1-basedness in t-minimal structures

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Zilber, CHL

- Totally categorical theories are not finitely axiomatizable.
- First key property: local modularity of the (unique, up to non-orthogonality) strongly minimal set.
- Second key property: locally modular non-trivial strongly minimal sets are "Module like".
- Cherlin-Harrington-Lachlan: $\aleph_0$-categorical $\aleph_0$-stable theories are coordinatized by locally modular strongly minimal types.

CHL moral: $\aleph_0$-categorical $\aleph_0$-stable theories share important structural properties with totally categorical theories.

Hrushovski

Locally modular regular types

- Non-trivial locally modular regular types are, essentially, linear spaces over a division ring.
- A group chunk theorem and a suitable group configuration theorems are needed for the construction.

With Cherlin ("Finite structures with few types"): the CHL moral extends beyond the stable setting to structures coordinatized by linear SU-minimal types.

Hrushovski-Pillay

1-based groups

1. Abelian by finite.
2. Rigid (no definable families of infinite subgroups).
3. Linear: definable subsets of G^n are finite boolean combinations of cosets of definable subgroups.
4. Wagner proves analogous results for 1-based simple groups.

Recall also (Bouscaren-Hrushovski): (roughly) in a 1-based theory any non-trivial type can be associated with a type definable group.

Peterzil

Linear o-minimal structures

1. O-minimal structures are geometric (so "local modularity makes sense").
2. There are non-locally modular o-minimal structures that are locally linear.
3. A new notion of linearity captures precisely those o-minimal structures where a group is type-definable, but no field is.
4. Maalouf: An analogous notion of linearity* produces a (type)-definable group in geometric C-minimal structures.

* The statement assumes local modularity, but the proof seems to go through for linear structures.

Peterzil-Loveys

The structure of linear o-minimal groups[1]

[1] I am unaware of analogous results in the C-minimal setting.

O-minimal linear groups are:

1. Abelian (well, all o-minimal groups are).
2. Have no definable families of definable subgroups.
3. Every definable subset of G^n is, locally – at a generic point – the generic of a coset of a definable subgroup.
4. Families of germs of (unary) functions are 1-dimensional.

In fact: a linear o-minimal local group is a reduct of an ordered vector space over an ordered division ring. This extends to the above type-def. groups.

Many notions of linearity out there

Welcome to the Zoo

- Local modularity, 1-basedness, (almost) canonical bases of definable plane curves are 1-dimensional, no large families of (almost) normal families of plane curves, no (complete) quasi-designs, no type-definable pseudo-planes, weak 1-basedness, weak local modularity... aimed to capture variants of linearity in various tame contexts.
- Much work went into understanding the relations between these notions in various settings.
- I am unaware of significant structural results associated with those notions of linearity (besides those already mentioned).

The goal

The big organizing idea: Zilber's Trichotomy is, essentially, a topological phenomenon.

- Find a uniform *topological* proof of the group construction theorems for the linear o-minimal and C-minimal setting.
- Prove structural results in the spirit of Hrushovski-Pillay and Peterzil-Loveys.
- Expand the proof to *non-geometric* topological settings (e.g., dense weakly o-minimal structures).

Larger aim: find an axiomatic topological framework where this can be carried out.

No need to look far

t-minimality

- L. Mathews (∼1995): A structure $\mathcal{M}$ is t-minimal if it admits a uniformly definable basis for a Hausdorff topology such that a definable $S \subseteq M$ is infinite if and only if $\mathrm{Int}(S) \neq \emptyset$.
- Examples include: dense (weakly) o-minimal structures, dense C-minimal structures, dp-minimal valued fields. 1-h-minimal fields...
- Problem: with no further assumptions, results are fairly weak.
- A t-minimal structure is *visceral* if the topology is *uniform*.
- Dolich-Goodrick: considerably stronger results for visceral theories (still with additional assumptions).

Not quite there yet

Visceral is too strong

- Dense o-minimal and C-minimal theories need not be visceral.
- The standard source of uniform structures:
  1. G a (definable) group.
  2. A (definable) neighbourhood base at e.
  3. Declare translations homeomorphisms.
- Primary goal: construct a group, not assume one exists to begin with!

Johnson: visceral theories without assumptions. Dimension theory works for general t-minimal.

Dimension is not enough

Example (The all purpose counter-example: the Sorgenfrey line)

Take $(\mathbb{R}, +, \cdot, <)$ with the topology generated by intervals of the form $[a, b)$. It is:

1. t-minimal.
2. geometric.
3. The function $x \mapsto -x$ is definable and nowhere continuous.

Definition

A t-minimal $\mathcal{M}$ has the *independent neighborhood property* if for any tuple $a \in M^n$, any parameter set A, and any neighborhood U of a, there is a neighborhood V of a such that $V \subset U$ and V is definable over a parameter t with $\dim(a/At) = \dim(a/A)$.

Many natural examples

Example

The following are t-minimal with the the independent neighbourhood property:

1. Visceral theories (Johnson).
2. Dense C-minimal and weakly o-minimal theories.

The INP implies, e.g., generic continuity, the local homeomorphism property...

A useful observation

The INP provides some replacement for exchange, supplying "enough" additive pairs satisfying

$$\dim(ab/A) = \dim(a/A) + \dim(b/Aa).$$

How are groups constructed?

In model theory, often, groups are constructed in two steps:

1. Construct a *group chunk*: multiplication is defined only generically.
2. Recover a group from the group chunk.

- Step (2) is standard: Weil, Hrushovski, van den Dries, Pillay, Eleftheriou...
- For step (1) use composition of definable bijections within definable families.
- For better results, work with *germs* of definable bijections.

Caution: In different settings germs may have different meanings.

The usual definition

Definition

Let $a \in M^n$.

1. The *germ* $g := \mathrm{germ}_a(X)$ of X at a is the equivalence class realized by X among all subsets of M^n modulo agreement in a neighborhood of a.
2. A *definable germ* at a is a germ at a realized by some definable subset of M^n.
3. If g is a definable germ at a, then $\dim(g)$, is the smallest value of $\dim(X)$ where X is a definable set realizing g.

Caution: This is **not** the dimension of g as an element of $\mathcal{M}^{eq}$.

Germs of types exist

Fact

If $\mathcal{M}$ is t-minimal then for all $a \in M^n$ and parameter set A $\mathrm{germ}_a(X)$ is eventually constant as X ranges over $\mathrm{tp}(a/A)$.

This is $\mathrm{germ}(a/A)$.

A crucial implication of the generic neighbourhood property is that germs characterize independence:

Fact

Let $a \in M^n$. Then $\dim(a/A) = \dim(a/B)$ iff $\mathrm{germ}(a/A) = \mathrm{germ}(a/B)$.

Some calculus of germs

Infinitesimal neighbourhoods

For $a \in M^n$ and a parameter set A we let $\mu(a/A) := \mathrm{tp}(a/A) \cap \mu(a)$, where $\mu(a)$ is the infinitesimal neighbourhood of a.

Fact

- $\mathrm{germ}_a(\mu(a/A)) = \mathrm{germ}(a/A)$.
- $\mu(ab/A)_b = \mu(a/Ab)$ *and* $\mathrm{germ}(ab/A)_b = \mathrm{germ}(a/Ab)$ *follows.*
- *A trace of exchange: If* $B \supset A$ *and* $\dim(ab/B) = \dim(ab/A)$ *then* $\dim(a/Bb) = \dim(a/Ab)$.

What does 1-basedness mean?

In the stable setting:

- Recall: a stable theory is 1-based if for any $a, A = \mathrm{acl}(A)$, $\mathrm{Cb(a/A)} \subseteq \mathrm{acl}(a)$.
- Any A-definable set where a is generic is determined uniquely "almost everywhere", by the fact that a belongs to it.
- Compare with: a point incident to a line of the form $ax + by - c = 0$ in $\mathbb{R}^2$.

This does not make sense already in o-minimal theories.

Solution

- Think of "X, Y agree generically" in the stable setting as "$X \cap U = Y \cap U$ for some Zariski open U" in AG.
- In Hausdorff topologies this reads: $\mathrm{germ}_a(X) = \mathrm{germ}_a(Y)$.

Topologically 1-based types

Definition

Let $a \in M^m$, $b \in M^n$, and A a parameter set. We say that $\mathrm{tp}(a/Ab)$ is *topologically 1-based over A* if

$$\dim(b/Aa) = \dim(b/A\,\mathrm{germ}(a/Ab)).$$

This reads as: b is independent from $\mathrm{germ}(a/Ab)$ over Aa.

Theorem (The main structure theorem)

TFAE for $a, b \subseteq M$ and any A:

1. $\mathrm{tp}(a/Ab)$ *is topologically 1-based over A.*
2. *The map* $y \mapsto \mathrm{germ}(a/Ay)$ *is constant for* $y \in \mu(b/Aa)$.
3. *Any two fibers in the projection* $\mu(ab/A) \to \mu(b/A)$ *are either equal or disjoint.*

topologically 1-based theories

Definition

A t-minimal $\mathcal{M}$ with the INP is topologically 1-based if $\mathrm{tp}(a/Ab)$ is for all $a, b \subseteq M$ and parameter set A.

Fact ($\mathcal{M}$ topologically 1-based)

1. *Stable under naming constants and admissible reducts (preserving the topology and INP).*
2. *No definable infinite fields in $\mathcal{M}$.*
3. *If $\mathcal{M}$ is geometric it is weakly 1-based (and vice versa).*
4. *So linear o-minimal and geometric C-minimal structures are topologically 1-based with INP.*

Summary of the results

Theorem

Let $\mathcal{M}$ be a sufficiently saturated t-minimal structure with the independent neighborhood property. Assume that $\mathcal{M}$ is non-trivial and topologically 1-based. Then there are a countable parameter set A, and an A-type-definable abelian group G, such that:

1. *G is open in $\mathcal{M}$.*
2. *G is a topological group with the topology inherited from M.*
3. *G is locally linear.*

Remark

The group existence theorem is local. For the structural part more than one non-trivial topologically 1-based type is needed.

Strategy of the proof

A combinatorial group(oid) configuration theorem

- A *groupoid spine* is a certain category (possibly missing some id_X arrows) whose objects are sets, whose morphisms are bijections, and with a linear order on its collection of objects.
- We assume that for all $i < j$ the set $Mor(X_i, X_j)$ is given (satisfying obvious assumptions), and ask when a groupoid spine can be extended to a groupoid.

Our combinatiral group configuration theorem is:

Theorem

A symmetric groupoid spine on more than two objects extends to a groupoid.

Rigidity

When can a groupoid spine be extended to a symmetric groupoid spine?

Definition

Let $(I, <, \mathcal{X}, R, Mor)$ be a groupoid spine. We say that $(I, <, \mathcal{X}, R, Mor)$ is *regular* if for all $(i, j) \in R$, and all $x \in X_i$ and $y \in X_j$, there is exactly one morphism $f \in \mathrm{Mor}(i, j)$ with $f(x) = y$.

Theorem

Let $(I, <, \mathcal{X}, R, Mor)$ be a regular groupoid spine. Then there is a symmetric set $S \supset R$ such that $(I, \mathcal{X}, R, Mor)$ extends to S. If the data is type-definable, the group action $(Mor(X_i, X_i), X_i)$ is type-definable for any object $X_i \in X$.

The construction of the group

Aim: construct a regular type-definable group-spine (assuming topological 1-basedness).

Definition

1. $\mathcal{M}$ is *non-trivial* if for some A, there are $a, b, c \in M$ each A-algebraic over the others and $\dim(abc/A) = 2$.
2. A 1-dimensional topological pre-group configuration (a, b, c, t, u) is a topological group configuration if $\text{tp}(ac/tu)$ is topologically 1-based.

Theorem

Let (a, b, c, t, u) be a 1-dimensional topological group configuration in $\mathcal{M}$. Then there is an $\mathcal{M}$-type-definable group structure on the set $\mu(a/\emptyset)$ with identity element a.

1. $\mu(abt/\emptyset)$ is the graph of a regular family of homeomorphisms $\mu(a/\emptyset) \to \mu(b/\emptyset)$.
2. Similarly for $\mu(b, c, u/\emptyset)$, $\mu(a, u, tc/\emptyset)$.
3. $\mu(a/\emptyset)$, $\mu(b/\emptyset)$ and $\mu(c/\emptyset)$ with the above families of homemorphisms form a regular groupoid spine.

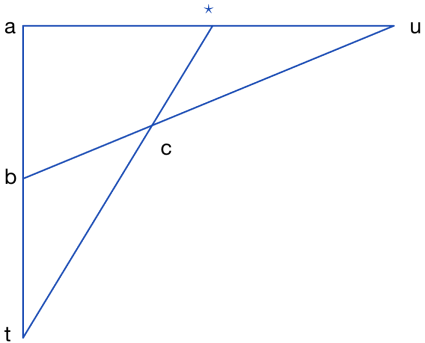

Figure: A pre-group configuration

The structural results

1. Topologize the group using Marikova's method.
2. We can replace the group to get one that is topological with the affine topology.
3. Local linearity and local commutativity are "topologized" adaptations of the analogous results from stability theory.

Remark: with a little more effort we can make the group visceral (in an appropriate sense).

Thank you!