dividing lines. The key is to look at *families of plane curves.*

Definition 6.15. (1) A *plane curve* is a one-dimensional definable set $X \subset M^2$.
(2) A plane curve $X \subset M^2$ is *non-trivial* if both projections $X \to M$ are finite-to-one.

Exercise 6.16. A stationary plane curve is *trivial* (= not non-trivial) if and only if it is almost equal to a horizontal or vertical line.

A stationary non-trivial plane curve is a generalization of a definable bijection $M \to M$. It is instead a definable finite-to-finite correspondence between cofinite subsets of M. But we still want the intuition of a family of such curves as 'acting on M'.

Consider all values $\dim(\mathrm{Cb}(X))$ where X is a stationary non-trivial plane curve. These values encode the possible sizes of faithful families of plane curves in M. For example:

Exercise 6.17. (1) In a pure set, every stationary non-trivial plane curve X is almost equal to the diagonal $y = x$, and thus $\dim(\mathrm{Cb}(X)) = 0$.
(2) In an F-vector space, every stationary non-trivial plane curve X is almost equal to the graph of an affine linear map $y = cx + v$ (where $c \in F$ and v is a fixed vector). In this case, v is a canonical base, so $\dim(\mathrm{Cb}(X)) \leq 1$.
(3) In ACF, $\dim(\mathrm{Cb}(X))$ can be arbitrarily large (consider the graph of a generic polynomial function of degree d).

Zilber (and Hrushovski) showed that these cases are exhaustive:

Fact 6.18 (Weak Trichotomy)**.** *Let S be the set of all values* $\dim(\mathrm{Cb}(X))$ *where X is a stationary non-trivial plane curve. Then either $S = \{0\}$, or $S = \{0, 1\}$, or S is unbounded; and if* $\mathrm{Th}(M)$ *is totally categorical, only the first two cases can occur.*

We will not prove this fact – it is extremely complicated. We will develop many of the ideas, though.

Interestingly, the proof of Fact 6.18 is another case of algebraic objects arising out of nowhere. Here is the idea. Suppose S is bounded. We get a family of plane curves of maximal dimension, say $\mathcal{F}$. Consider the 'composite family' $\mathcal{F} \circ \mathcal{F}$ of all 'compositions' of curves from $\mathcal{F}$ (this has to be carefully defined; if the curves are functions, it is just the usual composition of functions). The composite family looks bigger, but can't be: there have to be a lot of redundant compositions. This means that $\mathcal{F}$ looks a bit like a group acting on M. Using ideas we will develop, one can build an actual transitive group action differing only mildly from $\mathcal{F}$. Then one uses sophisticated group theory to classify the group. If $\mathcal{F}$ had dimension at least 2, then every possibility for the group ends up being an algebraic group over a model of ACF. In particular, out of nowhere, M ends up being closely related to a model (or rather expansion of a model) of ACF. In this case, as above, one can build large families of plane curves, and this ultimately contradicts the assumption that S is bounded.

The second clause (that totally categorical theories are locally modular) was proven in a very complicated way by Zilber, and later given a simplified proof by Hrushovski (though still a bit involved). Again, it is highly connected to algebra and geometry: the idea is to develop analogs (in M) of statements from the intersection theory of varieties (precisely *Bezout's Theorem*). Ultimately, the intersection theory inside M is 'too good' in a sense, and if one has access to large families of plane curves, one can contradict this 'too good Bezout's theorem' by building certain 'atypical' configurations of intersections. We sketch this argument in a separate sequence of exercises (at the end of the exercise document).

6.4. **The Zilber Trichotomy.** The three cases of Fact 6.18 are known as *Zilber's Trichotomy*:

Definition 6.19. Let S be as above.

(1) If $S = \{0\}$, M is *trivial*.
(2) If S is bounded, M is *locally modular* (thus $S = \{0, 1\}$ is called *non-trivial locally modular*).
(3) if S is unbounded, M is *not locally modular.*

('Locally modular' came from a previous equivalent definition that isn't used as much anymore.)

The trichotomy gives a very strong division of all strongly minimal structures into three levels: structures of different levels have very different behaviors. Accordingly, the levels of the trichotomy are hard to escape, in the sense that small manipulations to a structure don't change its level. For example:

Lemma 6.20. *Let A be a (small) set of parameters, and let M_A be M in the language naming elements of A as constants. Then M_A is trivial (resp. locally modular) if and only if M is.*

Proof. Adding parameters does not affect the collection of definable sets, so also does not affect the collection of dimensions of faithful families of plane curves (even though it may change the dimensions of *specific* canonical bases, the maximum values don't change). □

In fact, more can be said. Suppose X is any A-definable strongly minimal set in M^{eq}. We can view X as a strongly minimal structure in its own right, naming constants for A and then taking the induced structure on X. In this case, X will belong to the same level as M. We leave this as an additional exercise.

In general, the trichotomy has been hugely influential in the study of stable theories (and beyond). Often, one gains a strong understanding of a well-behaved mathematical structure by decomposing it into strongly minimal 'pieces' and studying the trichotomy levels of those pieces. This idea is at the heart of famous applications of model theory in geometry by Hrushovski (e.g. his work on the Mordell-Lang conjecture, [3]).

In the rest of the course, we will carefully study the lower two (= locally modular) levels of the trichotomy. Here the main results are accessible and give a clear and complete picture. The final level (non-locally modular) is much more complicated, and understanding it is still a major research area. The general theme of the non-locally modular case is to show that 'in natural settings' (whatever that means), one can only have models of ACF (and other things constructed from them). To clarify, there *are* exotic non-locally modular strongly minimal structures having nothing to do with ACF – they just don't seem to occur in the wild.

6.5. **Equivalences.** Let us begin with some equivalent characterizations. First, the trivial case can be characterized as follows:

Fact 6.21. *M is trivial if and only if for all $A \subset M$ and $b \in M$, if $b \in \operatorname{acl}(A)$ then $b \in \operatorname{acl}(a)$ for some $a \in A$.*

The proof of Fact 6.21 is not too crazy, but we don't have time. See the additional exercises.

Fact 6.21 roughly says that there can't be any interesting n-ary relations for $n \geq 3$. In particular:

Exercise 6.22. Show that if $(M, \cdot, ...)$ is an expansion of a group then M is not trivial.

Now we give stronger characterizations of local modularity. It might seem arbitrary that local modularity is defined only in the plane (i.e. why don't we care about the behavior of canonical bases in M^n?). In fact, the plane is enough to control everything:

Fact 6.23. *Suppose M is locally modular.*

(1) *Let X be a stationary d-dimensional definable subset of M^n. Then*
$$\dim(\operatorname{Cb}([X])) \leq n - d.$$
(2) *More generally, let Y be A-definable in M^{eq} of dimension n, and let X be a stationary d-dimensional definable subset of Y. Then* $\dim(\operatorname{Cb}([X])/A) \leq n - d$.
(3) *More generally, for all a and A we have* $\operatorname{Cb}(a/A) \in \operatorname{acl}(a)$.

We sketch a proof of this fact in the additional exercises.

In the above, (3) is a neat way to package (1) and (2) into a clean statement. The idea is the following: in the situation of e.g. (1), let c be a canonical base of X, let $Y \in [X]$ be c-definable, and let $a \in Y$ be generic over c. Then

$$\dim(ac) = \dim(a/c) + \dim(c) = d + \dim(c),$$

while also

$$\dim(ac) = \dim(c/a) + \dim(a) \leq \dim(c/a) + n.$$

So the assertion '$c \in \mathrm{acl}(a)$' really gives $d + \dim(c) \leq n$, which is (1).

In more general settings, property (3) above is known as '1-based' (as in, 1 point (almost) determines the canonical base).

6.6. **The Trivial Case.** The general picture of totally categorical theories is coming into focus: suppose $\mathrm{Th}(N)$ is totally categorical. Then it is prime over a totally categorical strongly minimal theory, say our $\mathrm{Th}(M)$. By Zilber's trichotomy results, M is either trivial or non-trivial and locally modular. We will show that:

- If M is trivial, then after slight modifications it becomes a pure set.
- If M is non-trivial and locally modular, then after slight modifications it becomes a vector space.

Here we do the first of these; the proof is a simplified preview of the second one.

Theorem 6.24. *Assume M is totally categorical and trivial. Then there is a $\emptyset$-definable equivalence relation E on M with all classes finite, such that M/E is a pure set: every definable subset of $(M/E)^n$ is definable in the language of equality.*

Proof. We let E be the relation 'x and y are interalgebraic'. This is usually not definable – but it is in this case because of $\aleph_0$-categoricity (there are only finitely many formulas that could witness the interalgebraicity). In particular, it follows that all classes are finite. For ease of notation, let us just replace M with M/E. So from now on, we assume that no two distinct elements of M are interalgebraic.

(Something subtle happened here: we are assuming that M/E is still strongly minimal, totally categorical, and trivial; this can be checked and we omit it).

Now for each n, we prove that every definable set $X \subset M^n$ is definable from equality. We work by induction on $d := \dim(X)$. If $d = 0$ then X is finite, and everything is clear.

Now assume $d \geq 1$. We may assume X is stationary: if not, break X into components and handle each separately. So assume X is stationary, and let $A \subset M$ be a small set such that X is A-definable. Let $a = (a_1, ..., a_n)$ be generic in X over A, and let b be a basis for a over A. Without loss of generality $b = (a_1, ..., a_d)$. For each $i > d$, $a_i \in \mathrm{acl}(Ab)$, and thus $a_i \in \mathrm{acl}(c)$ for some $c \in Ab$. Then one of two things happens:

- $a_i \in \mathrm{acl}(\emptyset)$.
- $a_i \notin \mathrm{acl}(\emptyset)$. In this case, a_i and c are interalegraic, and so $a_i = c$. Either $a_i \in A$ or $a_i \in \{a_1, ..., a_d\}$.

In either case, either a_i agrees with a basis coordinate, or it is in $\mathrm{acl}(A)$. Then we get a corresponding formula in the language of equality, say $\phi(x_1, ..., x_n)$ over $\mathrm{acl}(A)$, obtained by imposing no restrictions on $x_1, .., x_d$, and setting each higher coordinate to either be one of $x_1, ..., x_d$ or to be an element of $\mathrm{acl}(A)$ (the same as the analogous coordinate of a). Let Y be the solution set of ϕ. Then $\dim(Y) = d$, and Y is stationary; and moreover a generic element of X belongs to Y. It follows that X and Y are almost equal. Now X is a Boolean combination of Y, $X - Y$, and $Y - X$, and each of these is definable from equality (for the latter two, this is induction). Thus so is X. □

7. The Locally Modular Case

From now on, assume M is non-trivial and locally modular. Our main goal: prove that that there is a strongly minimal definable group in M^{eq}. This means there is a strongly minimal set G with a definable group operation $G^2 \to G$.

7.1. **The Germ Groupoid.**

Let us formally introduce the objects we will use to build a group. Recall that a *groupoid* is a category where every morphism is invertible.

Definition 7.1. The *germ groupoid* of M is the groupoid defined as follows:

- The objects are almost equality classes of strongly minimal sets in M^{eq}.
- For objects C, D, the morphisms $C \to D$ are almost equality classes of definable bijections between members of C and D.

Exercise 7.2. Check that this is a well-defined groupoid. In particular, this includes making sense of the phrase 'almost equality classes' for definable bijections between strongly minimal sets.

Note that whenever possible, we reserve X, Y etc. for definable sets, and C, D, etc. for objects in the germ groupoid (i.e. almost equality classes).

Suppose C, D are objects. Let us say that a *definable* collection of morphisms $C \to D$ is the quotient $F = H/\sim$, where H is a definable collection of bijections between members of C and D, and $\sim$ is the almost equality relation on H. If H is moreover A-definable for some set A, then we say F is A-definable. So a definable collection of morphisms is naturally identified as a definable set in M^{eq}, and thus we can speak of definable subsets of F^n and so on.

We will also use dimension theory for objects and morphisms; this is done by means of canonical bases. For example, if $f : C \to D$ is a morphism, then $\dim(f/CD)$ means $\dim(r/pq)$, where p, q, r are canonical bases of C, D, f, respectively (note that f is also strongly minimal, so it has its own canonical base). Similarly, we may speak of a set or tuple being e.g. 'f-definable'. This means 'definable over the canonical base of f'.

Local modularity gives the following:

Exercise 7.3. Let C, D be objects, and F a definable collection of morphisms $C \to D$. Then $\dim(F) \leq 1$. Thus, if $f : C \to D$ is a morphism, then $\dim(f/CD) \leq 1$.

We use throughout that composition of morphisms is 'definable in families':

Exercise 7.4. Suppose F, G, H are A-definable collections of morphisms from C to D, D to E, and C to E, respectively. Then $\{(f, g, h) \in F \times G \times H : g \circ f = h\}$ is A-definable.

7.2. **Summary.**

Our first main challenge is to find a reasonable supply of morphisms in the germ groupoid: namely, we will show that there are objects C and D and a one-dimensional definable family of morphisms $C \to D$. Constructing a family of morphisms is probably the most confusing thing we will do in this course; let us explain the general idea.

First, non-triviality gives us a one-dimensional family of *plane curves*, say $\{X_t\} \subset M^2$, which are finite-to-finite correspondences between cofinite subsets of M. We need to upgrade this to a family of *bijections*. Let us fix a generic curve X_t from the given family. Then there is a cheap way to turn X_t into a bijection: first we replace the right copy of M by $Z_t :=$ the collection of all finite sets of the form $X_t(x) = \{y : (x, y) \in X_t\}$ for $x \in M$. This turns X_t into a surjective finite-to-one function $X_t : M \to Z_t$. Then we replace the left copy of M with $Y_t := M/\sim$, where $\sim$ is the equivalence relation $X_t(x) = X_t(y)$ on M. So X_t now gives a bijection $Y_t \to Z_t$. Moreover, this whole process is (essentially) definable in M^{eq}: by uniform finiteness, there is n so that $X_t(x)$ has size n for almost all $x \in M$. Then up to finitely many points, we can view Z_t as a definable subset of $M^{(n)}$, the quotient of M^n by permutations of $\{1, ..., n\}$. Then Y_t is simply M modulo a definable equivalence relation, so it is trivially also M^{eq}-definable. Moreover, once we know they are definable, one can easily check that Y_t and Z_t are strongly minimal. Thus $[Y_t]$ and $[Z_t]$ are objects in the germ groupoid, and $[X_t]$ determines a morphism between them.

So out of a plane curve $X_t \subset M^2$, we cheaply construct a morphism $[Y_t] \to [Z_t]$ for some Y_t and Z_t. Of course, this doesn't really build an infinite family of morphisms between two objects, because the objects $[Y_t]$ and $[Z_t]$ vary with t. Our goal will be to construct a very specific scenario where $[Y_t]$ and $[Z_t]$ *do not* vary with t. This will involve replacing M with another object before we start (namely a curve in M^2).

7.3. **Composition Configurations.** In general, we want to encode the idea of *composing* two families of plane curves. Say we have stationary non-trivial plane curves $X, Y \subset M^2$. Define

$$Y \circ X := \{(x, z) : \text{ for some } y \text{ we have } (x, y) \in X \text{ and } (y, z) \in Y\}.$$

Then $Y \circ X$ is a plane curve, but is probably not stationary. So we have something like a multi-valued composition operation, sending $[X], [Y]$ to the set of components of $Y \circ X$. Now if we have two *families* of plane curves, say $\mathcal{X}, \mathcal{Y}$, then we could try setting $\mathcal{Y} \circ \mathcal{X}$ to be the set of all compositions $Y \circ X$ for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. This gets messy fast because compositions are not stationary. What ultimately works is doing everything on the level of tuples. In this language, the key notions will be certain kinds of *configurations* of tuples:

Definition 7.5. A *non-trivial configuration* in M is a triple $(a, b, t) \in M^{eq}$ such that:

(1) $\dim(a) = \dim(b) = \dim(t) = 1$.
(2) $\dim(ab) = \dim(at) = \dim(bt) = 2$ (i.e. any two of a, b, t are independent).
(3) Each of a, b, t is algebraic over the other two (thus $\dim(abt) = 2$).
(4) $t = \mathrm{Ct}(ab/t)$.

A non-trivial configuration is supposed to represent a one-dimensional family of plane curves. The idea is that $\mathrm{Loc}(ab/t)$ is a generic member X_t of a one-dimensional family of plane curves, and (a, b) is a generic point on X_t. Let us make this precise:

Exercise 7.6. Suppose $[X]$ is an almost equality class of stationary non-trivial plane curves, with canonical base t satisfying $\dim(t) = 1$. So $[X]$ contains some t-definable member – without loss of generality X itself. Now let $(a, b) \in X$ be generic over t. Show that (a,b,t) is a non-trivial configuration. Conversely, if (a,b,t) is any non-trivial configuration, then there is a stationary t-definable non-trivial plane curve X so that $(a,b) \in X$ is generic over t.

Exercise 7.7. Show that if (a,b,t) is a non-trivial configuration, then so is (b,a,t).

Hint: this just says that if X is a stationary non-trivial plane curve with canonical base t, then t is also a canonical base for $X^{-1} = \{(y,x) : (x,y) \in X\}$.

Definition 7.8. A *composition configuration* consists of a triple

$$s = (a_1, a_2, a_3, t_{12}, t_{23}, t_{13}) \in M^{eq},$$

such that:

(1) Each of (a_1, a_2, t_{12}) and (a_2, a_3, a_{23}) is a non-trivial configuration.
(2) $t_{13} = \mathrm{Cb}(a_1 a_3 / t_{12} t_{23})$.
(3) $\dim(s) = 3$.

The idea is as follows: we start with t_{12} and t_{23}, defining a generic pair of plane curves X and Y from two one-dimensional families $\mathcal{X}$ and $\mathcal{Y}$. Then we take a generic point $a_1 \in M$ over $t_{12}t_{23}$; we use t_{12} to generate a point a_2 with $(a_1, a_2) \in X$; and we use t_{23} to generate a point a_3 with $(a_2, a_3) \in Y$. Then we think of the 'composition' $Y \circ X$ as corresponding to $\mathrm{Loc}(a_1 a_3 / t_{12} t_{23})$. This is a stationary non-trivial plane curve, and we call its canonical base t_{13}.

Again, we make this precise:

Exercise 7.9. Let $[X]$ and $[Y]$ be almost equality classes of stationary non-trivial plane curves, with canonical bases t_{12} and t_{23} satisfying $\dim(t_{12}) = \dim(t_{23}) = 1$ and $\dim(t_{12}t_{23}) = 2$. Without loss of generality assume X is t_{12}-definable and Y is t_{23}-definable. Let $a_1 \in X$ be generic over $t_{12}t_{23}$. Show that there are a_2, a_3 with $(a_1, a_2) \in X$ and $(a_2, a_3) \in Y$, and that for any such a_2 and a_3, $s = (a_1, a_2, a_3, t_{12}, t_{23}, t_{13})$ is a composition configuration.

A key property of composition configurations is the following. This is analogous to the 'cancellation laws' in a group (if $xy = xz$ then $y = z$):

Lemma 7.10. *Suppose $(a_1, a_2, a_3, t_{12}, t_{23}, t_{13})$ is a composition configuration. Then each of t_{12}, t_{23}, t_{13} is algebraic over the other two.*

Proof. Let $[X] = \mathrm{Loc}(a_1 a_2 / t_{12})$, $[Y] = \mathrm{Loc}(a_2 a_3 / t_{23})$, and $[Z] = \mathrm{Loc}(a_1 a_3 / t_{12} t_{23}) = \mathrm{Loc}(a_1 a_3 / t_{13})$. We may assume X is t_{12}-definable, and similarly for Y and Z. Then $[Z]$ is a component of the $t_{12}t_{23}$-definable set $Y \circ X$, thus $t_{13} = \mathrm{Cb}(Z) \in \mathrm{acl}(t_{12}t_{23})$. Similarly, $t_{23} \in \mathrm{acl}(t_{12}t_{13})$ because $[Y]$ is a component of the $t_{12}t_{13}$-definable set

$$Z \circ X^{-1} = \{(y,z) : \text{ for some } x \text{ we have } (x,y) \in X \text{ and } (x,z) \in Z\}.$$

This is because (a_2, a_3) is generic in Y over $t_{12}t_{13}$ and belongs to $Z \circ X^{-1}$, so Y is almost contained in $Z \circ X^{-1}$. The fact that (a_2, a_3) is generic is because otherwise $(a_2, a_3) \in \mathrm{acl}(t_{12}t_{13})$; and then repeatedly using that each side is a non-trivial configuration, one gets that all six points are algebraic over $t_{12}t_{13}$. So $\dim(s) \leq 2$, a contradiction. A similar argument shows that $t_{12} \in \mathrm{acl}(t_{23}t_{13})$. □

Our main use of local modularity is the following:

Lemma 7.11. *Suppose $(a_1, a_2, a_3, t_{12}, t_{23}, t_{13})$ is a composition configuration.*

(1) $\dim(t_{13}) = 1$*, and thus (a_1, a_3, a_{13}) is a non-trivial configuration.*

(2) *In particular,* $(a_2, a_3, a_1, t_{23}, t_{13}, t_{12})$ *and* $(a_3, a_1, a_2, t_{13}, t_{12}, t_{13})$ *are also composition configurations.*

Proof. Local modularity gives $\dim(t_{13}) \leq 1$. If $\dim(t_{13}) = 0$ then $t_{13} \in \operatorname{acl}(\emptyset)$, so $t_{23} \in \operatorname{acl}(t_{12}t_{13}) = \operatorname{acl}(t_{12})$, contradicting that t_{12} and t_{23} are independent. The rest is clear. □

7.4. **Coordinate-wise Interalgebraicity.** The following is the main technical lemma we need about composition configurations. It is completely unmotivated, but is about to be crucial. The idea is that we are starting to control the equivalence relations we have to mod by to replace finite-to-finite correspondences with bijections.

Lemma 7.12. *Let* $s_a = (a_1, a_2, a_3, t_{12}, t_{23}, t_{13})$ *and* $s_b = (b_1, b_2, b_3, t_{12}, t_{23}, t_{13})$ *both be composition configurations. If any two of the pairs* (a_1, b_1)*,* (a_2, b_2)*,* (a_3, b_3) *are interalgebraic (meaning* a_i *is interalgebraic wth* b_i*), then so is the third.*

Proof. Let us assume (a_1, b_1) and (a_2, b_2) are interalgebraic, and prove the same for (a_3, b_3). The other cases follow after permuting (Lemma 7.11). For $i < j$, set $s_{ij} = (a_i, b_i, a_j, b_j, t_{ij})$ (the 'ij' part of the given data). We make two observations:

- First, $\dim(s_a s_b) = 3$, because every point of s_b is algebraic over s_a (it is enough to get a single b_i and fill in from there, and we are given that $b_1 \in \operatorname{acl}(a_1)$).
- Second, similarly, each $\dim(s_{ij}) = 2$, because it is algebraic over the non-trivial configuration (a_i, a_j, t_{ij}). Again it is enough to get a single b_i, and this is possible because i and j can't both be 3.
- Third, $s_a s_b \in \operatorname{acl}(s_{13}s_{23})$ (and the same for other i, j pairs, but we don't need them). Indeed, the only point from $s_a s_b$ not among the coordinates of $s_{13}s_{23}$ is t_{12}; and $t_{12} \in \operatorname{acl}(t_{13}t_{23})$.

Now suppose that a_3 and b_3 are not interalgebraic. Then $\dim(a_3 b_3) = 2$. Since $\dim(s_{13}) = \dim(s_{23}) = 2$, we get $s_{13}, s_{23} \in \operatorname{acl}(a_3 b_3)$, and thus $s_a s_b \in \operatorname{acl}(a_3 b_3)$. This is a contradiction because $\dim(s_a s_b) = 3$ while $\dim(a_3 b_3) \leq 2$. □

7.5. **Construction of a Family.** Now we can show:

Theorem 7.13. *There is a one-dimensional definable family of morphisms between two objects in the germ groupoid. In other words, there is a morphism* $f : C \to D$ *so that* $\dim(f/CD) = 1$*.*

Proof. By non-triviality and local modularity, there is a non-trivial configuration (a, b, t) with $a, b \in M$, which comes from a generic member X_t of a one-dimensional family of plane curves, say $\{X_v : v \in T\}$. Let X_u be another generic member of the same family with u independent from tab. Put another way, t and u are now independent generics in T, and (a, b) is generic in X_t over tu. It follows that there is c with $(b, c) \in X_u$; and then there is d with $(c, d) \in X_t$. Let $v = \operatorname{Cb}(ac/tu)$ and $w = \operatorname{Cb}(bd/tu)$. We now have two composition configurations:

$$s_{left} = (a, b, c, t, u, v), \qquad s_{right} = (b, c, d, u, t, w).$$

We will view X_t as fixed (as the new 'M'), and u as encoding a generic-in-a-family multivalued map in $X_t \to X_t$ (sending $(a, b) \mapsto (c, d)$).

As previously discussed, we may now view X_u as a bijection, say $f_u : Y_u \to Z_u$. Precisely, let $W_u \subset M^4$ be the set of pairs $((x, y), (z, w)) \in X_t^2$ with $(y, z) \in X_u$.

View W_u as a multivalued map $X_t \to X_t$. Then let Z_u be the collection of image sets $\{(z,w) : ((x,y),(z,w)) \in W_u\}$ for $(z,w) \in X_t$; so W_u gives a function (defined almost everywhere) $W_u : X_t \to Z_u$. Then let Y_u be X_t modulo the equivalence relation $W_u(x,y) = W_u(y,z)$.

One checks easily that Y_u and Z_u are strongly minimal sets, and W_u gives a bijection between cofinite subsets of Y_u and Z_u; we now call this $f_u : [Y_u] \to [Z_u]$, a morphism in the germ groupoid. The trick is to show that $[Y_u]$ and $[Z_u]$ only depend on t, not u; so if we fix t (as we are) and vary u, we indeed get infinitely many maps $[Y_u] \to [Z_u]$.

The trick here is Lemma 7.12. Let $h = W_u(a,b) \in Z_u$ – so h is a finite subset of X_t containing (c,d). Now suppose $(c',d') \neq (c,d)$ is another point in h. Then by Lemma 7.12 in s_{left}, c and c' are interalgebraic; and then by Lemma 7.12 in s_{right}, so are d and d'. That is, any two points in h are 'coordinate-wise interalgebraic'. Similarly, let $g = W^{-1}(h) \in Y_u$; then any two points of g are coordinate-wise interalgebraic (by the same argument in reverse). It follows that h is interalgebraic with (c,d), and g is interalgebraic with (a,b). So

$$\dim(g/t) = \dim(g/tu) = \dim(h/t) = \dim(h/tu) = 1.$$

Let G and H be one-dimensional t-definable sets containing g and h, respectively. So g and h remain generic in G and H over tu; it follows that $Y_u \cap G$ and $Z_u \cap H$ are infinite (as they contain g and h). So Y_u and Z_u are stationary components of G and H, respectively. So, letting $p := \mathrm{Cb}(Y_u)$ and $q := \mathrm{Cb}(Z_u)$, we have $p, q \in \mathrm{acl}(t)$. In particular,

$$\dim(u/tpq) = 1.$$

So infinitely many of the curves X_v from our original family will produce morphisms f_v between the same almost equality classes $[Y_u]$ and $[Z_u]$, which is basically what we wanted to show.

Precisely, let us check the exact statement in the theorem:

Claim 7.14. *Let* $r = \mathrm{Cb}(f_u)$. *Then* $\dim(r/pq) = 1$.

Proof. Local modularity gives $\dim(r/pq) \leq 1$. Assume $\dim(r/pq) = 0$. Since $p, q \in \mathrm{acl}(t)$, $\dim(r/t) = 0$, i.e. $r \in \mathrm{acl}(t)$. But $\dim(gh/r) = 1$, so $\dim(gh/t) \leq 1$, thus by interalgebraicity $\dim(abcd/t) \leq 1$. So $\dim(abcdt) \leq 2$, so in particular $\dim(abct) \leq 2$. But $u, v \in \mathrm{acl}(abc)$, so then $\dim(s_{left}) \leq 2$, contradicting that $\dim(s_{left}) = 3$. ☐

The theorem is now proved. ☐

7.6. **Construction of a Group.** The hard part is done. We now have objects C and D, and a morphism $f : C \to D$ with $\dim(f/CD) = 1$. In particular:

Assumption 7.15. For the rest of this section, we fix objects C and D, and a one-dimensional definable family F of morphisms $C \to D$.

For ease of notation, we will assume throughout that C, D, and F are $\emptyset$-definable in M^{eq}.

If we replace F by one of its stationary components, we may assume that F is strongly minimal. We assume this from now on.

To build a group, we need a collection of maps $C \to C$, not $C \to D$. We do that, roughly speaking, by composing the given maps from $C \to D$ with some maps $D \to C$. Let us consider options of how to do this.

Let's say we are in a vector space M, and $C = D = [M]$. Then F is the collection of (classes of) maps $\{f_t(x) = cx + t : t \in M\}$ for some scalar c – maybe plus or minus finitely many sporadic exceptions.

A first idea is to fix a generic $g \in F$ and replace F with the family of all compositions $f \circ g^{-1}$ for $f \in M$. This results in the family of translation maps $x \mapsto x+t$ (which is what we want), but still plus or minus finitely many exceptions. So how do we recognize and fix the exceptions?

Another idea is to instead consider $F \circ F^{-1}$, the family of *all compositions* $f \circ g^{-1}$ (for all $f, g \in F$). This will get *all translations* as desired (i.e. we fill all holes); but it makes the set of 'added' maps even worse: each original 'added map' now appears in infinitely many compositions. So we filled holes but also added a bunch of extra stuff we have to remove.

The trick: there is a definable way to pick the genuine translations out of $F \circ F^{-1}$: they are the maps belonging to $F \circ F^{-1}$ for 'infinitely many reasons'. That is, h is a translation if and only if there are infinitely many pairs $(f, g) \in F^2$ with $f \circ g^{-1} = h$. This is exactly what works in general too:

Definition 7.16. Let G be the set of morphisms $C \to D$ which can be expressed as a composition $f \circ g^{-1}$ for infinitely many pairs $(f, g) \in F^2$.

Note that G is $\emptyset$-definable, because F is $\emptyset$-definable and composition is definable in families. We use throughout:

Lemma 7.17. *Let $f, g \in F$. Then f and g are interdefinable over $f \circ g^{-1}$.*

Proof. Clear. □

Local modularity then gives:

Lemma 7.18. *G is strongly minimal.*

Proof. Let $Z \subset F^2$ be the set of pairs (f, g) with $f \circ g^{-1} \in G$. So we have a definable map $Z \to G$. For each $a \in G$, the fiber Z_a has dimension 1: it is infinite since $a \in G$, but it also has dimension at most 1 by Lemma 7.17.

We claim that $\dim(Z) = 2$, and thus Z is almost equal to F^2 (here we use that F^2 is stationary, because F is). Indeed, let $(f, g) \in F^2$ be generic – so $\dim(fg) = 2$ – and let $a = f \circ g^{-1}$. By local modularity, $\dim(a) \leq 1$, so $\dim(fg/a) \geq 1$. This shows that $(f, g) \in Z$, which shows that $\dim(Z) = 2$.

So $\dim(Z) = 2$, and we have a map $Z \to G$ with all fibers of dimension 1. It follows that $\dim(G) = 1$. In fact, it even follows that G is strongly minimal: if we could split G into two infinite definable sets G_1 and G_2, then the preimages Z_{G_1}, Z_{G_2} would split Z into two 2-dimensional definable subsets. But Z is stationary (since F^2 is), so this is a contradiction. □

Finally, we now show:

Theorem 7.19. *G is a group under composition.*

Proof. It is clear that G contains the identity, since $\text{id} = f \circ f{-1}$ for any $f \in F$. It is also clear that G is closed under inverses (since $(f \circ g^{-1})^{-1} = g \circ f^{-1}$ for $f, g \in F$).

The hard part is composition. Given any morphism $a : C \to C$, let a_{left} be the set of $f \in F$ so that $a = f \circ g^{-1}$ for some $g \in F$; then let a_{right} be the set of $g \in F$ so that $a = f \circ g^{-1}$ for some $f \in F$. Then a_{left} and a_{right} are definable, and thus each is finite or cofinite in F. The following is then clear (using Lemma 7.17):

Claim 7.20. *Let $a : C \to C$ be a morphism. Then the following are equivalent:*

(1) $a \in G$.
(2) a_{left} *is cofinite.*
(3) a_{right} *is cofinite.*

Proof. Each of (2) and (3) trivially implies (1). Conversely, if (1) holds, there is (f, g) with $a = f \circ g^{-1}$ and $\dim(fg/a) \geq 1$. Since f and g are interdefinable over a, they are both generic in F over a. This implies (2) and (3). □

Now let $a, b \in G$. We show that $a \circ b \in G$. Let $g \in F$ be generic over ab. Then $g \in a_{right} \cap b_{left}$, so there are $f, h \in F$ with $a = f \circ g^{-1}$ and $b = g \circ h^{-1}$. Then

$$a \circ b = (f \circ g^{-1}) \circ (g \circ h^{-1}) = f \circ h^{-1}.$$

So $f \in (a \circ b)_{left}$, and f is interdefinable with g over ab, thus is generic in F over ab. This shows that $(a \circ b)_{left}$ is cofinite, and thus $a \circ b \in G$. □

Since composition is definable in families, the group operation on G is definable. Thus, we have now shown:

Theorem 7.21. *If M is non-trivial and locally modular, then in M^{eq} there is a definable strongly minimal group.*

8. The Structure of the Group

We are still assuming that M is non-trivial and locally modular, and thus by Theorem 7.21, there is a definable strongly minimal group in M^{eq}. We now fix such a group $(G, \cdot)$. For ease of notation, we assume G and its group operation are $\emptyset$-definable.

Our new goal is to study the induced structure on G – that is, we want to understand the definable subsets of each G^n. The result will be that G is abelian, and every definable subset of each G^n is a Boolean combination of cosets of subgroups. This is an approximation of 'G is a vector space'. If we are not assuming total categoricity, it is the best we can do.

Throughout, we use the following basic and crucial properties of strongly minimal groups:

Exercise 8.1. (1) Let $H \leq G$ be a definable subgroup. Then either H is finite, or $H = G$.
(2) Let $f : G \to G$ be a definable endomorphism. Then either f is trivial, or f is surjective with finite kernel.

8.1. **Stationary Sets are Affine.** First we classify stationary sets up to almost equality. This is the main step. We need a couple preliminary facts. Note that if $H \leq G^n$ is a definable subgroup, then the coset space G^n/H is M^{eq}-definable.

Lemma 8.2. *Let $H \leq G^n$ be a definable subgroup. Then* $\dim(G/H) + \dim(H) = n$.

Proof. Each coset of H has dimension $\dim(H)$, as it is in definable bijection with H via a translation. Now the natural projection $G^n \to G/H$ is definable, surjective, and has all fibers of dimension $\dim(H)$. Thus

$$n = \dim(G^n) = \dim(G/H) + \dim(H).$$

□

Theorem 8.3. *Let $X \subset G^n$ be definable and stationary. Then there is a definable subgroup $H \leq G^n$ so that X is almost equal to a right coset of H.*

Proof. Let T be the collection of translates of $g \cdot X$ of X modulo almost equality. There is a natural transitive action of G on T by translation. Via this action, T is in definable bijection with the coset space $G^n / \operatorname{Stab}([X])$, where $\operatorname{Stab}([X])$ is the stabilizer of $[X]$. Note that $\operatorname{Stab}([X])$ is definable: it is precisely the elements $g \in G^n$ so that $g \cdot X$ is almost equal to X.

Now by the lemma above, we get $\dim(T) + \dim(\operatorname{Stab}([X])) = n$. On the other hand, by local modularity, $\dim(T) \leq n - \dim(X)$. So $\dim(\operatorname{Stab}([X])) \geq \dim(X)$.

Now let $a \in X$ be generic. We claim that X is almost equal to $\operatorname{Stab}(X) \cdot a$. First, if $g \in \operatorname{Stab}(X)$ is generic over a, then a is generic in X over g, thus $g \cdot a \in X$. Thus $\operatorname{Stab}(X) \cdot a$ is almost contained in X. But X is stationary and of dimension at most that of $\operatorname{Stab}(X)$, so this is only possible if the two sets are almost equal. □

Corollary 8.4. *Every definable subset $X \subset G^n$ is a Boolean combination of right cosets of definable subgroups.*

Proof. By induction on $d := \dim(X)$. Suppose we are given X of dimension d. We can break X into stationary components and handle each separately; thus we assume X is stationary. Then the theorem provides a coset C of a definable subgroup so that X and C are almost equal. Now X is a Boolean combination of C, $X - C$, and $C - X$, and these are all of the desired form (for the latter two, by the inductive hypothesis). □

Note that the same holds for left cosets, since $Ha = a(a^{-1}Ha)$.

8.2. **Few Subgroups.** We now do something a bit more confusing. We want to show that G has a 'few subgroups' property: for each n, there are essentially no infinite definable families subgroups of G^n. Unfortunately, this is hard to make precise. Let us start with a simpler case:

Lemma 8.5. *Suppose $\{f_t : t \in F\}$ is a definable family of endomorphisms $G \to G$. Then there are only finitely many distinct endomorphisms appearing among the f_t. In other words, every definable endomorphism of G is $\operatorname{acl}(\emptyset)$-definable.*

Proof. (Sketch) If $\{f_t\}$ is an infinite definable family of endomorphisms, then the maps $x \mapsto a \cdot f_t(x)$ (for all t and all $a \in G$) form a two-dimensional (or higher) family of curves in G^2, contradicting local modularity. □

Already, this is a bit imprecise. How exactly is the family of maps $a \cdot f_t(x)$ 'two-dimensional'? Strictly speaking, local modularity bounds the sizes of *faithful families* of curves (i.e. where any two have finite intersection). So we can't really say anything until we produce such a family. In the following two exercises, let us sketch how to do this precisely:

Exercise 8.6. Suppose X and Y are both cosets of definable subgroups of G^n. If X and Y are almost equal, then X and Y are equal.

Exercise 8.7. Suppose $\{f_t : t \in T\}$ is a definable family of endomorphisms of G, and infinitely many distinct maps occur among the f_t.

1. By modding T by an equivalence relation, show that there is such a family which is *faithful*: whenever $s \neq t$ the maps f_s and f_t agree in only finitely many points. Conclude in this case that $\dim(T) \geq 1$.

(2) Now assume $\{f_t\}$ is faithful, and consider the family $\{g_{t,a} : (t,a) \in T \times G\}$, where $g_{t,a}(x) = a \cdot f_t(x)$. Use Exercise 8.6 to show that $\{g_{t,a}\}$ is still faithful.
(3) Conclude that for generic $(t,a) \in T \times G$, the canonical base of the graph of $g_{t,a}$ has dimension at least 2, and thus we contradict local modularity.

We would like to do something general in G^n. Roughly speaking, if we had an infinite family of d-dimensional stationary subgroups of G^n, then the family of all of their cosets would have dimension at least $n - d + 1$, and this would contradict local modularity. Again, this is hard to make precise. In the end, the cleanest statement that works is:

Theorem 8.8. *Every stationary definable subgroup of G^n is definable over* $\operatorname{acl}(\emptyset)$.

The proof works most smoothly by exploiting the calculus of canonical parameters and canonical bases. We give the details in a series of exercises:

Exercise 8.9. Suppose $H \le G^n$ is a stationary definable subgroup of dimension d. Let c be a canonical parameter of H, so H is c-definable. Let $a \in G^n$ be generic over c, and let c_a be a canonical parameter of the coset $a \cdot H$.

(1) Use Exercise 8.6 to show that c_a is also a canonical base of $[a \cdot H]$.
(2) Show that $\dim(a/cc_a) = d$, by computing $\dim(ac_a/c)$ in two ways. Conclude that a is generic in $a \cdot H$ over c_a.
(3) Conclude that $\operatorname{Loc}(a/c_a) = [a \cdot H]$ and $\operatorname{Cb}(a/c_a) = c_a$, and thus by local modularity, $c_a \in \operatorname{acl}(a)$.
(4) Show that c is definable over (c_a, a) (hint: $H = a^{-1} \cdot (a \cdot H)$). Conclude that $c \in \operatorname{acl}(a)$.
(5) Finally, show that $c \in \operatorname{acl}(\emptyset)$ by computing $\dim(ac)$ in two ways. Conclude that H is $\operatorname{acl}(\emptyset)$-definable.

8.3. **Abelianity.**

Theorem 8.10. *G is abelian.*

Proof. Let $\operatorname{Inn}(G)$ be the group of inner automorphisms of G – the maps of the form $x \mapsto axa^{-1}$ for $a \in G$. By Lemma 8.5, $\operatorname{Inn}(G)$ is finite. Now there is a natural homomorphism $G \to \operatorname{Inn}(G)$, whose kernel is the center $Z(G)$. Since $\operatorname{Inn}(G)$ is finite, the kernel must be infinite. Thus $Z(G)$ is an infinite definable subgroup, and so $Z(G) = G$, i.e. G is abelian. □

Exercise 8.11. (Very Hard) Prove that *every* strongly minimal group is abelian (regardless of local modularity). A bit easier: just do it assuming there is an element of infinite order.

8.4. **The Totally Categorical Case.** We now prove arguably the main theorem of the course. Note that if $H \le G$ is a finite subgroup, then the quotient G/H is a group (by abelianity). Moreover, G/H is M^{eq}-definable (because H is finite), and is still strongly minimal. Now we will show:

Theorem 8.12. *Suppose M is totally categorical. Then there is a finite subgroup $H \le G$ so that G/H has precisely the structure of a vector space over a finite field. Namely, one can endow G/H with an F-vector space structure for some finite field F, so that the definable subsets of each $(G/H)^n$ are precisely those definable in the the language of F-vector spaces.*

Proof. Since G is abelian, let us write G additively from now on (so the identity is 0). We may still assume G is $\emptyset$-definable in M^{eq}, because adding finitely many constants preserves $\aleph_0$-categoricity.

By $\aleph_0$-categoricity, for each n, there are only finitely many $\emptyset$-definable subsets of G^n. Now let H be the set of *algebraic elements* – those elements $g \in G$ with $g \in \mathrm{acl}(\emptyset)$. Then H is a subgroup, and is finite (there are only finitely many $\emptyset$-definable finite subsets of G). We will show that G/H is precisely a vector space over a finite field. For convenience, in the rest of the proof we will replace G with G/H. Thus we assume the only algebraic element is 0.

Let F be the set of definable endomorphisms $G \to G$ (a priori, F is not a definable object). Note that F has a natural ring structure, and G is naturally a left F-module:

- Addition in F is given pointwise, i.e. $(f + g)(x) = f(x) + g(x)$.
- Multiplication is given by composition (so id is a multiplicative identity).
- If $f \in F$ and $x \in G$, then scaling is given by $f \cdot x = f(x)$.

Note that we have not assumed F is commutative. However, we do know two things about F:

Claim 8.13. *F is finite.*

Proof. Essentially, because by Lemma 8.5, every element of F is definable over $\mathrm{acl}(\emptyset)$, and by $\aleph_0$-categoricity, only finitely endomorphisms are $\mathrm{acl}(\emptyset)$-definable. But this is imprecise, because $\mathrm{acl}(\emptyset)$ includes elements of all sorts (so it could be infinite).

To make it precise: let $a \in G$ be generic, and consider the *evaluation homomorphism* $\mathrm{Eval}_a : F \to G$ given by $f \mapsto f(a)$. We show two things about this homomorphism:

- Eval_a is injective. Indeed, suppose $\mathrm{Eval}_a(f) = 0$ for some $f \in F$. Then $a \in \ker(f)$. Since f is $\mathrm{acl}(\emptyset)$-definable, so is $\ker(f)$. So since a is generic, $\ker(f)$ is infinite, and is thus all of G. So $f = 0$.
- Eval_a has finite image in G. Indeed, since each $f \in F$ is $\mathrm{acl}(\emptyset)$-definable, each $\mathrm{Eval}_a(f) = f(a) \in \mathrm{acl}(a)$. But by $\aleph_0$-categoricity, $\mathrm{acl}(a) \cap G$ is finite.

By the two points above, F is clearly finite. □

Claim 8.14. *Every non-zero element of F has an inverse. Thus, F is a division ring.*

Proof. Let $f \in F$ with $f \neq 0$. By Exercise 8.1, f is surjective with finite kernel, say $N = \ker(f)$. By Lemma 8.5, f is $\mathrm{acl}(\emptyset)$-definable, and thus so is N. So every element of N is algebraic, and thus $N = \{0\}$ (since we modded by the algebraic elements).

It follows that f is an isomorphism. Then the inverse f^{-1} is also a definable isomorphism, and is thus a multiplicative inverse of f in F. □

Now it follows that F is commutative after all: a famous theorem in algebra (Wedderburn's Theorem) says that every finite division ring is a field. So in fact, F is a field, and G is naturally an F-module – equivalently an F-vector space.

We now show that the definable subsets of each G^n are precisely those definable in the language of vector spaces. Clearly, every vector space-definable set is definable in M^{eq} (since addition and scaling are). We show the converse.

So let $X \subset G^n$ be definable. We show that X is vector space-definable, by induction on $\dim(X)$. We may break X into stationary components and handle each one separately – so assume X is stationary and we have handled all definable sets of dimension less than $\dim(X)$.

Claim 8.15. *We may assume that X is a subgroup of G^n.*

Proof. By Theorem 8.3, there is a definable subgroup $H \leq G^n$ so that X is almost equal to a coset of H. Replacing X with a translation, we may assume X is almost equal to H itself (this is allowed because translation is vector-space-definable). In fact, we may then replace X with H: if we know H is vector space definable, then X is a Boolean combination of H, $X - H$, and $H - X$ (and the latter two can be handled by induction). □

We have now reduced to that case that $X = H$ is a stationary definable subgroup of G^n. By Lemma 8.8, H is $\mathrm{acl}(\emptyset)$-definable.

Let $a = (a_1, ..., a_n)$ be generic in G, and let b be a basis – without loss of generality $(b_1, ..., b_d)$. Let $\pi : H \to G^d$ be the projection sending a to b, and $N \leq H$ the kernel of π. Then all fibers of π have dimension $\dim(N)$, so

$$d = \dim(H) = \dim(\pi(H)) + \dim(N) = d + \dim(N),$$

and thus N is finite.

Claim 8.16. *N is trivial.*

Proof. Since H is $\mathrm{acl}(\emptyset)$-definable, so is N. Thus every element of N is algebraic, and so N is trivial. □

So π is injective. Moreover, $\pi(H)$ is a d-dimensional subgroup of G^d; by Exercise 8.6, $\pi(H)$ has to be exactly G^d. In other words, π is a group isomorphism $H \to G^d$.

We may now view H as the graph of a group isomorphism $f : G^d \to G^{n-d}$ (where $f(x)$ is the element y so that $(x, y) \in H$). This map is given by an $(n - d) \times d$ matrix of endomorphisms f_{ij} of G. Namely, for $i \leq n - d$ and $j \leq d$, let f_{ij} be the map sending $x \in G$ to the ith coordinate of $f(0, ..., x, ..., 0)$ (where x is in the jth position). Then each f_{ij} is a definable endomorphism of G, so belongs to F; and then the whole map f is given matrix multiplication by $A = \{f_{ij}\}$, which is then definable over the F-vector space structure. □

References

[1] J. T. Baldwin and A. H. Lachlan. On strongly minimal sets. *Journal of Symbolic Logic*, 36(1):79–96, 1971.

[2] Ehud Hrushovski. Locally modular regular types. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Math.*, pages 132–164. Springer, Berlin, 1987.

[3] Ehud Hrushovski. The Mordell-Lang conjecture for function fields. *J. Amer. Math. Soc.*, 9(3):667–690, 1996.

[4] David Marker. Strongly minimal sets and geometry. In *Colloquium'95 (Haifa)*, pages 191–213, 2017.

[5] Michael Morley. Categoricity in power. *Transactions of the American Mathematical Society*, 114(2):514–538, 1965.

[6] Anand Pillay. *Geometric stability theory*, volume 32 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.