# What is Set Theory with Urelements?

Bokai Yao
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## 1 Urelements: What and Why

Urelements: members of sets that are not themselves sets.

Why urelements?

(a) Set theory with absolute generality: quantify over everything.

(b) A different mathematical theory: many equivalent statements come apart.

(c) Philosophy of set theory: how are the axioms of set theory justified in this broader context?

## 2 $\mathbf{ZFCU_R}$

### 2.1 The iterative conception of set

Sets are formed stage-by-stage. The universe of sets $V$ consists of all the stages.

The ZF set theory is commonly viewed as a standard theory for $V$.

### 2.2 Basic axioms

What is a standard theory for the set-theoretic universe which includes everything?

$\mathscr{A}(x)$ : $x$ is a urelement. $Set(x)$ : $\neg\mathscr{A}(x)$.

(Axiom $\mathscr{A}$) No urelement has members.

(Extensionality) Co-extensional *sets* are identical.

(Foundation) Every nonempty set has an $\in$-minimal element.

(Pairing) $\{x, y\}$ is a set for every $x$ and $y$.

(Union) $\bigcup x$ is a set for every set $x$.

(Powerset) $\{y \mid y \subseteq x \wedge Set(y)\}$ (i.e., $P(x)$) is a set for every set $x$.

(Separation)$\{y \mid y \in x \wedge \varphi(y, u)\}$ is a set for every $x$ and $u$.

(Infinity) An inductive set exists.

(Replacement) $\forall w, u(\forall x \in w\ \exists! y \varphi(x, y, u) \rightarrow \exists v \forall x \in w\ \exists y \in v\ \varphi(x, y, u))$

(AC) Every set is well-orderable.

**Definition 1.**
$\mathrm{ZFU_R}$ = Axiom $\mathscr{A}$ + Extensionality + Foundation + Pairing + Union + Powerset + Infinity + Separation + Replacement.
$\mathrm{ZFCU_R}$ = $\mathrm{ZFU_R}$ + AC.

## 2.3 The iterative conception with urelements

**Definition 2.** For every set $A \subseteq \mathscr{A}$,

$V_0(A) = A$;

$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup V_\alpha(A)$;

$V_\gamma(A) = \bigcup_{\alpha<\gamma} V_\alpha(A)$, where $\gamma$ is a limit;

$V(A) = \bigcup_{\alpha \in Ord} V_\alpha(A)$.

$\mathrm{ZFU_R}$ proves that $U($ i.e., $\{x : x = x\}) = \bigcup_{A \subseteq \mathscr{A}} V(A)$.

## 2.4 Collection and reflection

(Collection) $\forall w, u(\forall x \in w \exists y \varphi(x, y, u) \to \exists v \forall x \in w \exists y \in v \varphi(x, y, u))$.

(RP) For every $x$ there is a transitive set $t$ extending $x$ such that for every $v_1, ..., v_n \in t$, $\varphi(v_1, ..., v_n) \leftrightarrow \varphi^t(v_1, ..., v_n)$.

($\mathrm{RP}^-$) If $\varphi(v_1, ..., v_n)$, then there is a transitive set $t$ containing $v_1, ...v_n$ such that $\varphi^t(v_1, ..., v_n)$.

RP $\to$ ($\mathrm{RP}^-$ $\wedge$ Collection). $\mathscr{A}$ is a set $\to$ RP.

## 2.5 Urelement axioms

**Definition 3.** Let $A, B \subseteq \mathscr{A}$.

1. $x$ is *realized by* $A$ if $x$ and $A$ are equinumerous ($x \sim A$).
2. $B$ *duplicates* $A$ if $B$ and $A$ are disjoint and $A \sim B$.
3. $B$ is a *tail* of $A$ if $B$ is disjoint from $A$ and every $C \subseteq \mathscr{A}$ disjoint from $A$ injects into $B$.

(Plenitude) Every initial ordinal is realized.

(Closure) The supremum of a set of realized cardinals is realized.

(Duplication) Every set of urelements has a duplicate.

(Tail) Every set of urelements has a tail.

Plenitude $\to$ Closure $\wedge$ $\neg$Tail. Assuming AC, Plenitude $\to$ Duplication.

## 2.6 Interpreting $U$ in $V$

**Definition 4.** Let $V$ be a model of ZF and $X$ be a class of $V$. Let $\bar{\mathscr{A}} = \{0\} \times X$.

$$V[\![X]\!] := \bar{\mathscr{A}} \cup \{\bar{x} \in V : \exists x(\bar{x} = \langle 1, x \rangle \wedge x \subseteq V[\![X]\!])\}.$$

For every $\bar{x}, \bar{y} \in V[\![X]\!]$, $\bar{x} \mathrel{\bar{\in}} \bar{y}$ if and only if $\exists y(\bar{y} = \langle 1, y \rangle \wedge \bar{x} \in y)$.

**Theorem 5.**

1. $V[\![X]\!] \models \mathrm{ZFU_R} + \mathrm{RP}$.
2. $V[\![X]\!] \models \mathrm{AC}$ if and only if $V \models \mathrm{AC}$.
3. $V[\![\mathrm{Ord}]\!] \models$ Plenitude.

## 3 A hierarchy in $\mathrm{ZFCU_R}$

**Definition 6.** For every $x$, $ker(x)$ is the set of urelements in the transitive closure of $\{x\}$.

For every $x$, $x \in V(A)$ if and only if $ker(x) \subseteq A$.

**Definition 7.** If $\pi$ is a permutation of some $A \subseteq \mathscr{A}$, then the $\in$-extension of $\pi$ is the class function such that

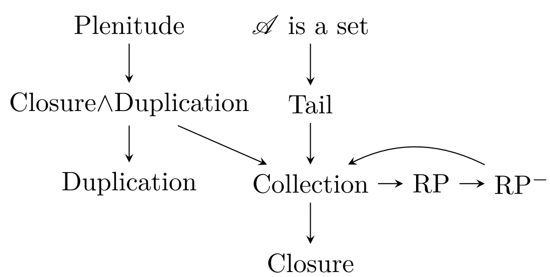
$$\pi x = \begin{cases} \pi x & \text{if } x \in A \\ x & \text{if } x \in \mathscr{A} - A \\ \{\pi y : y \in x\} & \text{otherwise} \end{cases}$$

$\pi$ is an automorphism of $U$. $\pi$ point-wise fixes $x$ if $\pi$ point-wise fixes $ker(x)$.

**Definition 8.** *Homogeneity holds over* $A$, if whenever $B \cup C \subseteq \mathscr{A}$ is disjoint from $A$ and $B \sim C$, there is an automorphism $\pi$ such that $\pi B = C$ while point-wise fixing $A$.

**Lemma 9** ($\mathrm{ZFCU_R}$)**.** Every $A \subseteq \mathscr{A}$ can be extended to some $D$ over which homogeneity holds.

**Theorem 10** ($\mathrm{ZFCU_R}$)**.** The following implication diagram holds and is complete.




**Corollary 10.1** ($\mathrm{ZFCU_R}$)**.** The following are equivalent.

1. RP
2. $\mathrm{RP}^-$
3. Collection
4. Plenitude $\vee$ Tail

**Definition 11** (**Small Kernel Model**)**.** A (definable) class $\mathscr{I}$ of sets of urelements is an *$\mathscr{A}$-ideal* if

1. $\mathscr{A} \notin \mathscr{I}$;
2. if $A, B \in \mathscr{I}$, then $A \cup B \in \mathscr{I}$;
3. if $A \in \mathscr{I}$ and $B \subseteq A$, then $B \in \mathscr{I}$; and
4. for every $a \in \mathscr{A}$, $\{a\} \in \mathscr{I}$.

$$U^{\mathscr{I}} := \{x : ker(x) \in \mathscr{I}\}.$$

**Theorem 12** ($\mathrm{ZFU_R}$)**.**

1. $U^{\mathscr{I}} \models \mathrm{ZFU_R}$ + "$\mathscr{A}$ is a proper class".
2. $U^{\mathscr{I}} \models \mathrm{AC}$ if $U \models \mathrm{AC}$.

**Theorem 13.** Over $\mathrm{ZFCU_R}$,

1. Duplication $\nrightarrow$ Closure.
2. Closure $\wedge$ Duplication $\nrightarrow$ Plenitude.
3. Collection $\nrightarrow$ Duplication.

# 4 $\mathbf{ZFU_R}$

What kind of hierarchy do the axioms form without AC?

**Definition 14** (**Permutation Model**)**.** Let $A$ be a set of urelements and $\mathcal{G}$ be a *group* of permutations of $A$. For every $x$, $sym(x) = \{\pi \in \mathcal{G} : \pi x = x\}$; $fix(x) = \{\pi \in \mathcal{G} : \pi y = y \text{ for all } y \in x\}$ if $x$ is a set. An ideal $I$ on $A$ is $\mathcal{G}$*-norma* if $I$ contains contains every urelement singleton and for every $E \in I$ and $\pi \in \mathcal{G}$, $\pi E \in I$. $x$ is *symmetric* (w.r.t. $I$ and $\mathcal{G}$) if $fix(E) \subseteq sym(x)$ for some $E \in I$ ($E$ is said to be a *support* of $x$).

$$W_{(\mathcal{G},I)} := \{x : x \text{ is hereditarily symmetric}\}.$$

**Theorem 15.**

1. $W_{(\mathcal{G},I)} \models \mathrm{ZFU_R}$.
2. $W_{(\mathcal{G},I)} \models$ Collection if $U \models$ Collection.

**Theorem 16.** Over $\mathrm{ZFU_R}$, Plenitude $\not\rightarrow$ (Duplication $\vee$ Collection).

This motivates a stronger version of Plenitude:

(Plenitude$^+$) *Every set* is realized.

**Lemma 17** ($\mathrm{ZFU_R}$)**.** Plenitude$^+$ $\rightarrow$ Duplication.

**Corollary 17.1.** Over $\mathrm{ZFU_R}$, Plenitude $\not\rightarrow$ Plenitude$^+$.

**Open Question 18.** Does $\mathrm{ZFU_R}$ prove any of the following?

1. Plenitude$^+$ $\rightarrow$ Collection
2. Tail $\rightarrow$ Collection
3. Collection $\rightarrow$ $\mathrm{RP}^-$
4. $\mathrm{RP}^-$ $\rightarrow$ Collection

# 5 Philosophical Questions

Is the iterative conception robust enough?

What justifies Collection?

If Collection $\not\rightarrow$ $\mathrm{RP}^-$ over $\mathrm{ZFU_R}$, what justifies $\mathrm{RP}^-$?

If $\mathrm{RP}^-$ $\not\rightarrow$ RP over $\mathrm{ZFU_R}$, are there two conceptions of reflection?