Metamathematical Methods in Descriptive Set Theory

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Background Motivating Question

When can we say two proofs really use different methods?

E.g., Halmos: nonstandard methods are just a matter of taste, no new mathematical insights.

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Classic philosophical references

- Dawson (2006) *Why do mathematicians re-prove theorems?*
- Later turned into a book: Dawson (2015) *Why Prove it Again?: Alternative Proofs in Mathematical Practice*

My goal with this work

1. To survey and classify proofs using metamathematical methods in DST (we will focus on this)
2. To determine about whether these really use different methods than classical proofs (won't do much of this today, but nice to think about)

How do these proofs work

Basic Tools

- Forcing
- Solovay-type characterizations
- Complexity calculation
- Borel codes
- Absoluteness

Definition

A formula $A(x)$ is Σ^1_1 iff $A(x)$ is equivalent to a formula of the form $\exists y \forall n R(x, y, n)$, where R is a computable relation, y ranges over $\{$subset of naturals, reals, functions from naturals to naturals, etc$\}$, and n ranges over naturals. It's Π^1_1 iff its negation is Σ^1_1.

Complexity in second-order arithmetic

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Example (Luzin, 1927)

Consider the space $(\omega \setminus \{0\})^\omega$. This is the space of sequences of positive integers. Define a subset A of the space as follows:

$$A(x) \Leftrightarrow \exists n_0 < n_1 < n_2 < ... x(n_i) \text{ divides } x(n_{i+1})$$

In other words, $x \in A$ iff there is some increasing $y \in (\omega \setminus \{0\})^\omega$ such that for all $i \in \omega$, we have $x(y(i))$ divides $x(y(i+1))$. This is Σ^1_1, because the relation "$y(m) > y(m+1) \wedge x(y(n)) \mid x(y(n+1))$", with free variables (x, y, m, n), is computable.

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Example (Well-founded trees)

The set of $f \in 2^\omega$ coding well-founded trees or well-orderings is Π^1_1: "f codes a tree and every attempt g to trace a infinite descending path in f fails".

Fact

A Σ^1_1 sentence is true if and only if a particular tree is ill-founded.

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Well-foundedness is Δ_1 (in the language of set theory):

$$\begin{aligned} R \text{ is well-founded on } X &\leftrightarrow (\forall Y \subseteq X)(Y \text{ has a minimal element}) \\ &\leftrightarrow (\exists f : \mathrm{Ord} \to X)(f \text{ is order-preserving with respect to } R) \end{aligned}$$

Fact

A Σ^1_1 sentence is true if and only if a particular tree is ill-founded.

Corollary (Mostowski Absolutenesss)

Σ^1_1 and Π^1_1 are absolute between transitive models of enough set theory.

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Σ^1_1 and Π^1_1 are absolute between transitive models of enough set theory.

A philosophical question

This above follows from having a Δ_1 characterization of well-foundedness. The Σ_1 part depends crucially the ability to express the notion of an ordinal in a Δ_0 way. This relies on having the von Neumann definition of an ordinal. But in principle (according to the structuralist) it shouldn't matter what the ordinals *really* are. So here's a challenge: can a structuralist recover the mathematical content in Mostowski Absoluteness?

A warmup

Theorem

There are incomparable Turing degrees.

Proof.

First observe that total comparability of Turing degrees implies the continuum hypothesis: $(\mathbb{R}, \leq_T)$ would be a linear order with only countable initial segments. This makes $|\mathbb{R}| = \omega_1$.

Now force to get $\neg$CH. In $V[G]$ we have incomparable reals. But "there exists $x, y \in \mathbb{R}$ s.t. $x \not\leq_T y \wedge y \not\leq_T x$" is Σ^1_1, and so it is absolute and holds in V too. $\square$

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How to prove something is true by proving it is consistent

Reference: Kunen (2013). IV.5. The metamathematics of forcing

The ctm method

1. Take a large enough $H_\theta \prec_{1000} V$ and a countable $M \prec H_\theta$.
2. Force over M to get $M[G]$.
3. Use absoluteness between M and $M[G]$ to show that a statement is true in M.
4. And use elementarity to go all the way back to V.

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The syntactic method

1. Define a relation $\Vdash^*$.
2. Show that the relation satisfies all logical rules.
3. For each formula $\varphi(\vec{x})$ known to be absolute, show:
4. for every p and all sets $\vec{a}$: $p \Vdash^* \varphi(\vec{a})$ iff $1 \Vdash^* \varphi(\check{\vec{a}})$ iff $\varphi(\vec{a})$.

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The Boolean-valued method ("the naturalist account")

For any complete Boolean algebre $\mathbb{B}$, there is a definable elementary embedding $j : (V, \in) \preceq (\bar{V}, \bar{\in})$, such that there is in V a $\bar{V}$-generic filter G for $j(\mathbb{B})$. So we have: $V \preceq \bar{V} \subseteq \bar{V}[G]$.
Then prove that absolute statements are still absolute across $\bar{V} \subseteq \bar{V}[G]$ (which might not be transitive).

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2. Force over M to get $M[G]$.
3. Use absoluteness between M and $M[G]$ to show that a statement is true in M.
4. And use elementarity to go all the way back to V.

We adopt the ctm method for simplicity.

Incomparable Turing degrees, proof 2

Proof 2.

Force to add two mutually generic Cohen reals c, d. Obviously c, d are Turing-incomparable, because otherwise (say) $c \leq_T d$ would imply that $c \in V[d]$, contradicting mutual genericity. And so the extension has incomparable Turing degrees. Again, this is absolute to models of set theory, and so it holds to begin with. $\square$

Computability relative to large sets

Definition (Topological notion of largeness)

- A set $A \subseteq \mathbb{R}$ is nowhere dense iff it's not dense in any open interval. Equivalently, $(\mathrm{cl}(A))^{\mathrm{o}} = \emptyset$
- A is meager iff it is a countable union of nowhere dense sets.
- A is comeager iff its complement is meager.

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Measure-Category Duality

meager :: measure zero

comeager :: full measure (measure 1 in the case of $[0, 1]$ or Cantor space)

Property of Baire :: Lebesgue measurable

non-meager Borel :: positive measure

Reference: Oxtoby, *Measure and Category*

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Theorem

If x is computable relative to a comeager set of reals (i.e., its Turing cone $\{y \mid x \leq_T y\}$ is comeager), then it is computable.

Proof of Lemma.

If x is computed by the Turing program Φ_e^c, then this fact also holds true in $M[c]$, and so by the forcing theorem this is forced by some condition p. That is,

$$p \Vdash \text{ the } \check{e}\text{th Turing program in the oracle } \dot{c} \text{ computes } \check{x}$$

For any $i \in \omega$ we compute $x(i)$ as follows: run $\Phi_e^s(i)$ for all the s extending p.

Computability relative to large sets

Lemma (Blass)

Let M be a countable transitive model of enough of ZFC, and let $x \in M \cap \mathbb{R}$ and c a Cohen real over M. If $x \leq_T c$, then x is computable.

Proof of Lemma.

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$$p \Vdash \text{ the } \check{e}\text{th Turing program in the oracle } \dot{c} \text{ computes } \check{x}$$

For any $i \in \omega$ we compute $x(i)$ as follows: run $\Phi_e^s(i)$ for all the s extending p.

As soon as any of these computations halt, the output will be the correct value of $x(i)$.

This is because: if s_0, s_1 are two different nodes extending p and $\Phi_e^{s_0}(i) = 0 \neq 1 = \Phi_e^{s_1}(i)$, then we can build two different filters G_0 and G_1 containing s_0, s_1 respectively.

Now $M[G_0]$ and $M[G_1]$ will both think x is computed by Φ_e^c (since both filters contain p.) (Note that they will interpret c differently; but that doesn't matter.) So $M[G_0]$ thinks that $x(i) = 0$ and $M[G_1]$ thinks $x(i) = 1$.

But whatever $x(i)$ is, this is an absolute fact about $x \in M$, so it should be answered in the same way by all transitive models extending M.

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Computability relative to large sets

Theorem

If x is computable relative to a comeager set of reals (i.e., its Turing cone $\{y \mid x \leq_T y\}$ is comeager), then it is computable.

Lemma (Blass)

Let M be a countable transitive model of enough of ZFC, and let x be a real in M and c a Cohen real over M. If x is computable relative to c, then x is computable.

Fact (Solovay characterization of genericity)

Let M be a transitive model of enough set theory. Then c is Cohen-generic over M iff it is not in any meager F_σ set coded in M. (Recall: every Borel B set has a Borel code c_B. The property of being a Borel code is Π^1_1.)

Fact

When M is a ctm, the set of reals Cohen over M is comeager.

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Proof of Theorem.

Let x be a real whose Turing cone is comeager. Let $M \ni x$ be a ctm of enough set theory. Since the comeager sets form a filter, every comeager set must contain Cohen reals over M, and so x is computable from a Cohen real. By Blass's lemma, it is computable. $\square$

More absoluteness...

Definition

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Theorem (Shoenfield Absoluteness)

Σ^1_2 and Π^1_2 are absolute across models with the same countable ordinals.

Very sketchy proof.

Via a Suslin representation, truth of Σ^1_2 is again reduced to the well-foundedness of certain trees. □

Mycielski's Perfect Set Theorem

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Let $R \subseteq X^2$ be a Borel equivalence relation on a Polish space X, such that each equivalence class is meager. Then there exists a perfect set of pairwise inequivalent elements.

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Corollary

$\mathbb{R}$ *injects into* $\mathbb{R}/\mathbb{Q}$, $\mathbb{R}/Tur$, *etc...*

Special Case of $\mathbb{R}/\mathbb{Q}$

Proof:
Force to add a perfect set of mutually generic Cohen reals.
In the extension, the perfect set of Cohen reals are all Vitali-inequivalent.
But "There is a perfect tree whose branches are pairwise Vitali-inequivalent" is Σ^1_2. By Shoenfield absoluteness this holds to begin with.

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Proof of The General Case of Mycielski's Theorem

Force to add a perfect set of mutually generic Cohen reals.

In any extension, the interpretation of the Borel code of R is still an equivalence relation with meager equivalence classes.

Why? Because being an equivalence relation is a Π^1_1 property, and the equivalence classes being meager is equivalent to the relation itself being a meager subset of X^2, which is a Σ^1_2 property about c_R.

Fact (Kuratowski-Ulam Theorem)

If $R \subseteq X^2$ has the property of Baire and each section R_x is meager, and R is a meager subset of X^2

Fact

"Meager(R)" $\Leftrightarrow$ $\exists$closed sets $C_1, C_2, \ldots$ each nowhere dense, such that $R \subseteq \bigcup_i C_i$.

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"Meager(R)" $\Leftrightarrow$ $\exists$closed sets $C_1, C_2, \ldots$ each nowhere dense, such that $R \subseteq \bigcup_i C_i$.

Proof continued

Now consider an arbitrary Cohen real c in that perfect set.

Already in the intermediate extension $V[c]$, the equivalence class $[c]_R$ is meager. Also $[c]_R \subseteq F$ for an F_σ meager set F, by the usual properties of Baire category.

Since F is coded in $V[c]$, any Cohen real over $V[c]$ will not be F, by Solovay's characterization of Cohen-genericity.

This includes all Cohen reals on that perfect tree added by the forcing (by mutual genericity). Therefore, any two such Cohen reals are R-inequivalent, and there is a perfect set of them.

Finally, the statement that there is a perfect tree, any two branches of which are R-inequivalent, is Σ^1_2 in the code of R and hence absolute to V.

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Now consider an arbitrary Cohen real c in that perfect set.
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More applications in Borel equivalence relations

Theorem

There is no Borel function $F : 2^\omega \to 2^\omega$ *such that* $xE_0y \Leftrightarrow F(x) = F(y)$*, where* E_0 *is the equivalence relation of being different in only finitely many places.*

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There is no Borel function $F : 2^\omega \to 2^\omega$ such that $xE_0y \Leftrightarrow F(x) = F(y)$, where E_0 is the equivalence relation of being different in only finitely many places.

In the language of Borel equivalence relations: $E_0 \not\leq_B =$. ($= \leq_B E_0$ follows from Mycielski above.)

Proof.

Suppose towards a contradiction that F is a Borel reduction ($xE_0y \Leftrightarrow F(x) = F(y)$). and let b_F be its Borel code

Now force to add a Cohen real c. In $V[c]$, the function F^* coded by b_F still has the same properties as in the assumption of the theorem, by Π^1_1-absoluteness.

But now in $V[c]$, the image $w = F^*(c)$ of the Cohen real under this map remains the same regardless finite changes to c, which implies the value of w is already decided by the weakest condition. (For each n, $1 \Vdash w(n) = 0$ or $1 \Vdash w(n) = 1$)

Why? Suppose not, then pick two incomparable conditions s, t of equal length such that $s \Vdash w(n) = 0$, $t \Vdash w(n) = 1$. For any Cohen real extending s, the same tail extending t is another Cohen real. But these two Cohen reals will be mapped to different images, contradicting that F maps finitely-different reals to the same real.

So w is already in the ground model, and so its pre-image $F^{-1}(w)$ will contain a real that differs from a Cohen real in only finitely many places. But this is impossible, as the Cohen real is generic over the ground model. □

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Theorem (Sierpiński 1917, the first result of Borel non-reducibility)

There is no Borel function $F : \mathbb{R} \to \mathbb{R}$ *such that* $x - y \in \mathbb{Q} \Leftrightarrow F(x) = F(y)$.

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Soit maintenant x un nombre réel donné. Designons par $E(x)$ l'ensemble de tous les nombres $x+r$, r étant un nombre rationnel quelconque: on voit sans peine que ce sera un ensemble dénombrable et que nous aurons toujours $E(x) = E(x')$ pour $x - x'$ rationnel et $E(x) \neq E(x')$ pour $x - x'$ irrationnel.

A tout nombre réel donné x correspondra donc un nombre réel $\varphi(x) = f[E(x)]$, et il suit des propriétés de $E(x)$ et $f(E)$ que nous aurons $\varphi(x) = \varphi(x')$ pour $x - x'$ rationnel et $\varphi(x) \neq \varphi(x')$ pour $x - x'$ irrationnel.

Or, je dis que toute fonction $\varphi(x)$ jouissant de cette propriété est non mesurable [1].

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Same proof as before, except we prove that 1 decides all the rational intervals of $w = F^*(c)$.

One more application in Borel equivalence relations

Theorem (Friedman-Stanley jump of $=$)

There is no uniform Borel diagonalizer. That is, there is no Borel function $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, such that for all $g \in \mathbb{R}^{\mathbb{N}}$, and all $n \in \omega$, we have $F(f) \neq f(n)$; and that if $\operatorname{ran} f = \operatorname{ran} g$, then $F(f) = F(g)$.

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In words: Cantor's diagonalization cannot be performed in a Borel way that respects permutations of the given sequece. Or in slightly imprecise words, there's no Borel way to diagonalize out of any given countable *set* of reals (because $\operatorname{ran}(f) = \operatorname{ran}(g)$ means f and g enumerate the same set).

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Suppose towards a contradiction that there is such a Borel map F. Forcing with $\mathrm{Col}(\omega, \mathbb{R})$ to make the ground model reals countable, let f and g be mutually generic. In $V[f][g]$, the re-interpreted map F^* still satisfies the assumption by absoluteness. But since f and g enumerate the same set of reals (i.e., the ground model reals), we have that $F^*(f) = F^*(g)$, which implies that $z := F^*(f) = F^*(g)$ belongs to both $V[f]$ and $V[g]$. By Solovay's lemma on intersection of extensions from mutual generics, we obtain that $z \in V$, which is a contradiction since F is suppose to diagonalize out of the ground model reals. □

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Remark

The non-forcing proof uses Baire category theorem, but with $\underset{\sim}{R}^{\mathbb{N}}$, where *undertildeR* is the reals with discrete topology. This is somewhat artificial and unnatural.

Theorem

If $A \subseteq \mathbb{R}$ is $\underset{\sim}{\Sigma}^1_1$, then A is measurable.

Proof.

For notational simplicity, we work with the Boolean-value approach to forcing. Force with the (separative quotient of) $\mathcal{B}/\mathrm{Null}$.

Suppose $A := \{x \in \mathbb{R} \mid \varphi(x, a)\}$, where φ is Σ^1_1 and $a \in \mathbb{R}$. Let X be a G_δ such that its equivalence class $[X]$ in the random forcing algebra is equal to the Boolean value $[\![\varphi(\dot{r}, \check{a})]\!]$. ($X$ can be assumed to be G_δ because of general properties of Lebesgue measure.)

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Proof continued

Claim: $\mu(X \triangle A) = 0$. (This is just the equivalent formulation of the measurability of A.)

To see the claim, assume towards a contradiction that, say, $B = A \setminus X$ has positive outer measure (the case where $X \setminus A$ has positive outer measure is similar).

Then there is a real $r \in B$ random over some countable elementary submodel M of some V_κ large enough (so that it reflects the relevant facts and that $V_\kappa \models [\mathbb{R} \setminus X] \Vdash \neg\varphi(\dot{r}, \check{a})$), with $a, A, X \in M$. Notice that $[B]$ is a stronger condition than $[\mathbb{R} \setminus X]$.

Now, letting N be the transitive collapse of M, we have $N[r] \models \varphi(r, a)$, since $r \in A$ by assumption and Σ^1_1 formulas are absolute between V and $N[r]$. But this last fact contradicts that $M \models [B] \Vdash \neg\varphi(\dot{r}, \check{a})$, because with $r \in B$ we would also have $N[r] \models \neg\varphi(r, a)$.

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Theorem

Let WO *be the set of reals coding well-orderings. Let* $A \subseteq 2^\omega$ *be a choice set from the following partition on* WO*:*

$$xEy \Leftrightarrow x, y \text{ code well-orderings of the same ordertype}$$

Then A *is measurable. In fact* A *has measure zero.*

Metamathematical Proof, Fenstad-Normann 1972.

Let M be an arbitrary countable transitive model of (enough of) ZFC. So $A = W_0 \cup W_1$, where W_0 codes the ordinals in M and W_1 codes those not in M. Now, W_0 is a countable set of reals, and hence has measure zero. Next we show W_1 can be covered by a countable union of measure zero sets, which implies that A has measure zero.

Consider random forcing over M. We claim that any real $r \in W_1$ will be non-random over M. If it were, then $M[r]$ is a generic extension of M, which would have the same ordinals as M, and hence the ordinal coded by r is in M, contradicting that $r \in W_1$.

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Metamathematical proof, continued.

Now since each $r \in W_1$ fails to be random over M, by Solovay's characterization of random-genericity, r belongs to a measure zero G_δ set coded in M. But there can be only countably many such sets, so W_1 is covered by a countable union of measure zero sets. □

Classical Proof, Luzin & Sierpiński 1918.

This proof was originally phrased in the theory of sieves and constituents. First notice that $\mathrm{WO} = \bigcup_{\alpha<\omega_1} P_\alpha$, where P_α is the (Borel) set of reals coding well-ordering of type α. Second, since WO is Π^1_1, it is measurable. And by usual properties of Lebesgue measure, $\mathrm{WO} = \bigcup_{n\in\omega} N \cup M_n$, where N has measure zero and each M_n is closed.

By Σ^1_1-boundedness, each M_n is bounded in WO. Write α_n as the least upper bound of (the ordinals coded in) M_n. Note that this implies that for all $\beta > \alpha_n$, we have $M_n \cap P_\beta = \emptyset$. In other words, $M_n = \bigcup_{\alpha<\alpha_n} M_n \cap P_\alpha$. But now observe that, since $P_\alpha \cap A$ only has a single element, $M_n \cap A$ is at most countable and hence measure zero. Therefore,

$$\begin{aligned} A &= A \cap \mathrm{WO} \\ &= \bigcup_{n\in\omega} (A\cap N) \cup (A \cap M_n) \end{aligned}$$

This writes A as a countable union of measure zero sets, and hence A has measure zero. □

3 philosophical questions

1. Are the metamathematical proofs really different from the classical proofs?
2. For proofs crucially using absoluteness, can a structuralist (“I don’t care what ordinals *really* are”) recover the mathematical content?
3. Some proofs make substantial use of countable transitive models. Can a non-ctm understanding of forcing recover the same results?