Array Noncomputability: A Modular Approach

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Simple permitting

Theorem 1

For any noncomputable c.e. set C there is a simple set $A \leq_T C$.

Let $\{C_s\}_{s\in\omega}$ be a computable enumeration of C. We construct a coinfinite c.e. set A to satisfy for all e the following requirement:

$$\mathcal{R}_e\colon\ |W_e| = \infty \quad \Rightarrow \quad \Big(W_e \cap A \neq \emptyset \quad \text{or} \quad \neq C \equiv_T \emptyset\Big).$$

Construction:

- Let $A_0 = \emptyset$.
- At stage $s+1$, given A_s find the least e such that

$$W_{e,s} \cap A_s = \emptyset \quad \text{and} \quad \exists x > 2e\Big(x \in W_{e,s} \wedge C_{s+1} \restriction x \neq C_s \restriction x\Big).$$

  Then choose the least such e and enumerate the least corresponding x into A. If there is no such e go to stage $s+2$.

Verification:

- $\bar{A}$ is infinite.
- $A \leq_T C$.
- $|W_e| = \infty \Rightarrow W_e \cap A \neq \emptyset$.

Typical permitting construction

Permitting is a basic technique for constructing a c.e. set B which is Turing reducible to a given c.e. set A.

- We want to construct a c.e. set B Turing below a given c.e. set A such that B has certain properties which (in part) can be ensured by meeting positive requirements $\mathcal{R}_e (e \geq 0)$ of the following type:
- In order to meet $\mathcal{R}_e$ it suffices to pick a follower x (becoming a witness for the fact that $\mathcal{R}_e$ will be met). The follower may become "realized" at some stage. In this case the follower (or a certain greater number) has to be enumerated into B.
- In the presence of permitting, once x is realized we wait that A permits x to enter B (and, if so, we put x into B). While waiting, we iterate the attack on $\mathcal{R}_e$ with a new follower $x' > x$ (and so on).

Multiple permitting

- Simple permitting is achieved by enumerating a new number x into B at a stage $s+1$ only if (for a given enumeration of A) a number $\leq x$ (or, more generally, $\leq f(x)$ for some computable function f) enters A at stage $s+1$.
- More involved positive requirements or settings require stronger forms of permitting in order to perform the construction below a given c.e. set A.
- Multiple permitting is the case where any follower x of a requirement $\mathcal{R}_e$ is associated with an entourage of $\leq f(x)$ numbers (f is a computable function) all of which need permitting after becoming realized.
- It was discussed by Downey, Jockusch and Stob in 1990 (DJS1990) where it is argued that the sets and degrees giving this type of permitting are the array noncomputable (a.n.c.) sets and their degrees, respectively.

Array noncomputablity - definition and basic properties

- A sequence $\mathcal{F} = \{F_n\}_{n\in\omega}$ of finite sets is a *very strong array (v.s.a.)* if the following hold.
  1. There is a computable function f such that $f(n)$ is the canonical index of F_n;
  2. $\bigcup_{n\in\omega} F_n = \mathbb{N}$;
  3. $F_n \cap F_m = \emptyset$ for $n \neq m$;
  4. $0 < |F_n| < |F_{n+1}|$ for all $n \in \omega$.
- A c.e. set A is $\mathcal{F}$*-a.n.c.* if it is $\mathcal{F}$*-similar* to any c.e. set V, i.e.,

$$\exists^\infty n \left(A \cap F_n = V \cap F_n\right)$$

- A c.e. set A is *a.n.c.* if it is $\mathcal{F}$-a.n.c. for some v.s.a. $\mathcal{F}$.
- A c.e. degree $\mathbf{a}$ is *array noncomputable* if it contains a a.n.c. c.e. set.

Theorem 2 (DJS1990)

Let $\mathbf{d}$ *be an array noncomputable c.e. degree and let* $\mathcal{F}$ *be a very strong array. There is a c.e. set* $D \in \mathbf{d}$ *such that* D *is* $\mathcal{F}$*-a.n.c.*

Standard multiple permitting constrcution by a.n.c. sets

- If (the entourage of) a follower x has to be permitted up to $f(x)$ times then we choose a v.s.a. $\mathcal{F}$ of intervals F_n such that $|F_n| \geq f(\min F_n)$ and choose the numbers $x = \min F_n$ as followers.
- Then, whenever a member x_m of the entourage of x needs permitting, we enumerate the corresponding element y_m of F_n into a trigger set V.
- While waiting, we iterate the attack on $\mathcal{R}_e$ with a new follower $x' > x$ in almost all interval F_n.
- For any n such that A and V agree on F_n, the number y_m has to enter A later thereby giving the required permitting.

c.e. ℓ-properties

Definition 3

Let ℓ be a strictly increasing computable function. A property $\mathcal{P}$ is a *c.e. ℓ-property* if there are uniformly c.e. sets $\{B_a\}_{a\geq 0}$ with $B_a \subseteq [a, a+\ell(a)]$ such that for any set C

$$\exists^\infty a(C \cap [a, a+\ell(a)] = B_a) \;\Rightarrow\; C \in \mathcal{P}. \tag{1}$$

Lemma 4

Let ℓ be a strictly increasing computable function, let $\mathcal{P}$ be a c.e. ℓ-property, let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a.i. dominating ℓ, and let A be an $\mathcal{F}$-a.n.c. set. Then $A \in \mathcal{P}$.

- Let $B = \bigcup_{n\geq 0} B_{\min F_n}$. Then B is c.e.
- Since A is an $\mathcal{F}$-a.n.c. set, for infinitely many n, $A \cap [\min F_n, \min F_n + l(\min F_n)] = B \cap [\min F_n, \min F_n + l(\min F_n)] = B_{\min F_n}$.
- Thus $A \in \mathcal{P}$.

Lemma 5

Let ℓ be a strictly increasing computable function and let $\mathbf{d}$ *be an array non-computable c.e. degree. There is a c.e. set $A \in$* $\mathbf{d}$ *such that $A \in \mathcal{P}$ for all c.e. ℓ-properties $\mathcal{P}$.*

Modular approach for a.n.c. degrees

- In a typical wait-and-see or finite injury priority construction of a c.e. set A we ensure that A has a desired property $\mathcal{P}$ by meeting an infinite list $\mathcal{R}_e$ of requirements.
- For a fixed strictly increasing computable (length) function ℓ, we assign a c.e. ℓ-property $\mathcal{P}_e$ to each requirement $\mathcal{R}_e$ such that any set $A \in \mathcal{P}_e$ meets requirement $\mathcal{R}_e$.
- By Lemma 4, for any v.s.a.i. $\mathcal{F}$ dominating ℓ, any $\mathcal{F}$-a.n.c. set A has property $\mathcal{P}_e$.
- By Lemma 5 any c.e. a.n.c. Turing degree contains a c.e. set with property $\mathcal{P}$.

Modular approach for a.n.c. degrees

In order to show that the construction of a c.e. set with a certain property $\mathcal{P}$ can be adapted to show that sets with this property exist in all a.n.c. c.e. degrees, it suffices to analyze the individual requirements forcing $\mathcal{P}$ in isolation and to show that any requirement corresponds to a c.e. ℓ-property where the function ℓ does not depend on the requirement.

(We show an example using this approach later.)

Array noncomputability for left-c.e. reals - a failure attempt

Theorem 6

Given any left-c.e. real $\alpha < 1$, there exists a left-c.e. real $\beta < 2$ such that

$$\forall n \, \exists i \in \{0,1\} \, \big(\alpha(2n+i) \neq \beta(2n+i)\big).$$

To construct β, for $s > 0$, let $\beta_s = \alpha_s + \sum_{i=1}^{s} 2^{-2s}$, i.e.,

$$\beta_s = \alpha_s + 0.\underbrace{0101\ldots01}_{s \text{ times } 01}.$$

Thus for all $i \leq s$,

$$\text{number}(\beta_s \restriction [2i-2, 2i)) - \text{number}(\alpha_s \restriction [2i-2, 2i)) \equiv 1 \text{ or } 2 \pmod 4.$$

Corollary 7

Suppose $\{F_n\}_{n\in\mathbb{N}}$ is a very strong array of interval. Given any left-c.e. real α, there exists a left-c.e. real β such that $\forall i \, (\alpha \restriction F_i \neq \beta \restriction F_i)$.

Array noncomputability for locally l.c.e. sets

Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a very strong array of intervals.

- A left-c.e. approximation $\{\alpha_s\}_{s\geq 0}$ is $\mathcal{F}$*-compatible* if $\alpha_s \upharpoonright F_n \leq_{lex} \alpha_{s+1} \upharpoonright F_n$ for all $n, s \geq 0$ and $\alpha_s(x) \leq \alpha_{s+1}(x)$ for all $s \geq 0$ and $x \notin \bigcup_{n\geq 0} F_n$.
- A real α is $\mathcal{F}$*-compatibly left-c.e.* ($\mathcal{F}$*-l.c.e.* for short) if it has a $\mathcal{F}$-compatible left-c.e. approximation.

- An $\mathcal{F}$-l.c.e. real α is $\mathcal{F}$*-l.c.e.-a.n.c.* if it is $\mathcal{F}$-similar to any $\mathcal{F}$-l.c.e. real β, i.e.,

$$\exists^\infty n \; (\alpha \upharpoonright F_n = \beta \upharpoonright F_n).$$

Theorem 8

Let $\mathbf{d}$ *be an array noncomputable c.e. degree and let* $\mathcal{F}$ *be a very strong array of intervals. There is a left-c.e. real* $\alpha \in \mathbf{d}$ *such that* α *is* $\mathcal{F}$*-l.c.e.-a.n.c.*

l.c.e. ℓ-property

Definition 9

Let ℓ be a strictly increasing computable function. A property $\mathcal{P}$ is an *l.c.e. ℓ-property* if there are uniformly left-c.e. reals $\{\beta_a\}_{a\in\omega}$ with $\beta_a \subseteq [a, a+\ell(a)]$ such that for any real γ

$$\exists^\infty a \left(\gamma \upharpoonright [a, a+\ell(a)] = \beta_a\right) \ \Rightarrow\ \gamma \in \mathcal{P}. \tag{2}$$

Lemma 10

Let ℓ be a strictly increasing computable function, let $\mathcal{P}$ be a l.c.e. ℓ-property, let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a.i. dominating ℓ, and let α be an $\mathcal{F}$-l.c.e.-a.n.c. real. Then $\alpha \in \mathcal{P}$.

Lemma 11

Let ℓ be a strictly increasing computable function and let **d** *be an array non-computable c.e. degree. There is a left-c.e. real $\alpha \in$* **d** *such that $\alpha \in \mathcal{P}$ for all l.c.e. ℓ-properties $\mathcal{P}$.*

Example:

Theorem 12 (Merkle & F.)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable nondecreasing function such that $\sum_n 2^{-g(n)} = \infty$. Every array noncomputable c.e. degree $\mathbf{d}$ *contains a left-c.e. real that is not $(\mathrm{id}+g)$-bounded Turing reducible to any left-c.e. Martin-Löf random real.*

Lemma 13 (Merkle & F.)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable nondecreasing function such that $\sum_n 2^{-g(n)} = \infty$. There is a strictly increasing computable function ℓ such that the following hold. For any pair $\langle \Phi, \gamma \rangle$, where Φ is a Turing functional with use function bounded by $\mathrm{id} + g$ and γ is a left-c.e. real in $[0,1)$, uniformly in $\langle \Phi, \gamma \rangle$ and in all $a \in \mathbb{N}$, there are a left-c.e. real $\alpha_a \subseteq [a, a+\ell(a)]$ and a c.e. set E_a of strings with $\mu(E_a) < 2^{-a}$ such that

$$\exists\, x \in [a, a+\ell(a)]\ \alpha_a(x) \neq \Phi^{\gamma}(x) \quad \textit{or} \quad \gamma \in [E_a]. \tag{3}$$

Example: (continued)

Lemma 14

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable nondecreasing slow-growing function. There is a strictly increasing computable function ℓ such that, for any pair $\langle \Phi, \gamma \rangle$, where Φ is a Turing functional with use function bounded by $\mathrm{id}+g$ and γ is a left-c.e. real in $[0,1]$, the property

$$\mathcal{P}_{\Phi,\gamma} = \{\alpha\colon \alpha \neq \Phi^{\gamma} \text{ or } \gamma \text{ is not random}\}$$

is an almost-c.e. ℓ-property.

Proof.

- Let ℓ be the function as given by Lemma 13. Then for any pair $\langle \Phi, \gamma \rangle$, there are $\{\alpha_a\}_{a \geq 0}$ and $\{E_a\}_{a \geq 0}$ as stated in Lemma 13.
- Given any left-c.e. real $\tilde{\alpha}$, suppose $\exists^{\infty} a$ such that $(\tilde{\alpha} \upharpoonright [a, a+\ell(a)] = \alpha_a)$.
- Suppose $\tilde{\alpha} = \Phi^{\gamma}$. Then it suffices to show that γ is not random.
- By (3), it follows that $\exists^{\infty} a$ such that $\gamma \in [E_a]$.
- Let $U_e = \bigcup_{a>e} [E_a]$. Then $\gamma \in U_e$ for all $e \geq 0$.
- On the other hand, U_e are uniformly c.e., and $\mu(U_e) \leq \sum_{a>e} \mu(E_a) < \sum_{a>e} 2^{-a} \leq 2^{-e}$.
- Thus, $\{U_e\}_{e \in \omega}$ is a Martin-Löf test containing γ.

□

Example: (continued)

Theorem 12

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable nondecreasing function such that $\sum_n 2^{-g(n)} = \infty$. Every array noncomputable c.e. degree $\mathbf{d}$ contains a left-c.e. real that is not $(\mathrm{id}+g)$-bounded Turing reducible to any left-c.e. Martin-Löf random real.

Proof.

- Let $\{\Phi_i, \gamma_i\}_{i\in\omega}$ be an effective enumeration of all pairs of a Turing functional with use function bounded by $\mathrm{id}+g$ and a left-c.e. real in the unit interval.
- By Lemme 14, there is a strictly increasing computable function ℓ such that, for all i the properties $\mathcal{P}_i = \{\alpha\colon \alpha \neq \Phi_i^{\gamma_i}$ or γ_i is not random$\}$ are almost-c.e. ℓ-properties.
- By Lemma 11, there is a left-c.e. real $\alpha \in \mathbf{d}$ such that $\alpha \in \mathcal{P}_i$ for all i, which implies it is not $(\mathrm{id}+g)$-bounded Turing reducible to any left-c.e. Martin-Löf random real.

□

Thank you for your attention!