Groups definable in o-minimal theories

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Tarski Seidenberg (1948?)

Decidability of and quantifier elimination in $Th(\mathbb{R}, +, \cdot, <)$.

We will talk about a class of “tame” extensions of $Th(\mathbb{R}, +, \cdot, <)$.

Key idea (van den Dries?): Avoid infinite discrete subsets.

Definition of o-minimality

Definition

We will always assume there is an ordering in the language and work with totally ordered structures.
An ordered structure $\mathcal{R}$ will be *o-minimal* if any definable subset of $\mathcal{R}$ is a finite union of intervals and points.

Fact (Wilkie)

$(\mathbb{R}, +, \cdot, <, e^x)$ is o-minimal. We will call this structure $\mathbb{R}_{exp}$.

It is still open (I think) whether $(\mathbb{R}, +, \cdot, <, e^x)$ is decidable. Nonetheless, because of o-minimality we have a good control over definable sets:

Why o-minimality?

Fact

O-minimal structures have a good behaviour:

- *Every definable set is a finite union of sets, each of which homeomorphic to an open subsets (in fact, open boxes) of $\mathcal{R}^n$. This has several implications.*
- *Every definable set has finitely many connected components.*
- *We can define notions like dimension, Euler characteristic.*
- *We can do differential calculus. Derivatives are definable, every function is piecewise continuous (differentiable, C^k).*

Definable and Lie categories

Pillay

Every group definable in an o-minimal expansion of $(\mathbb{R}, +, \cdot)$ is a Lie group. (i.e. it can be equipped with a smooth manifold structure so that the group operations are smooth).

Any definable homomorphism between definable groups is a Lie homomorphism.

So the definable category is a subcategory of the Lie group category. It is natural to ask what Lie groups and morphisms belong to the definable category.

Categorical questions. Not as subsets:

$$\begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

This is isomorphic to the additive group, but not definable.

Objects of the definable category

Question

What real Lie groups are Lie isomorphic to groups definable in o- minimal expansions of the real field?

Question

What is the structure of groups definable in o-minimal expansions of a field?

Question

If two groups H and G are definable in some $\mathcal{R}$ and there is a Lie isomorphism ϕ between G and H,
When is ϕ definable in an o-minimal expansions of a field?
When does adding ϕ to $\mathcal{R}$ preserve o-minimality?

The structure of a Lie group

If G is definable in an o-minimal expansion of $(\mathbb{R}, +, \cdot)$, then G/G^0 is finite, where G^0 is the connected component.

The connected component

G^0 has a maximal connected solvable normal subgroup R and $S := G^0/R$ is semisimple (no infinite normal abelian subgroup).

Example 1

In $GL_2(\mathbb{R})$ the connected component are matrices with positive determinant $GL_2^+(\mathbb{R})$.
The center Z of $GL_2^+(\mathbb{R})$ is

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R}^{>0} \right\}$$

.
$GL_2^+(\mathbb{R})/Z = PSL_2(\mathbb{R})$, which is semisimple.

$GL_2^+(\mathbb{R})$ is *not* the direct product of Z and $PSL_2(\mathbb{R})$.

$GL_2^+(\mathbb{R})$ *is* the product of Z and $SL_2(\mathbb{R})$. But these intersect.

For G connected Lie group we get an exact sequence

$$0 \to R \to G \to S \to 0$$

with R solvable and S semisimple.

If G is definable, both R and S must be definable.

In general, it doesn't split.

To understand the structure of Lie groups, one usually finds the "Levi subgroup" L and then use $G = RL$.

To understand the structure (and to build a definable copy), it is convenient to find H "definable" such that $G = RH$.

Examples 2

Let $S := SL_2(\mathbb{R})$ and $\widetilde{S}$ be its universal cover.

$Z(\widetilde{S})$ is $\mathbb{Z}$ so $\widetilde{S}$ is semisimple, but not definable.

$0 \to \mathbb{Z} \to \widetilde{S} \to S \to 0$ doesn't split.

We need the center of the group to have finitely many connected components for the group to have a definable copy.

However, $B := \widetilde{S} \times_{\mathbb{Z}} \mathbb{R}$ is a central extension of the definable $SL_2(\mathbb{R})$, so always definable (HPP).

The Levi subgroup of B is $\widetilde{S}$. So even if a group is definable, its Levi subgroup might not be.

Strategy

Let G be a Lie group, R its solvable radical and H a subgroup of G projecting in G/R. Then G is isomorphic to $(H \ltimes_\gamma R)/(H \cap R)$ where γ is conjugation.

If we can build:

- a definable copy R_{def} of R,
- a definable copy H_{def} of H
- both with definable subgroups isomorphic to $R \cap H$, and
- a definable action γ of R_{def} on H_{def} copying the action of H on R by conjugation, then

$(H_{def} \ltimes_\gamma R_{def})/(H_{def} \cap R_{def})$ is isomorphic to G.

Finding H

For G connected Lie group we get an exact sequence

$$0 \to R \to G \to S \to 0.$$

The group $S = G/R$ is semisimple and linear (adjoint representation) so always semialgebraic.

Let L be the Levi subgroup L of G.
$H := L \cdot Z(G)$ is a central extension of $L/(Z(G) \cap L) = S$ so it is "definable".

$G = HR$ with $H \cap R = Z(G)$ and "definability" of G comes down to definability of R and the action.

Definability of R.

The solvable radical

- If R is a definable solvable group, then there is a maximal torsion free (solvable) subgroup U and $K := R/U$ is compact and solvable (so abelian).
- Compact abelian are just copies of the torus

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, \in \mathbb{R}, a^2 + b^2 = 1 \right\}$$

Furthermore,

$$0 \to U \to R \to K \to 0$$

splits. So $R = K \ltimes_\eta U$.

Definability of R comes down to finding a definable copy of U (with η definable).

Examples 3

In the definable category, U is unique. This will play an important role later.

If

$$K := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, c > 0 \right\}$$

and $G = K \times \mathbb{R}$, then there is a unique definable torsion free subgroup $\{Id\} \times \mathbb{R}$. But it has a torsion free *Lie* subgroup $\{(e^{it}, t) \mid t \in \mathbb{R}\}$.

Definability of U.

Definability of U

We can construct a linear group G isomorphic to $\mathbb{R} \ltimes_\gamma \mathbb{R}^2$ where γ is the action of $\mathbb{R}$ on $\mathbb{R}^2$ which rotates t radians.

$$0 \to (\mathbb{R}, +) \times (\mathbb{R}, +) \to G \to (\mathbb{R}, +) \to 0$$

This is, in escence, the only obstruction.

Supersolvable Groups

Definition

A torsion free group is Super solvable, completely solvable or triangular if it admits a series of normal subgroups of G $\{e\} \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_d = G$ *such that* N_{i+1}/N_i *is isomorphic ro* $(\mathbb{R}, +)$.

$$0 \to \mathbb{R} \to G \to \mathbb{R} \times \mathbb{R} \to 0,$$

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem

A torsion free solvable Lie group U has a definable copy if and only if U is completely solvable.

Torsion free definable groups (Necessary)

Lemma

[Conversano, Starchenko, O.] Let U be a torsion free group definable in an o-minimal expansion of a real field. Then U has a one dimensional normal abelian subgroup.

Theorem

[Conversano, Starchenko, O.] Let U be a torsion free group definable in an o-minimal expansion of a real field. Then U is supersolvable.

Classification of "definable" torsion free solvable Lie groups (Sufficient)

Theorem (Conversano, Starchenko, O.)

$\mathcal{R}$ be an o-minimal theory expanding and let U be a torsion free (solvable) Lie group, and $\mathfrak{t}$ its Lie algebra. Then:

- *(Ado)$\mathfrak{t}$ is supersolvable, so it is isomorphic to a subalgebra of upper triangular matrices.*
- *(Dixmier) The image under Exp of a vector in $\mathfrak{t}$ is a subgroup in $GL_n(\mathcal{R})$ isomorphic to U.*
- *Exp restricted to upper triangular matrices is definable in $\mathcal{R}$.*

Characterizing “definable” Lie groups.

Classification of definable connected Lie groups

Theorem

If G is connected, $Z(G)$ has finitely many connected components and its maximum solvable torsion free normal subgroup U is supersolvable, then G has a definable copy.

Building R_{def}

Classification of solvable connected "definable" groups

- A solvable R with torsion free group U is semidirect product of $K := R/U$ and U.
  Find U upper triangular. Automorphisms of U are in definable correspondance with automorphisms of the Lie algebra (linear). So the automorphisms of U are definable in $\mathbb{R}_{exp}$ (clear).
- If we choose K linear (which we can), then the action is algebraic.
- So we can build a linear group R_{def} definable in $\mathbb{R}_{exp}$.

Building G_{def}

Sketch

- Let H, R as above (so H central extension of the linear semisimple G/R). Find definable copies H_{def} and R_{def}.
- G is then isomorphic to a semidirect product of $H := Z \cdot S$ and R mod out by the intersection.
- The corresponding action from H_{def} to R_{def} factors through S and is then definable.

Definability of isomorphisms.

Can we add Lie isomorphisms between definable groups?

Let G and H be definable in $\mathcal{R}$.
Let $\phi : G \to H$ be a Lie isomorphism.

Can we add ϕ to the language and preserve o-minimality?

Fact (Speissegger)

If $\mathcal{R}$ is any o-minimal expansion of the real field, there is an o-minimal expansion $\mathcal{R}_{Pfaff}$.

$\mathcal{R}_{Pfaff}$ includes in particular the exponential. So this strengthens Wilkie's.

Can we add Lie isomorphisms between definable groups?

Let G and H be definable in $\mathcal{R}$, let H_G, U_G and K_G the components of G: Let $\phi : G \to H$ be a Lie isomorphism between $\mathcal{R}$-definable groups. Two obstructions (opportunities):

1. ϕ must send the definable components of G to definable components of H.
2. The restriction of ϕ to H_G, U_G and K_G each definable component must be definable in an o-minimal expansion.

Can we add Lie isomorphisms between definable groups. (About (1)):

- H_G and H_H are always definable and characteristic. ϕ sends H_G to H_H.
- K_G and $\phi(K_G)$ need not be definable, not always definable, but they are definable in $\mathcal{R}_{Pfaff}$. So we can add them and keep o-minimality.
- There is no reason why $\phi(U_G) = U_H$:

Can we add Lie isomorphisms between definable groups? (About (2))

Let G and H be definable in $\mathcal{R}$, let H_G, U_G and K_G the components of G.

We can reduce (2) to understanding the restriction of definable maps between definable abelian subgroups of G and H.

- $H_G/Z(H_G) = S_G$ is definably linear and semisimple. The induced map on S_G is always definable. Only need the isomorphism between $Z(G)$ and $Z(H)$ to be definable.
- K_G and K_H are abelian and compact and so is the isomorphism between them (all definable in $\mathcal{R}_{Pfaff}$).
- Both U_G and U_H are products of 1-dimensional extensions of $Z(G)$ and $Z(H)$ so they are products of abelian subgroups.

Can we add Lie isomorphisms between definable groups?

Theorem

If A, B are $\mathcal{R}$-definable abelian groups and ϕ is a Lie isomorphism between A and B, then ϕ is definable in $\mathcal{R}_{Pfaff}$.

Theorem

Let G and H be definable in $\mathcal{R}$, let R_G, R_H be their solvable radicals and let U_G, U_H be the maximal torsion free normal subgroups of R_G and R_H, respectively.

Let ϕ be a Lie isomorphism between G and H.

Then ϕ is definable in $\mathcal{R}_{Pfaff}$ if and only if $\phi(U_G) = U_H$.

Without going to $\mathcal{R}_{Pfaff}$, is ϕ definable?

We can again restrict to abelian subgroups

- Between compact abelian, not always.
- Additive and multplicative, depends on definability of e^x.
- Between other one dimensional torsion free groups?
- If G is abelian and

$$0 \to (\mathbb{R}, +) \xrightarrow{f} G \xrightarrow{g} (\mathbb{R}, +) \to 0$$

with f, g definable is G isomorphic to $(\mathbb{R}, +)^2$?

Thanks