Fudan notes on UA

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Abstract

The subject of this tutorial is the Ultrapower Axiom...

1 Ultrapowers of models of set theory

Tarski set the stage for modern large cardinal theory by posing the following questions:

- Is the least inaccessible cardinal less than the least weakly compact cardinal?
- Is the least weakly compact cardinal less than the least measurable cardinal?
- Is the least measurable cardinal less than the least strongly compact cardinal?

Recall that an uncountable cardinal κ is:

- *inaccessible* if it is regular and for all $\lambda < \kappa$, $2^\lambda < \kappa$.
- *weakly compact* if every tree $T \subseteq 2^{<\kappa}$ containing sequences of arbitrary length below κ has a branch of length κ.[1]
- *measurable* if there is a κ-complete non-principal ultrafilter on κ.
- *strongly compact* if every κ-satisfiable theory in the infinitary logic $\mathcal{L}_{\kappa,\omega}$ is satisfiable.

We have deliberately chosen these definitions to emphasize the diverse subjects in which large cardinals arise; e.g., cardinal arithmetic, combinatorics, measure theory, and model theory. Note also that all these properties hold of the cardinal $\aleph_0$, except that $\aleph_0$ is countable.

The modern approach to Tarski's questions involves unifying these disparate concepts into the framework of ultrapowers of models of set theory.

For an example of this approach, let us prove that the least measurable cardinal is greater than the least weakly compact cardinal. Here is the idea. Let U be a κ-complete non-principal ultrafilter on κ. Let M_U be the ultrapower of the universe of sets $(V, \in)$ by U, i.e., $M_U = V^\kappa/U$, and let $j_U : V \to M_U$ denote the canonical elementary embedding. (More details on this construction are provided below.)

- The structure M_U turns out to be *well-founded*, which means it is isomorphic to a proper class transitive model of ZFC.
- Identifying M_U with this inner model, the κ-completeness of U implies that the embedding j_U fixes every element of V_κ.

[1] Here $2^{<\kappa}$ denotes the set of $\{0, 1\}$-valued sequences of length less than κ.

- The non-principality of U implies j_U sends κ to an ordinal strictly above κ.

How does this help compare the least measurable cardinal with the least weakly compact cardinal? Fix a tree $T \subseteq 2^{<\kappa}$ containing sequences of arbitrary length below κ. Then $j_U(T)$ is a subtree of $2^{<j(\kappa)}$ containing sequences of arbitrary length below $j_U(\kappa)$. In particular, there is some $t \in j_U(T)$ of length exactly κ. For $\alpha < \kappa$, we have $t \restriction \alpha \in V_\kappa$, and so

$$j_U(t \restriction \alpha) = t \restriction \alpha \in j_U(T)$$

By the elementarity of j_U, it follows that $t \restriction \alpha \in T$. Thus t is a branch of T of length κ. It follows that κ is weakly compact, but moreover, κ is weakly compact *in* M. This is because $2^\kappa \subseteq M$. Therefore M satisfies that $j_U(\kappa)$ is larger than the least weakly compact cardinal. It follows by elementarity that κ is larger than the least weakly compact cardinal (in V).

Let us now present the details of the ultrapower construction. Suppose U is an ultrafilter on a set X. Then M_U denotes the usual model-theoretic ultrapower of the universe of set theory V, viewed as a structure $(V, \in)$ in the language of set theory. This is the quotient V^X/U of the class V^X of all functions on X under the equivalence relation $=_U$ on V^X defined by

$$f =_U g \iff \{x \in X : f(x) = g(x)\} \in U$$

Moreover M_U is a structure in the language of set theory when equipped with the binary relation $\in_U$ defined by

$$[f]_U \in_U [g]_U \iff \{x \in X : f(x) \in g(x)\} \in U$$

Here $[f]_U$ denotes the equivalence class in V^X/U of the function $f \in V^X$. (Check that $\in_U$ is well-defined in the sense that it is independent of the choice of f and g.)

One can define an elementary embedding $j_U : V \to M_U$ by

$$j_U(a) = [c_a]_U$$

where $c_a : X \to V$ is the constant function with value a. The map j_U is an *elementary embedding*: if $\varphi(v_1, \ldots, v_n)$ is a formula in the language of set theory and $a_1, \ldots, a_n \in V$, then

$$V \vDash \varphi(a_1, \ldots, a_n) \iff M_U \vDash \varphi(j_U(a_1), \ldots, j_U(a_n))$$

This is a consequence of Łoś's theorem, the "fundamental theorem of ultrapowers":

Theorem 1.1 (Łoś). *For any* $f_1, \ldots, f_n \in V^X$,

$$M_U \vDash \varphi([f_1]_U, \ldots, [f_n]_U) \iff \{x \in X : V \vDash \varphi(f_1(x), \ldots, f_n(x))\} \in U$$

Exercise 1. Prove Łoś's theorem. (Note that it is technically a theorem scheme.)

An ultrafilter U is *κ-complete* if U is closed under intersections of size less than κ; that is, for all $\sigma \subseteq U$ with $|\sigma| < \kappa$, $\bigcap \sigma \in U$. Thus every ultrafilter is $\aleph_0$-complete, and every principal ultrafilter is κ-complete for all κ. A nonprincipal ultrafilter on ω is never $\aleph_1$-complete, for the same reason that finite sets do not carry nonprincipal ultrafilters. More generally, if U is a nonprincipal ultrafilter on a set X, the *completeness of* U is the cardinal

$$\kappa_U = \sup\{\kappa \leq |X| : U \text{ is } \kappa\text{-complete}\}$$

Thus κ_U is the largest cardinal κ such that U is κ-complete or equivalently, the least cardinal κ such that U is not κ^+-complete.

Exercise 2. The structure $(M_U, \in_U)$ is well-founded if and only if U is $\aleph_1$-complete, or in other words closed under countable intersections.

By Łoś's theorem, M_U is extensional, so $(M_U, \in_U)$ is well-founded, then the Mostowski collapsing lemma yields a unique isomorphism from $(M_U, \in_U)$ to $(N, \in)$ where N is a transitive proper class. A transitive proper class N such that $(N, \in) \vDash \text{ZF}$ is called an *inner model*.

If U is $\aleph_1$-complete, we will *identify* M_U with the inner model to which it is isomorphic, so $[f]_U$ will for us denote an element of N. The statement of the following exercise is an example of this identification in practice:

Exercise 3. If U is an $\aleph_1$-complete nonprincipal ultrafilter, then κ_U is the least cardinal κ such that $j_U(\kappa) \neq \kappa$.

If $j : Q \to M$ is an elementary embedding between transitive classes, the *critical point of* j is the least ordinal κ such that $j(\kappa) > \kappa$.

Exercise 4 (Scott)**.** Show that an uncountable cardinal κ is measurable if and only if it is the critical point of an elementary embedding from the universe of sets into an inner model.

If you are struggling with the converse direction, see Section 3.

Here is another example of the utility of the transitive collapse:

Exercise 5. Show that if U is a κ-complete nonprincipal ultrafilter on κ, then $2^\kappa < j_U(\kappa) < (2^\kappa)^+$.

Exercise 6. Show that the theory ZFC + a weakly compact does not imply the existence of a measurable cardinal.

The proof is identical to the proof that ZFC does not imply the existence of an inaccessible cardinal.

Exercise 7. Prove in detail that if κ is measurable, then the set of weakly compact cardinals less than κ is unbounded.

2 Exercises on weakly compact cardinals

The exercises in this subsection answer one of Tarski's questions by showing that the least weakly compact cardinal is a limit of inaccessible cardinals.

Suppose X is a set. An *algebra of subsets of* X is a family $\mathcal{A} \subseteq P(X)$ that contains X and is closed under intersections and complements. An ultrafilter on $\mathcal{A}$ is a subset $U \subseteq \mathcal{A}$ obeying the ultrafilter axioms, or if you prefer, the preimage of 1 under a homomorphism $h : \mathcal{A} \to \{0,1\}$, where $\{0,1\}$ is endowed with operations of intersection and complement in the natural way. An ultrafilter U on $\mathcal{A}$ is κ-complete if for every $\sigma \subseteq U$ with $|\sigma| < \kappa$, $\bigcap \sigma \neq \emptyset$. (Note that we cannot require $\bigcap \sigma \in U$ since we are not assuming that $\mathcal{A}$ is closed under infinite intersections.)

Exercise 8. A cardinal κ is weakly compact if and only if for every algebra $\mathcal{A}$ of subsets of κ with $|\mathcal{A}| = \kappa$, there is a κ-complete nonprincipal ultrafilter on $\mathcal{A}$.

The forwards direction, which is all we need to answer Tarski's question, is a straightforward combinatorial proof involving the following tree T. Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be an algebra of subsets of κ. Then T is the tree of all $s \in 2^{<\kappa}$ such that for some $\beta > |s|$, $\beta \in \bigcap_{\alpha \in s} A_\alpha$.

The converse can be shown by repeating the proof that measurable cardinals are weakly compact, which requires generalizing the ultrapower construction to smaller algebras of sets. Suppose Q is a transitive model of $\text{ZFC}^- = \text{ZFC} - \text{Powerset}$.[2] If $X \in Q$, a family U of subsets of X is an *Q-ultrafilter on X* if U is an ultrafilter on the

[2] In formulating this theory, one should be careful to use the collection schema rather than the replacement schema.

algebra of sets $P(X) \cap Q$. The ultrapower of Q by U, denoted by M_U^Q, is the quotient of $Q^X \cap Q$, the class of functions $f : X \to Q$ with $f \in Q$, under the equivalence relation $=_U$ defined by

$$f =_U g \iff \{x \in X : f(x) = g(x)\} \in U$$

Thus we consider the usual model-theoretic ultrapower *except that only functions that belong to the model M are used*. We turn M_U^Q into a structure in the language of set theory by equipping it with the relation $\in_U$ defined by

$$[f] \in_U [g] \iff \{x \in X : f(x) \in g(x)\} \in U$$

Exercise 9. State and prove Łoś's theorem for M_U^Q.

Exercise 10. Show that κ is weakly compact if and only if for every model $Q \vDash \text{ZFC}^-$ such that $|Q| = \kappa$, there is a transitive model M and an elementary embedding $j : Q \to M$ with critical point κ.

Exercise 11. Show that every measurable cardinal is an inaccessible cardinal and a limit of inaccessible cardinals.

The easiest way to do this is to use the characterization of measurable cardinals as critical points of elementary embeddings $j : V \to M$.

Exercise 12. Show that every weakly compact cardinal is an inaccessible cardinal and a limit of inaccessible cardinals.

The easiest way to do this is to generalize the solution to the previous exercise to smaller models of ZFC^-.

Exercise 13. Conclude that ZFC + a proper class of inaccessible cardinals does not imply the existence of a weakly compact cardinal.

3 Derived ultrafilters

If Q and M are models of ZFC^- and $j : Q \to M$ is an elementary embedding, then for each $X \in Q$ and $a \in j(X)$, one obtains an associated Q-ultrafilter on X called the *Q-ultrafilter derived from j using a*. This is the ultrafilter U consisting of all $A \in P(X) \cap Q$ such that $a \in j(A)$.

Exercise 14. There is a unique elementary embedding $k : M_U^Q \to M$ such that $k \circ j_U = j$ and $k([\text{id}]_U) = a$.

Hint: define $k([f]_U) = j(f)(a)$ and use Łoś's theorem.

An elementary embedding $j : Q \to M$ is an *ultrapower embedding* if there is some $X \in Q$ and $a \in j(X)$ such that every element of M is definable in M from parameters in $j[Q] \cup \{a\}$.

Exercise 15. Show that $j : Q \to M$ is an ultrapower embedding if and only if there is an ultrafilter $U \in Q$ and an isomorphism $k : M_U^Q \to M$ such that $k \circ j_U = j$.

Thus there is a correspondence between $\aleph_1$-complete ultrafilters U on X and pairs (j, a) such that $j : V \to M$ is elementary and $a \in j(X)$ witnesses that j is an ultrapower embedding.

In particular, an ultrafilter U on X is determined by the ultrapower embedding j_U and the point $[\text{id}]_U$.

The diagonal intersection of a sequence $\langle A_\alpha \rangle_{\alpha<\kappa}$ of subsets of κ is the set

$$\triangle_{\alpha<\kappa} A_\alpha = \{\beta < \kappa : \forall \alpha < \beta \, \beta \in A_\alpha\}$$

A filter on κ is *normal* if it is closed under diagonal intersections.

We remind you of the key example of a normal filter: if κ is regular, then the closed unbounded filter $\mathcal{C}_\kappa$ on κ is a κ-complete normal filter. Recall that $C \subseteq \kappa$ is *closed* if it is closed in the order topology on κ, or, more concretely, if for all $\alpha < \kappa$, $\sup(C \cap \alpha) \in C$. For any ordinal κ, $\mathcal{C}_\kappa$ denotes the family of all subsets of κ that contain a closed unbounded subset of κ.

Exercise 16. Assume $\text{cf}(\kappa) \geq \aleph_1$. Show that $\mathcal{C}_\kappa$ is $\text{cf}(\kappa)$-complete and *weakly normal* in the sense that $\mathcal{C}_\kappa$ is closed under *decreasing* diagonal intersections.

If F is a filter on X, a set S is *F-positive* if it intersects every set $A \in F$. The collection of F-positive subsets of X is denoted by F^+. A function f on a set of ordinals S is *regressive* if for all $\alpha \in S$, $f(\alpha) < \alpha$.

Exercise 17. Prove that a filter F on κ is normal if and only if it satisfies *Fodor's lemma*: if $f : S \to \kappa$ is a regressive function on an F-positive set S, then there is an F-positive set $T \subseteq S$ on which f is constant.

Exercise 18. Show that a κ-complete ultrafilter on κ is normal if and only if $[\text{id}]_U = \kappa$.

Combined with the derived ultrafilter construction, this yields a clearer view of the objects associated with measurability:

Exercise 19 (Scott)**.** Show that κ is measurable if and only if there is a κ-complete nonprincipal normal ultrafilter on κ.

Going forward, we will use the term *normal ultrafilter on κ* to abbreviate the clumsier phrase "κ-complete nonprincipal normal ultrafilter on κ."

Exercise 20. Show that if U is a normal ultrafilter on κ, then the set of weakly compact cardinals less than κ belongs to U.

This shows that a measurable cardinal is not only larger than the first weakly compact cardinal, but moreover is the limit of a "large set" of weakly compact cardinals.

4 Exercises on strong compactness

Tarski's question on strongly compact cardinals is much more subtle than the other questions.

Exercise 21. Show that every strongly compact cardinal is measurable.

Here is a hint. First, assume that κ is a regular cardinal. For each $\alpha < \kappa$, let T_α be the $\mathcal{L}_{\kappa,\omega}$-theory of the structure $(P(\kappa), \alpha, \in)$ with parameters from $P(\kappa)$. Let

$$T_* = \lim_{\alpha \to \kappa} T_\alpha = \bigcup_{\beta < \kappa} \bigcap_{\alpha \in (\beta, \kappa)} T_\alpha$$

Use the regularity of κ to prove that T_* is κ-satisfiable, and show that any model of T_* gives rise to a κ-complete ultrafilter on κ.

To remove the assumption that κ is regular, show by a similar argument that if κ were singular and strongly compact, then κ^+ would be measurable, contradicting that measurable cardinals are regular.

Exercise 22. Show that an uncountable cardinal κ is strongly compact if and only if every κ-complete ultrafilter on an algebra $\mathcal{A}$ of subsets of a set X can be extended to a κ-complete ultrafilter on X.[3]

[3]The exercise highlights an issue with the standard terminology for ultrafilters: an ultrafilter on an algebra $\mathcal{A}$ is a certain kind of subset of $\mathcal{A}$, whereas an ultrafilter on a set X is a subset of $P(X)$, more specifically, an ultrafilter on the algebra $P(X)$. This causes ambiguity when we say something like "an ultrafilter on $P(X)$," if we do not specify whether we view $P(X)$ as an algebra or as a set. Such ambiguities are usually easily resolved in context, but some authors prefer to reserve the term "ultrafilter on $\mathcal{A}$" for ultrafilters on algebras, and "ultrafilter over X" for ultrafilters on the algebra $P(X)$.

Another way to state this is that every κ-complete filter (on any set) can be extended to a κ-complete ultrafilter.

Solovay introduced the concept of a *supercompact* cardinal in an attempt to answer Tarski's question.

Definition 4.1. A cardinal κ is λ-*supercompact* if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and every $A \subseteq M$ with $|A| \leq \lambda$ belongs to M. The cardinal κ is *supercompact* if it is λ-supercompact for all λ.

Exercise 23. Show that every measurable cardinal κ is κ-supercompact.

Suppose Y is a family of subsets of X such that $\bigcup Y = X$. An ultrafilter $\mathcal{U}$ on the set Y is *fine* if for every $x \in X$, $\{\sigma \in Y : x \in \sigma\}$ belongs to $\mathcal{U}$.

If $\langle A_x \rangle_{x \in X}$ is a sequence of subsets of Y, its diagonal intersection is the set

$$\triangle_{x \in X} A_x = \{\sigma \in Y : \forall x \in \sigma \, \sigma \in A_x\}$$

The ultrafilter $\mathcal{U}$ is *normal* if it is closed under diagonal intersections.

Exercise 24. Show that κ is λ-supercompact if and only if there is a κ-complete normal fine ultrafilter on $P_\kappa(\lambda)$, where $P_\kappa(\lambda)$ denotes the set of all $\sigma \subseteq \lambda$ such that $|\sigma| < \kappa$.

Definition 4.2. A cardinal κ is λ-*strongly compact* if there is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$.

Exercise 25. Show that κ is λ-strongly compact if and only if there is an elementary embedding $j : V \to M$ with critical point κ such that every set $A \subseteq M$ with $|A| \leq \lambda$ is included in a set $B \in M$ with $|B|^M < j(\kappa)$.

Exercise 26. Show that κ is strongly compact if and only if κ is λ-strongly compact for all cardinals λ.

Solovay conjectured that every strongly compact cardinal is supercompact, which if true would have settled Tarski's question positively:

Exercise 27. Show that if κ is 2^κ-supercompact, then κ is larger than the least measurable cardinal.

But in fact, such a simple answer is not possible:

Theorem 4.3 (Magidor)**.** *If it is consistent that there is a strongly compact cardinal, then it is consistent that the least measurable cardinal is the least strongly compact cardinal.*

On the other hand, if it is consistent that there is a supercompact cardinal, then it is consistent that the least strongly compact is the least supercompact cardinal.

We will return to this problem in the context of the Ultrapower Axiom (Proposition 12.2 and Theorem 13.3).

5 Scott's theorem

The constructible universe L is an inner model discovered by Gödel in the 1940s in the course of his proof of the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis. The construction plays a major role in modern set theory.

The inner model L is built up by recursively closing under the operations required to satisfy the ZFC axioms. For this reason, L turns out to be the minimum inner model of ZF. If M is a transitive set, let $\text{def}(M)$ denote the set of all subsets of M that are definable over $(M, \in)$ from parameters. The *constructible hierarchy* is the sequence of transitive sets L_α defined by transfinite recursion for all ordinals α by setting $L_0 = \emptyset$, $L_{\alpha+1} = \text{def}(L_\alpha)$, and for limit ordinals λ by $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

The constructible universe is the class $L = \bigcup L_\alpha$. Gödel proved that L is an inner model that satisfies the Axiom of Choice and the Generalized Continuum Hypothesis. Since he proved this in ZF alone, he was able to demonstrate that if ZF is consistent, so is ZFC + GCH.

The *Axiom of Constructibility* states that every set belongs to L, so the universe of sets V is equal to the constructible universe L. Thus the Axiom of Constructibility is often abbreviated by $V = L$.

The Axiom of Constructibility is true in L and so it is consistent with ZFC. When Gödel first demonstrated this, it was an open question whether $V = L$ was independent of ZFC. Gödel conjectured that it was, and this was later established by Cohen. Several years earlier, however, Scott proved the independence of $V = L$ from ZFC assuming the consistency of measurable cardinals.

Theorem 5.1 (Scott)**.** *If there is a measurable cardinal, then $V \neq L$.*

Proof. Assume towards a contradiction that $V = L$. Let κ be the smallest measurable cardinal. Let $j : V \to M$ be an elementary embedding with critical point κ from the universe of sets V into an inner model M. Since $V = L$ and L is the minimum inner model of ZF, $M = L = V$. Therefore M satisfies that κ is the smallest measurable cardinal. But since j is an elementary embedding, M satisfies that $j(\kappa)$ is the smallest measurable cardinal. But by the definition of a critical point, $j(\kappa) \neq \kappa$! □

Corollary 5.2. *If* ZFC *is consistent with a measurable cardinal, then* ZFC *does not prove $V = L$.*

While the constructible universe contains no measurable cardinals, it does contain some large cardinals. For example, every inaccessible cardinal is inaccessible in L by a simple absoluteness argument. (Inaccessibility is Π_1 expressible in the Lévy hierarchy.) The following is a significantly more subtle example of the absoluteness of large cardinals to the constructible universe:

Exercise 28. Show that if κ is weakly compact, then κ is weakly compact in L.

Let's also include this quote from Gödel:

> The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. In this connection it is important that the consistency-proof for A does not break down if stronger axioms of infinity (e.g., the existence of inaccessible numbers) are adjoined to T. Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase.

Here Gödel denotes by A the hypothesis $V = L$ and by T the class theory NBG, but you might as well think about ZFC instead.

6 Minimal models of measurability

Scott's theorem raises the question: are measurable cardinals consistent with the Axiom of Choice and the Generalized Continuum Hypothesis, assuming measurable cardinals are consistent at all? More vaguely, is there an extension of ZFC + a measurable cardinal that naturally completes this theory in the sense that the Axiom of Constructibility naturally completes ZFC?

The inner model $L[U]$ was devised to answer these questions. If A is a set, the inner model $L[A]$ denotes the class of all sets constructible relative to A, the smallest inner model M such that $A \cap M \in M$. The existence of this model is proved by defining the constructible hierarchy relative to A, which is exactly like the constructible hierarchy except that the def operation is replaced with the operation def_A which maps a transitive set M to the set of all subsets of M definable from parameters over the structure $(M, \in, A \cap M)$. Thus $L_0[A] = \emptyset$, $L_{\alpha+1}[A] = \text{def}_A(L_\alpha[A])$, and for limit ordinals λ, $L_\lambda[A] = \bigcup_{\alpha<\lambda} L_\alpha[A]$. Finally, $L[A] = \bigcup_{\alpha\in\text{Ord}} L_\alpha[A]$.

The inner model $L[U]$ is just the special case where $A = U$ for some normal ultrafilter U on κ.

Exercise 29. Prove in ZF that if κ is measurable and W is a κ-complete nonprincipal ultrafilter on κ, then κ is measurable in $L[W]$. Conclude that ZF plus a measurable cardinal is equiconsistent with ZFC plus a measurable cardinal.[4]

We will show that the inner model $L[U]$ is canonical in the sense that it does not depend on the choice of U. In fact, let us prove something stronger. We say a pair (M, U) is a *minimal model with a measurable cardinal* if M is a transitive model of ZFC^-, U is an element of M, M satisfies that U is a normal ultrafilter on κ for some $\kappa \in \text{Ord}^M$, and $M \vDash V = L[U]$.

Exercise 30. If (M, U) is a minimal model with a measurable cardinal, then M satisfies that κ_U is the unique measurable cardinal.

Hint: By generalizing Scott's theorem, one can show that if $M \vDash V = L[A]$ where $A \in V_\alpha^M$, then there are no measurable cardinals above α in M. This easily shows that if (M, U) is a minimal model with a measurable cardinal, then there are no measurable cardinals in M above κ_U. The proof that there are no measurable cardinals below κ_U in M is a bit trickier. The following fact is useful here: if D is an $\aleph_1$-complete ultrafilter on λ and U is a normal ultrafilter on $\kappa > \lambda$, then $j_D(U) = U \cap M_D$. Another (harder) proof of this exercise can be obtained by considering the following main theorem of this section.

Theorem 6.1 (Kunen)**.** *If (M, U) and (N, W) are minimal inner models with measurable cardinals and $\kappa_U \leq \kappa_W$, then N is a definable inner model of M. Moreover, there is an elementary embedding from M to N that is definable over M (using κ_W as a parameter).*

In particular, this shows that if $\kappa_U = \kappa_W$, then $M = N$ and $U = W$. In any case, (M, U) and (N, W) have the same first-order theory.

The general technique from which this theorem follows is the method of *comparison by iterated ultrapowers*. If M is a model of ZFC^- and M satisfies that $U \in M$ is an $\aleph_1$-complete ultrafilter, the iterated ultrapower of M by U is the linear directed system

$$\langle \text{Ult}_\alpha(M, U), j_{\xi\alpha} : \xi \leq \alpha \in \text{Ord} \rangle$$

defined by recursively taking ultrapowers by U and its images:

- $\text{Ult}_0(M, U) = (M, U)$.
- $j_{\alpha\alpha} = \text{id}$ for all ordinals α.
- $U_\alpha = j_{0\alpha}(U)$.
- For $\xi \leq \alpha$, $j_{\xi\alpha+1} = j_{U_\alpha}^{\text{Ult}_\alpha(M,U)} \circ j_{\xi\alpha}$.
- $\text{Ult}_{\alpha+1}(M, U) = (M_{U_\alpha}^{\text{Ult}_\alpha(M,U)}, U_{\alpha+1})$.

[4]Note that ZF does not prove that every measurable cardinal κ carries a κ-complete nonprincipal normal ultrafilter, yet the exercise shows that there is an inner model where such an ultrafilter exists!

- For λ a limit, $\mathrm{Ult}_\lambda(M,U) = \lim\langle \mathrm{Ult}_\alpha(M,U), j_{\xi\alpha}\rangle_{\xi\le\alpha<\lambda}$,
- For λ a limit and $\xi < \lambda$, $j_{\xi\lambda} : \mathrm{Ult}_\xi(M,U) \to \mathrm{Ult}_\lambda(M,U)$ is the direct limit map.

If $\mathrm{Ult}_\alpha(M,U)$ is well-founded, we identify it with its transitive collapse. We say (M,U) is *iterable* if for all $\alpha \in \mathrm{Ord}$, $\mathrm{Ult}_\alpha(M,U)$ is well-founded.

Theorem 6.2 (Kunen)**.** *If M is an inner model of* ZFC *and M satisfies that $U \in M$ is an $\aleph_1$-complete ultrafilter, then (M,U) is iterable.*

Exercise 31. Prove Kunen's iterability theorem.

Hint: consider the least ordinal α such that $\mathrm{Ult}_\alpha(M,U)$ is illfounded, and note that α is a limit ordinal. Let η be the least ordinal such that $j_{0\alpha}(\eta)$ is illfounded. Fix $\nu < j_{0\alpha}(\eta)$ such that ν is illfounded. Find $\xi < \alpha$ such that $\nu \in \mathrm{ran}(j_{\xi\alpha})$. Show that $j_{0\xi}(\eta)$ is the least ordinal β such that $j_{\xi\alpha}(\beta)$ is illfounded, and use $\bar\nu = j_{\xi\alpha}^{-1}(\nu)$ to contradict the minimality of $j_{0\xi}(\eta)$.

Exercise 32. Suppose M is an iterable model of ZFC^- and M satisfies that $U \in M$ is a κ-complete ultrafilter on κ. Let $\kappa_\xi = \kappa_{U_\xi}$. Show that the class $\{\kappa_\xi : \xi \in \mathrm{Ord}\}$ is closed under suprema. Use this to show that if $\delta > 2^\kappa$ is a regular cardinal, then $j_{0\delta}(\kappa) = \delta$.

Hint: for the second part, use Exercise 5.

The following is Kunen's key lemma, showing that every normal measure "looks like" a restriction of the closed unbounded filter:

Lemma 6.3. *Suppose (M,U) is an iterable model with a measurable cardinal κ and $\delta > 2^\kappa$ is a regular cardinal. If $(M_\delta, U_\delta) = \mathrm{Ult}_\delta(M,U)$, then $U_\delta = \mathcal{C}_\delta \cap M_\delta$ where $\mathcal{C}_\delta$ denotes the closed unbounded filter on δ.*

Sketch. For $\xi < \delta$, let $\kappa_\xi = \kappa_{U_\xi}$. For $A \subseteq \delta$, we have $A \in U_\delta$ if and only if $j_{\xi\delta}^{-1}[A] \in U_\xi$ for all sufficiently large $\xi < \delta$, or equivalently $\kappa_\xi \in j_{\xi+1\delta}^{-1}[A]$, or equivalently, $\kappa_\xi \in A$. So $A \in U_\delta$ if and only if for some $\xi < \delta$, $\{\kappa_\nu : \nu \in [\xi,\delta)\} \subseteq A$.

By Exercise 32, $\{\kappa_\nu : \nu < \delta\}$ is a closed unbounded subset of δ. It follows that U_δ is contained in $\mathcal{C}_\delta \cap M_\delta$. Since U_δ is an M_δ-ultrafilter, it is maximal among M_δ-filters, and this implies $U_\delta = \mathcal{C}_\delta \cap M_\delta$. □

The proof yields:

Exercise 33 (Comparison lemma)**.** Suppose (M,U) and (N,W) are iterable minimal models with measurable cardinals. Show that for all some ordinal α, either $\mathrm{Ult}_\alpha(M,U) \in \mathrm{Ult}_\alpha(N,W)$, $\mathrm{Ult}_\alpha(N,W) \in \mathrm{Ult}_\alpha(M,U)$, or $\mathrm{Ult}_\alpha(M,U) = \mathrm{Ult}_\alpha(M,W)$.

Conclude that if (M,U) and (N,W) are minimal inner models with measurable cardinals, then for all sufficiently large ordinals α, $\mathrm{Ult}_\alpha(M,U) = \mathrm{Ult}_\alpha(N,W)$

We now use the comparison lemma to establish Kunen's theorem.

Exercise 34. Suppose (M,U) is a minimal inner model with a measurable cardinal. Suppose H is a proper class elementary substructure of (M,U) such that $\kappa_U \subseteq H$. Then the transitive collapse of $(H, U\cap H)$ is (M,U).

Exercise 35. If (M,U) and (N,W) are minimal inner models with measurable cardinals and

$$\langle \mathrm{Ult}_\alpha(M,U), j_{\xi\alpha} : \xi \le \alpha \in \mathrm{Ord}\rangle$$
$$\langle \mathrm{Ult}_\alpha(N,W), i_{\xi\alpha} : \xi \le \alpha \in \mathrm{Ord}\rangle$$

are their iterated ultrapowers. Suppose α is large enough that $\mathrm{Ult}_\alpha(M,U) = \mathrm{Ult}_\alpha(N,W)$. Then $\mathrm{ran}(j_{0\alpha}) \cap \mathrm{ran}(i_{0\alpha})$ is a proper class elementary substructure of $\mathrm{Ult}_\alpha(M,U)$.

We can now prove Kunen's theorem.

Proof of Theorem 6.1. Let us use the notation of Exercise 35. Let

$$H = \operatorname{ran}(j_{0\alpha}) \cap \operatorname{ran}(i_{0\alpha})$$

Let $H_M = j_{0\alpha}^{-1}[H]$ and let $\pi_M : H_M \to M$ be the transitive collapse. Let H_N be the class of all sets definable in N with parameters in $i_{0\alpha}^{-1}[H] \cup \kappa_W$, and let $\pi_N : H_N \to N$ be the transitive collapse. We define an elementary embedding $k : M \to N$ by setting

$$k = \pi_N \circ i_{0\alpha}^{-1} \circ j_{0\alpha} \circ \pi_M^{-1}$$

Note that every element of N is definable in N from parameters in $\operatorname{ran}(k) \cup \kappa_W$. Moreover, we have the following characterization of k on subsets of κ_U:

Exercise 36. For all $A \subseteq \kappa_U$, $k(A) = j_{0\alpha}(A) \cap \kappa_W$.

It follows that $k = j_{0\xi}$ where ξ is the least ordinal such that $j_{0\xi}(\kappa_U) \geq \kappa_W$. To see this, define a map $e : N \to \operatorname{Ult}_\xi(M, U)$ by $e(k(f)(\nu)) = j_{0\xi}(f)(\nu)$ whenever $f : \kappa \to M$ is in M and $\nu < \kappa_W$. Then check that e is well-defined and elementary, and $e \circ k = j_{0\xi}$. Since every element of $\operatorname{Ult}_\xi(M, U)$ is definable in $\operatorname{Ult}_\xi(M, U)$ from parameters in $\operatorname{ran}(j_{0\xi}) \cup \{\kappa_{U_\beta} : \beta < \xi\} \subseteq \operatorname{ran}(e)$, the embedding e is surjective. It follows that e is the identity, which implies $k = j_{0\xi}$, as desired. □

The argument sketched at the end of the proof of Theorem 6.1 is related to the theory of extenders. Suppose M and N are transitive models of ZFC and $j : M \to N$ is an elementary embedding. If $\nu \in \mathrm{Ord} \cap N$ and $\kappa \in \mathrm{Ord} \cap M$ is the least ordinal such that $j(\kappa) \geq \nu$, the *extender of length* ν *derived from* j is the function $E : P(\kappa) \cap M \to N$ defined by $E(A) = j(A) \cap \nu$. We say j is ν*-generated* if every element of N is of the form $j(f)(\alpha)$ for some $f : \kappa \to M$ in M and some $\alpha < \nu$.

Exercise 37. If M and N are inner models and $j_0 : M \to N_0$ and $j_1 : M \to N_1$ are ν-generated elementary embeddings and their derived extenders of length ν are equal, then $N_0 = N_1$ and $j_0 = j_1$.

Exercise 38. Suppose (M, U) is a minimal inner model with a measurable cardinal. Suppose $j : M \to N$ is an elementary embedding with critical point κ_U and j is $j(\kappa_U)$-generated. Then for some ordinal ξ, $(N, j(U)) = \operatorname{Ult}_\xi(M, U)$ and $j = j_{0\xi}$.

Exercise 39. Suppose M is an inner model and M satisfies that for some κ, there is a κ-complete ultrafilter W on κ such that $M = L[W]$. Then there is a normal ultrafilter $U \in N$ such that (M, U) is a minimal model with a measurable cardinal.

Exercise 40. Show that any minimal inner model with a measurable cardinal satisfies the generalized continuum hypothesis.

Hint: the proof that for $\lambda \geq \kappa_U$, $M \vDash 2^\lambda = \lambda^+$ is the same as the proof in L. For $\lambda < \kappa_U$, the proof appeals to Exercise 33. Fix $\theta > 2^\kappa$ such that $(H(\theta) \cap M, U)$ is a minimal model with a measurable cardinal. For each $A \subseteq \lambda$, let H_A be the set of definable elements of $(H(\theta) \cap M, U)$, allowing parameters in $\lambda \cup \{A\}$. Let M_A be the transitive collapse of H_A, and note that M_A is a minimal inner model with a measurable cardinal. For $A, B \subseteq \lambda$, set $A \preceq B$ if $A \in M_B$. Using Exercise 33, show that in M, $(P(\lambda), \preceq)$ is a linear preorder each of whose initial segments has cardinality at most λ. Conclude that M satisfies $2^\lambda = \lambda^+$.

7 The comparison lemma

Inner model theory is the subfield of set theory concerned with the construction and analysis of inner models generalizing L and $L[U]$. Mostly the field is concerned with inner models containing large cardinals far beyond a measurable. The goal is to build models of large cardinal hypotheses that are *canonical* in the same sense that L is canonical: e.g., all natural set theoretic statements can be settled in the models.

Current inner model theory provides a deep analysis of canonical models with many Woodin cardinals [**?**, **?**], with many applications to higher descriptive set theory and the theory of the Axiom of Determinacy [**?**, **?**, **?**].

Beyond the level of Woodin cardinals, there are proposals [**?**, **?**] for canonical models containing cardinals κ that are κ^{+n}-strongly compact for any number $n < \omega$.[5] It is open whether it is possible to construct such models, but assuming the models exist, one can develop their theories at roughly the same level of detail as the smaller canonical models.[6] At the level of a cardinal κ that is $\kappa^{+\omega}$-strongly compact, not even such conditional results are known, and there are hints that the inner models at this level look significantly different from the models known today.

The one constant in inner model theory, from a measurable cardinal onwards, is the comparison lemma (Exercise 33). For more complicated models, this typically asserts: any two canonical models M and N have iterated ultrapowers M_* and N_* such that either $M_* \in N_*$, $N_* \in M_*$, or $M_* = N_*$. The idea is that the existence of such a process shows that the models are determined solely by their position in the large cardinal hierarchy, certifying that the models really are canonical.

The question of constructing canonical models containing strongly compact cardinals has been open since th 1960s. Given the difficulties involved, it seems reasonable to doubt that strongly compact cardinals are compatible with the comparison methodology that succeeds at the level of Woodin cardinals. The Ultrapower Axiom was originally formulated to as a precise way to probe this skepticism.

8 The Ultrapower Axiom

The Ultrapower Axiom (UA) is a structural principle in set theory that governs the theory of countably complete ultrafilters. (See Definition 8.1.) The principle is true in every known canonical model of set theory, and in this way it resembles the classical combinatorial principles $\diamondsuit$ and $\square$. The purpose of UA, is quite different from these other principles. The idea is that UA must hold in any model that is subject to any form of the Comparison Lemma, and as a consequence, by studying UA in the context of large cardinals beyond the current canonical models, one can get a glimpse of models beyond the reach of current inner model theory. Moreover, if one can refute UA from some large cardinal hypothesis, one can conclude that this large cardinal has no canonical inner model, at least as the term is currently conceived.

Recall the notion of an *ultrapower embedding* from Exercise 15.

Definition 8.1. The Ultrapower Axiom (UA) is the assertion that for any ultrapower embeddings $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there exist ultrapower embeddings $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$, definable over M_0 and M_1 respectively, such that $k_0 \circ j_0 = k_1 \circ j_1$.

Exercise 41. UA holds if and only if for any $\aleph_1$-complete ultrafilters U_0 and U_1, the following objects exist:

- A $\aleph_1$-complete ultrafilter $W_0 \in M_{U_0}$
- A $\aleph_1$-complete ultrafilter $W_1 \in M_{U_1}$

such that $M_{W_0}^{M_{U_0}} = M_{W_1}^{M_{U_1}}$ and $j_{W_0}^{M_{U_0}} \circ j_{U_0} = j_{W_1}^{M_{U_1}} \circ j_{U_1}$.

[5]This means that every κ-complete filter on a set of size κ^{+n} extends to a κ-complete ultrafilter.

[6]More precisely, the constructions and their analysis can be carried out assuming *iterability hypotheses* which concern the well-foundedness of certain more complicated iterated ultrapowers of the universe of sets.

It is often useful to view an ultrafilter U on a set X as a generalized quantifier over X. If P is a unary predicate on X, we write $Ux\, P(x)$ to mean that the set of all x such that $P(x)$ belongs to U.

The following exercise reformulates UA in entirely combinatorial terms.

Exercise 42. Show that the following are equivalent:

- UA
- For any $\aleph_1$-complete ultrafilters U on X and W on Y, there exist $\aleph_1$-complete ultrafilters $\langle W_x\rangle_{x\in X}$ and $\langle U_y\rangle_{y\in Y}$ such that for any binary relation $A \subseteq X \times Y$,

$$Ux\, W_x y\, A(x,y) \iff Wy\, U_y x\, A(x,y)$$

Exercise 43. Suppose $j_0 : M \to N$ and $j_1 : M \to N$ are elementary embeddings that are definable from parameters over M. Then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Proposition 8.2. *If (M, U) is a minimal model of measurability, then M satisfies* UA*.*

Proof. Suppose U_0 and U_1 are $\aleph_1$-complete ultrafilters of M. Let $j_0 : M \to M_0$ and $j_1 : M \to M_1$ be their respective ultrapowers. Let $i_0 : M_0 \to P$ and $i_1 : M_1 \to P$ be the iterated ultrapowers coming from Exercise 33. We have $i_0(j_0(U)) = i_1(j_1(U))$ $i_0 \circ j_0 \restriction \mathrm{Ord} = i_1 \circ j_1 \restriction \mathrm{Ord}$. Since every element of M is definable in M from parameters in $\mathrm{Ord} \cup \{U\}$, it follows that $i_0 \circ j_0 = i_1 \circ j_1$. Finally, we must replace i_0 and i_1 with ultrapower embeddings, rather than iterated ultrapowers. Let $H = \mathrm{Hull}^M(i_0[M_0] \cup i_1[M_1])$. Let N be the transitive collapse $e : N \to H$. Let $k_0 = e^{-1} \circ i_0$ and let $k_1 = e^{-1} \circ i_1$.

Exercise 44. Show that k_0 and k_1 are definable ultrapower embeddings.

Hint: every element of N is definable in N from parameters in $k_0[N] \cup \{k_1([\mathrm{id}]_{U_1})\}$.

□

9 The linearity of the Mitchell order

Suppose U and W are κ-complete ultrafilters on κ. Then U lies below W in the *Mitchell order*, denoted $U \lhd W$, if U belongs to the ultrapower M_W of the universe by W.

Exercise 45 (Mitchell)**.** The Mitchell order is a well-founded partial order.

Hint: just show transitiveity and show well-foundedness. For the latter, show that if $U \lhd W$ then $j_U(\kappa) < j_W(\kappa)$. This is related to Exercise 5.

The rank of the Mitchell order restricted to normal ultrafilters on κ is denoted by $o(\kappa)$.

Exercise 46.

- $o(\kappa) \leq (2^\kappa)^+$.
- If $o(\kappa) > 1$, then κ is a limit of measurable cardinals.

Theorem 9.1. *Assuming* UA*, the Mitchell order well-orders the set of normal ultrafilters on any measurable cardinal.*

Proof. Suppose U_0 and U_1 are normal ultrafilters on κ and $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are their respective ultrapower embeddings. Let $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ witness UA.

Case 1. $k_0(\kappa) = k_1(\kappa)$.

Exercise 47. Show that in this case $U_0 = U_1$.

Case 2. $k_0(\kappa) < k_1(\kappa)$.

Let D be the ultrafilter on κ derived from k_1 using $k_0(\kappa)$.

Exercise 48. Show that $D = U_0$ and conclude that $U_0 \lhd U_1$.

Case 3. $k_0(\kappa) > k_1(\kappa)$.

Similarly to the previous case, $U_1 \lhd U_0$. □

10 The Ketonen order

The proof of the previous theorem suggests the following partial order on ultrafilters. If U_0 and U_1 are $\aleph_1$-complete ultrafilters on an ordinal κ, then U_0 precedes U_1 in the *Ketonen order,* denoted $U_0 <_{\Bbbk} U_1$, if there are elementary embeddings $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ such that $k_0 \circ j_0 = k_1 \circ j_1$, k_1 is a definable ultrapower embedding of M_1, and $k_0([\mathrm{id}]_{U_0}) < k_1([\mathrm{id}]_{U_1})$.

Lemma 10.1. *The Ketonen order is transitive.*

Proof. Suppose $U_0 <_{\Bbbk} U_1 <_{\Bbbk} U_2$. Let $j_i : V \to M_i$ be the ultrapower associated to U_i. Let $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ witness $U_0 <_{\Bbbk} U_1$. Let $i_1 : M_1 \to P$ and $i_2 : M_2 \to P$ witness $U_1 <_{\Bbbk} U_2$.

Exercise 49. Show that $i_1 \circ k_0 : M_0 \to i_1(N)$ and $i_1(k_1) \circ i_2 : M_2 \to i_1(N)$ witness $U_0 <_{\Bbbk} U_2$. □

Hint: draw the commutative diagram. Another proof follows from Exercise 55.

Exercise 50. Show that if U_0 and U_1 are normal ultrafilters, then $U_0 <_{\Bbbk} U_1$ if and only if $U_0 \lhd U_1$.

Exercise 51. $U_0 <_{\Bbbk} U_1$ if and only if there is a sequence $\langle U_\alpha : \alpha < \kappa\rangle$ of $\aleph_1$-complete ultrafilters on κ such that for all $A \subseteq \kappa$, $A \in U$ if and only if $W\alpha\,(A \cap \alpha \in U_\alpha)$.

Note that almost all of the ultrafilters U_α must concentrate on α.

For every set X and every cardinal ν, let $\beta_\nu(X)$ denote the ν-complete ultrafilters on X.

Exercise 52. Show that if $\kappa_0 \le \kappa_1$, then $(\beta_{\aleph_1}(\kappa_0), <_{\Bbbk})$ is isomorphic to an initial segment of $(\beta_{\aleph_1}(\kappa_1), <_{\Bbbk})$.

Exercise 53. Show that the Ketonen order is strict.

Hint: assume towards a contradiction that $U <_{\Bbbk} U$, fix $\langle U_\alpha\rangle_{\alpha<\kappa}$ witnessing this, and build a set $A \subseteq \kappa$ by recursion such that $U\alpha\,(\alpha \in A \iff A \notin U_\alpha)$.

Exercise 54. Suppose U is an ultrafilter, $i, k : M_U \to N$ are elementary embeddings, and i is a definable ultrapower embedding of M_U. Show that $i([\mathrm{id}]_U) \le k([\mathrm{id}]_U)$.

Challenge: show that if M and N are inner models, $i, k : M \to N$ are elementary embeddings, and i is definable over M, then for all $\alpha \in \mathrm{Ord}$, $i(\alpha) \le k(\alpha)$.

Theorem 10.2. *The Ketonen order is well-founded.*

We use the following fact:

Exercise 55. Suppose $j : V \to M$ is an elementary embedding and $U_0 <_{\Bbbk} U_1$. If $U_1^* \in j(\beta_{\aleph_1}(\kappa))$ is such that $U_1 = j^{-1}[U_1^*]$, then there is some $U_0^* \in j(\beta_{\aleph_1}(\kappa))$ with $U_0^* <_{\Bbbk} U_1^*$ such that $U_0 = j^{-1}[U_0^*]$.

Proof of Theorem 10.2. Suppose that for all ordinals $\alpha < \kappa$, the Ketonen order on $\beta_{\aleph_1}(\alpha)$ is well-founded. We will show that the same holds for κ. Fix $U \in \beta_{\aleph_1}(\kappa)$, and we will show that the Ketonen order is well-founded below U. Let S be a nonempty set of Ketonen predecessors of U. Let $j : V \to M$ be the ultrapower associated to U and let $\alpha = [\mathrm{id}]_U$. Let

$$\mathcal{B} = \{U \in j(\beta_{\aleph_1}(\kappa)) : \alpha \in U\}$$

By Exercise 52, $(\mathcal{B}, <_{\Bbbk}^M)$ is isomorphic to $(\beta_{\aleph_1}(\alpha), <_{\Bbbk})^M$, and hence $(\mathcal{B}, <_{\Bbbk}^M)$ is well-founded. Consider the set $S^* = \{W^* \in \mathcal{B} : j^{-1}[W^*] \in S\}$. Let W^* be a $<_{\Bbbk}^M$-minimal element of S^*. Let $W = j^{-1}[W^*]$. Then by Exercise 55, W is a $<_{\Bbbk}$-minimal element of S, as desired. □

Exercise 56. Assume UA.

- Show that the Ketonen order is linear: for any ultrafilters U_0 and U_1 on δ, either $U_0 <_{\Bbbk} U_1$, $U_1 <_{\Bbbk} U_0$, or $U_0 = U_1$.
- Show that in the definition of the Ketonen order, one can take both embeddings to be definable ultrapower embeddings.

Theorem 10.3. *The following are equivalent:*

- *The Ultrapower Axiom holds.*
- *The Ketonen order is linear.* □

11 Irreducible ultrafilters

If U and W are $\aleph_1$-complete ultrafilters, U is below W in the *Rudin-Frolík order*, denoted $U \leq_{\mathrm{RF}} W$, if there is a definable ultrapower embedding $i : M_U \to M_W$ such that $i \circ j_U = j_W$. We write $U <_{\mathrm{RF}} W$ if $U \leq_{\mathrm{RF}} W$ but $W \not\leq_{\mathrm{RF}} U$. A nonprincipal $\aleph_1$-complete ultrafilter W is *irreducible* if for every $U <_{\mathrm{RF}} W$ is principal.

Proposition 11.1. *The following are equivalent:*

- $U \leq_{\mathrm{RF}} W$ *and* $W \leq_{\mathrm{RF}} U$*.*
- $M_U = M_W$ *and* $j_U = j_W$*.*

Proof. Let $i : M_U \to M_W$ and $k : M_W \to M_U$ be definable ultrapower embeddings such that $i \circ j_U = j_W$ and $k \circ j_W = j_U$. By the uniqueness of definable embeddings, $i \circ k \restriction \mathrm{Ord}$ and $k \circ i \restriction \mathrm{Ord}$ are the identity. Therefore both these maps are the identity, and hence i and k are the identity. It follows that $M_U = M_W$ and $j_U = j_W$. □

In the situation of the previous proposition, we say U and W are *equivalent*, denoted $U \equiv W$.

Exercise 57. If $U \leq_{\mathrm{RF}} W$, then there is a unique definable ultrapower embedding $i : M_U \to M_W$ such that $i \circ j_U = j_W$.

Lemma 11.2. *If κ is the least measurable cardinal, then every irreducible ultrafilter on κ is equivalent to the unique normal ultrafilter on κ.*

If W is an ultrafilter on X and $U \leq_{\mathrm{RF}} W$, we let $t_U(W)$ denote the unique ultrafilter $W^* \in M_U$ such that $M_{W^*}^{M_U} = M_W$, $j_{W^*}^{M_U} \circ j_U = j_W$, and $[\mathrm{id}]_{W^*}^{M_U} = [\mathrm{id}]_W$.

Lemma 11.3. *If W is an $\aleph_1$-complete ultrafilter on an ordinal κ and $U \leq_{\mathrm{RF}} W$, then $t_U(W)$ is $<_{\Bbbk}^{M_U}$-minimal among $\aleph_1$-complete ultrafilters W^* on $j_U(\kappa)$ in M_U such that $j_U^{-1}[W^*] = W$.*

Proof. Suppose W^* is an $\aleph_1$-complete ultrafilter on $j_U(\kappa)$ in M_U such that $j_U^{-1}[W^*] = W$. Suppose $k : M_{W^*}^{M_U} \to N$ is an elementary embedding and $i : M_W \to N$ is a definable ultrapower embedding such that $k \circ j_{W^*}^{M_U} = i \circ j_{t_U(W)}^{M_U}$. We must show that $k([\mathrm{id}]_{W^*}) \geq i([\mathrm{id}]_{t_U(W)})$. Let $e : M_W \to M_{W^*}$ be the factor embedding, with $e([\mathrm{id}]_W) = [\mathrm{id}]_{W^*}$. Then $k([\mathrm{id}]_{W^*}) = k(e([\mathrm{id}]_W)) \geq i([\mathrm{id}]_{t_U(W)})$. The final inequality comes from Exercise 54. □

Proposition 11.4. *If W is an $\aleph_1$-complete ultrafilter and $U \leq_{\mathrm{RF}} W$ is nonprincipal, then $t_U(W) \neq j_U(W)$.*

Proof. Suppose $U \leq_{\mathrm{RF}} W$ and $t_U(W) = j_U(W)$. Then $j^{M_U}_{j_U(W)} \circ j_U = j_W$ and $[\mathrm{id}]^{M_U}_{j_U(W)} = [\mathrm{id}]_W$. But $j^{M_U}_{j_U(W)} \circ j_U = j_U \circ j_W$ and $[\mathrm{id}]^{M_U}_{j_U(W)} = j_U([\mathrm{id}]_W)$. It follows that j_U is the identity on $j_W[V] \cup \{[\mathrm{id}]_W\}$. Since every element of M_W is definable from parameters in $j_W[V] \cup \{[\mathrm{id}]_W\}$, we have that $j_U \restriction M_W$ is the identity. Therefore j_U has no critical point, which implies U is principal. □

Theorem 11.5 (UA). *For any ultrafilter W, there is no infinite increasing sequence $U_0 <_{\mathrm{RF}} U_1 <_{\mathrm{RF}} U_n <_{\mathrm{RF}} \cdots$ of predecessors of W in the Rudin-Frolík order.*

Proof. Without loss of generality, we may assume that W is an ultrafilter on an ordinal κ. Let M_n be the ultrapower of V by U_n. Let $i_{nn+1} : M_n \to M_{n+1}$ witness $U_n \leq_{\mathrm{RF}} U_{n+1}$ and let $k_{n\infty} : M_n \to M_W$ witness that $U_n \leq_{\mathrm{RF}} W$. Note that i_{nn+1} is the ultrapower embedding associated to $Z_n = t_{U_n}(U_{n+1})$, and so $j_{Z_n} \circ j_{U_n} = j_{U_{n+1}}$.

Let $W_n = t_{U_n}(W)$. Then

$$W_{n+1} = t_{U_{n+1}}(W) = t_{Z_n}(t_{U_n}(W)) = t_{Z_n}(W_n)$$

Applying Lemma 11.3 and Proposition 11.4 in M_n, we have that $W_{n+1} <_{\Bbbk} i_{nn+1}(W_n)$ in M_{n+1}. Let $W^\infty_n = k_{n\infty}(W_n)$. Then for all $n < \omega$, $W^\infty_{n+1} <_{\Bbbk} W^\infty_n$. This contradicts that the Ketonen order of M_W is well-founded. □

Corollary 11.6 (UA). *Every ultrapower embedding $j : V \to M$ can be decomposed as a finite iterated ultrapower*

$$V \xrightarrow{i_0} M_1 \xrightarrow{i_1} M_2 \xrightarrow{i_2} \dots \xrightarrow{i_{n-1}} M_n = M$$

such that $i_{n-1} \circ \cdots \circ i_2 \circ i_1 \circ i_0 = j$ and for all $k < n$, i_k is the ultrapower associated to an irreducible ultrafilter of M_k.

12 Strongly compact cardinals

An ultrafilter U on a set X is *uniform* if every set in U has the same cardinality as X. A cardinal λ is *Fréchet* if it carries an $\aleph_1$-complete uniform ultrafilter. For each Fréchet cardinal λ, assuming UA, there is a Ketonen minimum uniform ultrafilter on λ, which we denote by $\mathscr{K}_\lambda$.

An ultrafilter on a regular cardinal is *weakly normal* if it is closed under decreasing diagonal intersections. Note that U on λ is weakly normal if and only if $[\mathrm{id}]_U = \sup j_U[\lambda]$.

Exercise 58. If λ is a regular cardinal, and U is a Ketonen minimal uniform ultrafilter on λ, then U is weakly normal and concentrates on the set of ordinals $\alpha < \lambda$ such that $\mathrm{cf}(\alpha)$ is not Fréchet.

Hint: show that any Ketonen minimal ultrafilter on λ is weakly normal.

Exercise 59. Show that if $k : M \to N$ is an ultrapower embedding and $e : N \to N'$ is an elementary embedding such that $e \circ k$ is a definable ultrapower embedding, then k is a definable ultrapower embedding.

Theorem 12.1 (UA). *Suppose λ is a regular Fréchet cardinal and let $j : V \to M$ be the ultrapower associated with $\mathscr{K}_\lambda$. Suppose $k : M \to N$ is an ultrapower embedding that is continuous at* $\sup j[\lambda]$. *Then k is definable over M.*

Proof. Apply the Ultrapower Axiom to the ultrapower embeddings j and $k \circ j$. We obtain definable ultrapower embeddings $\ell : M \to P$ and $i : N \to P$ such that $\ell \circ j = i \circ k \circ j$.

Exercise 60. Show that $\ell(\sup j[\lambda]) = i \circ k(\sup j[\lambda])$.

Since every element of M is definable in M from parameters in $j[V] \cup \{\sup j[\lambda]\}$, $\ell = i \circ k$. Therefore k is definable over M by Exercise 58 □

Proposition 12.2 (UA). *The least strongly compact cardinal is larger than the least measurable cardinal.*

Proof. Let κ be the least strongly compact cardinal. Let $U = \mathscr{K}_{\kappa^+}$. Note that $\kappa_U \leq \kappa$. Let D be a normal ultrafilter on κ_U. Then $j_D \restriction M_U$ is definable over M_U. Therefore $D \in M_U$. Therefore κ_U is measurable in M_U. It follows that M_U satisfies that $j_U(\kappa_U)$ is larger than the least measurable cardinal, and hence κ_U is larger than the least measurable cardinal. □

Exercise 61. Following the notation of the previous proposition, show that the Mitchell order on κ_U has rank greater than 2^{κ_U}.

13 Further results

Theorem 13.1 (UA). *If κ is strongly compact, there is a set $A \subseteq \kappa$ such that every set is definable in $(V, \in)$ from A and ordinal parameters.*

In particular, there is a definable class well-order of the universe of sets.

This uses the following facts:

Exercise 62 (UA). Every $\aleph_1$-complete ultrafilter on an ordinal is definable from an ordinal.

Exercise 63. If κ is strongly compact, then there is a set $A \subseteq \kappa$ such that every set is definable in $(V, \in)$ from A and a κ-complete ultrafilter on an ordinal.

Hint: Assume κ is strongly compact. We will need the concept of a *κ-independent family of sets.* If X is a set, a family $\mathscr{F}$ of subsets of X is κ-independent if for any disjoint subfamilies $\sigma, \tau \subseteq \mathscr{F}$, $\bigcap_{A \in \sigma} A \cap \bigcap_{A \in \tau}(X \setminus A)$ is nonempty.

- Show that if $\mathscr{F}$ is a κ-independent family of subsets of an ordinal η and $|\mathscr{F}| = \lambda$, then every subset of λ is definable from $\mathscr{F}$ and a κ-complete ultrafilter on η.
- Show that for any $\lambda \geq \kappa$, there is a κ-independent family $\mathscr{F}$ of subsets of

$$P_\kappa(\lambda) = \{A \subseteq \lambda : |A| < \kappa\}$$

  such that $|\mathscr{F}| = \lambda$.
- Show that if κ is strongly compact and $S \subseteq \kappa$ is such that $V_\kappa \subseteq L[S]$, then for any λ, there is a κ-complete ultrafilter U and a well-order of $P_\kappa(\lambda)$ definable from $j_U(S)$, U, and λ.
- Conclude that for every $A \subseteq \lambda$ can be defined using S along with two κ-complete ultrafilters on ordinals.
- Finish by showing that any pair of κ-complete ultrafilters on ordinals is definable from a single one.

We mention two more results that we will not have time to cover. Both of them can be proved using the method of independent families.

Theorem 13.2 (UA). *If κ is strongly compact, then for all $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.*

Theorem 13.3 (UA). *If κ is strongly compact, either κ is supercompact or κ is a limit of supercompacts.*