Groups definable in o-minimal theories

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Key idea (van den Dries?): Avoid infinite discrete subsets.

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It is still open (I think) whether $(\mathbb{R}, +, \cdot, <, e^x)$ is decidable. Nonetheless, because of o-minimality we have a good control over definable sets:

Why o-minimality?

Fact

O-minimal structures have a good behaviour:

- *Every definable set is a finite union of sets, each of which homeomorphic to an open subsets (in fact, open boxes) of $\mathcal{R}^n$. This has several implications.*

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- *Every definable set has finitely many connected components.*
- *We can define notions like dimension, Euler characteristic.*
- *We can do differential calculus. Derivatives are definable, every function is piecewise continuous (differentiable,* C^k*).*

Definable and Lie categories

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Any definable homomorphism between definable groups is a Lie homomorphism.

So the definable category is a subcategory of the Lie group category. It is natural to ask what Lie groups and morphisms belong to the definable category.

Categorical questions. Not as subsets:

$$\begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

This is isomorphic to the additive group, but not definable.

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When is ϕ definable in an o-minimal expansions of a field?

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If two groups H and G are definable in some $\mathcal{R}$ and there is a Lie isomorphism ϕ between G and H,
When is ϕ definable in an o-minimal expansions of a field?
When does adding ϕ to $\mathcal{R}$ preserve o-minimality?

The structure of a Lie group

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The connected component

G^0 has a maximal connected solvable normal subgroup R and $S := G^0/R$ is semisimple (no infinite normal abelian subgroup).

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$GL_2^+(\mathbb{R})$ is *not* the direct product of Z and $PSL_2(\mathbb{R})$.

$GL_2^+(\mathbb{R})$ *is* the product of Z and $SL_2(\mathbb{R})$. But these intersect.

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$$0 \to R \to G \to S \to 0$$

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To understand the structure (and to build a definable copy), it is convenient to find H "definable" such that $G = RH$.

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However, $B := \widetilde{S} \times_{\mathbb{Z}} \mathbb{R}$ is a central extension of the definable $SL_2(\mathbb{R})$, so always definable (HPP).

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However, $B := \widetilde{S} \times_{\mathbb{Z}} \mathbb{R}$ is a central extension of the definable $SL_2(\mathbb{R})$, so always definable (HPP).

The Levi subgroup of B is $\widetilde{S}$. So even if a group is definable, its Levi subgroup might not be.

Strategy

Let G be a Lie group, R its solvable radical and H a subgroup of G projecting in G/R. Then G is isomorphic to $(H \ltimes_{\gamma} R)/(H \cap R)$ where γ is conjugation.

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If we can build:

- a definable copy R_{def} of R,
- a definable copy H_{def} of H
- both with definable subgroups isomorphic to $R \cap H$, and
- a definable action γ of R_{def} on H_{def} copying the action of H on R by conjugation, then

$(H_{def} \ltimes_\gamma R_{def})/(H_{def} \cap R_{def})$ is isomorphic to G.

Finding H

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$H := L \cdot Z(G)$ is a central extension of $L/(Z(G) \cap L) = S$ so it is "definable".

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$G = HR$ with $H \cap R = Z(G)$ and "definability" of G comes down to definability of R and the action.

Definability of R.

The solvable radical

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Definability of R comes down to finding a definable copy of U (with η definable).

Examples 3

In the definable category, U is unique. This will play an important role later.

If

$$K := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, c > 0 \right\}$$

and $G = K \times \mathbb{R}$, then there is a unique definable torsion free subgroup $\{Id\} \times \mathbb{R}$. But it has a torsion free *Lie* subgroup $\{(e^{it}, t) \mid t \in \mathbb{R}\}$.

Definability of U.

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We can construct a linear group G isomorphic to $\mathbb{R} \ltimes_\gamma \mathbb{R}^2$ where γ is the action of $\mathbb{R}$ on $\mathbb{R}^2$ which rotates t radians. This can't happen in a definable group despite having

$$0 \to (\mathbb{R},+) \times (\mathbb{R},+) \to G \to (\mathbb{R},+) \to 0$$

This is, in escence, the only obstruction.

Supersolvable Groups

Definition

A torsion free group is Super solvable, completely solvable or triangular if it admits a series of normal subgroups of G $\{e\} \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_d = G$ *such that* N_{i+1}/N_i *is isomorphic ro* $(\mathbb{R}, +)$.

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$$0 \to \mathbb{R} \to G \to \mathbb{R} \times \mathbb{R} \to 0,$$

One can find two groups G can fit such a sequence. $\mathbb{R}^3$ and the Heisenberg group

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem

A torsion free solvable Lie group U has a definable copy if and only if U is completely solvable.

Torsion free definable groups (Necessary)

Lemma

[Conversano, Starchenko, O.] Let U be a torsion free group definable in an o-minimal expansion of a real field. Then U has a one dimensional normal abelian subgroup.

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Classification of "definable" torsion free solvable Lie groups (Sufficient)

Theorem (Conversano, Starchenko, O.)

$\mathcal{R}$ be an o-minimal theory expanding and let U be a torsion free (solvable) Lie group, and $\mathfrak{t}$ its Lie algebra. Then:

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- *Exp restricted to upper triangular matrices is definable in $\mathcal{R}$.*

So a solvable torsion free Lie group is isomorphic to a group definable in an o-minimal expansion of $\mathbb{R}$ if and only if it is supersolvable.

Characterizing “definable” Lie groups.

Classification of definable connected Lie groups

Theorem

If G is connected, $Z(G)$ has finitely many connected components and its maximum solvable torsion free normal subgroup U is supersolvable, then G has a definable copy.

Building R_{def}

Classification of solvable connected “definable” groups

- A solvable R with torsion free group U is semidirect product of $K := R/U$ and U.
  Find U upper triangular. Automorphisms of U are in definable correspondance with automorphisms of the Lie algebra (linear).
  So the automorphisms of U are definable in $\mathbb{R}_{exp}$ (clear).

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  So the automorphisms of U are definable in $\mathbb{R}_{exp}$ (clear).
- If we choose K linear (which we can), then the action is algebraic.
- So we can build a linear group R_{def} definable in $\mathbb{R}_{exp}$.

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- G is then isomorphic to a semidirect product of $H := Z \cdot S$ and R mod out by the intersection.
- The corresponding action from H_{def} to R_{def} factors through S and is then definable.

Definability of isomorphisms.

Can we add Lie isomorphisms between definable groups?

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Let $\phi : G \to H$ be a Lie isomorphism.

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Fact (Speissegger)

If $\mathcal{R}$ is any o-minimal expansion of the real field, there is an o-minimal expansion $\mathcal{R}_{Pfaff}$.

$\mathcal{R}_{Pfaff}$ includes in particular the exponential. So this strengthens Wilkie's.

Can we add Lie isomorphisms between definable groups?

Let G and H be definable in $\mathcal{R}$, let H_G, U_G and K_G the components of G: Let $\phi : G \to H$ be a Lie isomorphism between $\mathcal{R}$-definable groups. Two obstructions (opportunities):

1. ϕ must send the definable components of G to definable components of H.
2. The restriction of ϕ to H_G, U_G and K_G each definable component must be definable in an o-minimal expansion.

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- K_G and $\phi(K_G)$ need not be definable, not always definable, but they are definable in $\mathcal{R}_{Pfaff}$. So we can add them and keep o-minimality.
- There is no reason why $\phi(U_G) = U_H$:
  (Maximal torsion free solvable normal Lie groups are not unique. Definable ones are.)

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- K_G and K_H are abelian and compact and so is the isomorphism between them (all definable in $\mathcal{R}_{Pfaff}$).

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- K_G and K_H are abelian and compact and so is the isomorphism between them (all definable in $\mathcal{R}_{Pfaff}$).
- Both U_G and U_H are products of 1-dimensional extensions of $Z(G)$ and $Z(H)$ so they are products of abelian subgroups.

Can we add Lie isomorphisms between definable groups?

Theorem

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Theorem

Let G and H be definable in $\mathcal{R}$, let R_G, R_H be their solvable radicals and let U_G, U_H be the maximal torsion free normal subgroups of R_G and R_H, respectively.

Let ϕ be a Lie isomorphism between G and H.

Then ϕ is definable in $\mathcal{R}_{Pfaff}$ if and only if $\phi(U_G) = U_H$.

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- If G is abelian and

$$0 \to (\mathbb{R}, +) \xrightarrow{f} G \xrightarrow{g} (\mathbb{R}, +) \to 0$$

with f, g definable is G isomorphic to $(\mathbb{R}, +)^2$?

Thanks