Pre-history of invariant descriptive set theory

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Descriptive Set Theory

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Let me give a quick A- historical overview DST's history following this motivation, as is commonly told (Kanamori, 1995; Moschovakis, 2009).

- (1898-1905) Borel, Baire, Lebesgue[1] studied the Borel sets and their measure/category properties, incorporating Cantor's countable ordinals.

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  - "Au cours de cette étude je formulais cet énoncé: la projection d'un ensemble mesurable B est toujours un ensemble mesurable B. La démonstration était simple, courte, mais fausse. M. Lusin, alors professeur débutant, et M. Souslin, l'un de ses premiers élèves, aperçurent la faute et entreprirent de la réparer." (Lebesgue, in the preface to Luzin, 1930)

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  - (1917) Suslin discovered projection of Borel sets (i.e., the analytic sets) need not be Borel, correcting an error in Lebesgue 1905. Luzin and Suslin proved they have the usual regularity properties enjoyed by Borel sets.
  - (1925) Taking projection as a basic operation, Luzin and Sierpiński defined and investigated the hierarchy of projective sets, lifting some of Lebesgue's work on Borel sets to higher levels.
  - (1930) Luzin's Paris lectures (Luzin, 1930) were published (more on that later).

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$\mathbb{R}$ or $\mathcal{N}$	ω
continuous functions	recursive functions
Borel sets	hyperarithmetical sets
analytic sets	Σ^1_1 sets
projective sets	analytical sets.

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    - Slogan: in a large universe, constructively defined sets are nice; in a small universe, bad sets show up at a very concrete level.
    - Today, DST is an integral part of inner model theory. [1]

[1] See MathOverflow: Why does inner model theory need so much descriptive set theory (and vice versa)?

What's wrong with that story?

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1. Harrington, L. A., Kechris, A. S., & Louveau, A. (1990). *A Glimm-Effros dichotomy for Borel equivalence relations.* Journal of the American mathematical society, 3(4), 903-928.
2. Foreman, M., Rudolph, D. J., & Weiss, B. (2011). *The conjugacy problem in ergodic theory.* Annals of mathematics, 1529-1586.
3. Marks, A. S., & Unger, S. T. (2017). *Borel circle squaring.* Annals of Mathematics, 186(2), 581-605.
4. Sabok, M. (2016). *Completeness of the isomorphism problem for separable C*-algebras.* Inventiones mathematicae, 204(3), 833-868.
5. Conley, C., & Miller, B. (2017).*Measure reducibility of countable Borel equivalence relations.* Annals of Mathematics, 185(2), 347-402.

Kanovei (2008), *Borel Equivalence Relations: Structure and Classification*

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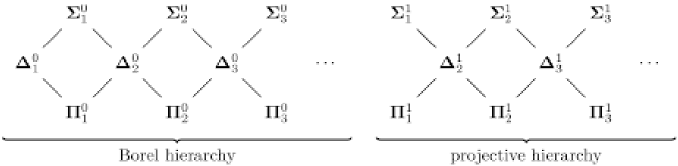
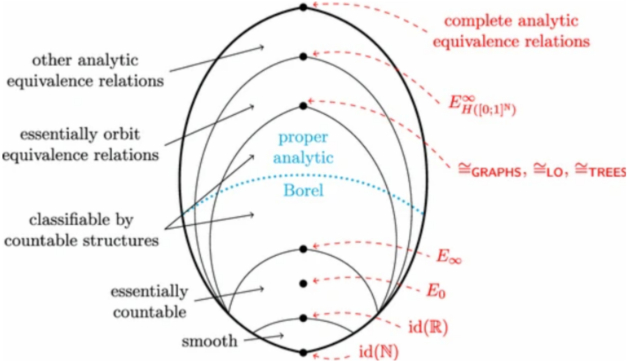
Motivation

The obvious question: what happened?

How did it go from this:



To this:


complete analytic
equivalence relations
other analytic
equivalence relations
$E^{\infty}_{H([0;1]^{\mathbb{N}})}$
essentially orbit
equivalence relations
proper
analytic
$\cong_{\text{GRAPHS}}, \cong_{\text{LO}}, \cong_{\text{TREES}}$
Borel
classifiable by
countable structures
E_∞
E_0
essentially
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$\mathrm{id}(\mathbb{R})$
smooth
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a.k.a. (?) Borel equivalence relations theory

Abstract study of the hierarchy of classification problems.

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Definition

Let E_1, E_2 be a Borel equivalence relation on standard Borel spaces X_1, X_2, respectively. We say E_1 is *Borel reducible* to E_2 (written as $E_1 \leq_B E_2$) iff there is a Borel function $F : X_1 \to X_2$ such that $uE_1v \Leftrightarrow f(u)E_2f(v)$. Such a function F is called a *Borel reduction* of E_1 to E_2.

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Intuition

A Borel reduction $F : (X_1, E_1) \to (X_2, E_2)$ associates, in a reasonably concrete way, each $x \in X_1$ with a **complete invariant** $y \in X_2$. This way, to know whether $u, v \in X_1$ fall in the same classification, we can just check if they get assigned equivalent invariants. In a number of familiar cases, E_2 is just Identity on some Polish space.

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Toy example: $= \leq_B E_0$

The identity relation $=$ on Cantor space 2^ω is Borel reducible to the relation E_0 of eventual equality on 2^ω. Indeed, fix any computable encoding $s : 2^{<\omega} \leftrightarrow \omega$, the function $F : 2^\omega \to 2^\omega$ mapping each $f \in 2^\omega$ to the characteristic function of $\{s(f \restriction n) \mid n \in \omega\}$ is a Borel reduction of $=$ to E_0.

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Toy example: $E_0 \not\leq_B =$

E_0 is not Borel reducible to $=$. If it were, then the preimage of each basic open set is a tail set, so by Kolmogorov's zero-one law it must have measure 0 or 1. Letting $I(k)$ be the unique 0-1 sequence of length k such that the basic open set determined by it has a preimage of measure one, it follows that any Borel reduction must map a measure one set to the singleton $\bigcup_{k \in \omega} I(k)$, which is impossible because any such reduction must be countable-to-one.

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Why the definability restriction?

This is because with the axiom of choice, questions about reduction are trivial: a choice function is a reduction from any equivalence relation to the identity relation. $E \leq_{\text{unrestricted}} Id$ for any E.

Birth of IDST: Some Pivotal Publications

- Friedman, H., & Stanley, L. (1989). *A Borel reductibility theory for classes of countable structures.* The Journal of Symbolic Logic, 54(3), 894-914.

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- Becker, H., & Kechris, A. S. (1996). *The descriptive set theory of Polish group actions* (Vol. 232). Cambridge University Press.
- Kechris, A. S. (1999). *New directions in descriptive set theory.* Bulletin of Symbolic Logic, 5(2), 161-174. (The 1998 Gödel Lecture)

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- Kechris, A. S. (1999). *New directions in descriptive set theory.* Bulletin of Symbolic Logic, 5(2), 161-174. (The 1998 Gödel Lecture)
- Hjorth's 1998 ICM Lecture, and his two-part Tarski Lectures (2010) (and also numerous other monographs by Hjorth.)

Glimm-Effros dichotomy

Theorem (Harrington, Kechris, Louveau)

If E is a Borel equivalence relation, then one of the following holds:

1. $E \leq_B =_{2^\omega}$
2. $E_0 \leq_B E$

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Origins in operator algebra

- Glimm, J. (1961). *Locally compact transformation groups*, Transactions of the America ematical Society, vol. 101, 124-138
- Effros, E. G. (1965). *Transformation groups and C^*-algebras*, Annals of Mathematics, vol. 81, pp. 38-55

Friedman-Stanley

Friedman, H., & Stanley, L. (1989). *A Borel reductibility theory for classes of countable structures.* The Journal of Symbolic Logic, 54(3), 894-914.

Abstract. We introduce a reducibility preordering between classes of countable structures, each class containing only structures of a given similarity type (which is allowed to vary from class to class). Though we sometimes work in a slightly larger context, we are principally concerned with the case where each class is an invariant Borel class (i.e. the class of all models, with underlying set $= \omega$, of an $L_{\omega_1\omega}$ sentence; from this point of view, the reducibility can be thought of as a (rather weak) sort of $L_{\omega_1\omega}$-interpretability notion). We prove a number of general results about this notion, but our main thrust is to situate various mathematically natural classes with respect to the preordering, most notably classes of algebraic structures such as groups and fields.

Friedman-Stanley

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The motivation for formulating and investigating these notions comes from several sources. First, part of the material of [14] can be viewed, after the fact, as providing a certain number of Borel embeddings which preserve additional information. Second, there is the (rather tenuous) connection to Vaught's conjecture, noted above. Most importantly however, in our view, these notions are a mathematically natural attempt to provide a notion of the complexity of a (reasonable) class of countable structures. We feel that this view is borne out by our results. Particularly significant, we feel, is the pattern emerging from the juxtaposition of Theorem 2 and Theorem 9, on one hand, and Theorem 1 and Theorem 10 on the other. We shall discuss this further following the statement of our results, below. Before we describe the results in detail, we should note that on occasion we work in a somewhat larger context: we are given an invariant Borel class and an equivalence relation (analytic, frequently Borel) on it, other than isomorphism, e.g., subsets of ω mod finite sets. We investigate Borel embeddability of the quotient class into nonquotiented invariant Borel classes, and vice versa, and in particular whether the quotient class is $\sim_B$ to any invariant Borel class. We turn, now, to the description of

Model-theoretic motivations, 70's

Vaught's Conjecture

A first-order theory in a countable language (or sometimes also a $L_{\omega_1\omega}$ theory) has either countably many or perfectly many non-isomorphic models.

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Topological Vaught's Conjecture (special case), $TVC(S_\infty)$

The orbit equivalence relations induced by the logic action of S_∞ has either countably many or perfectly many equivalence classes.

Group-theoretic paradigm

Becker, H., & Kechris, A. S. (1996). *The descriptive set theory of Polish group actions* (Vol. 232). Cambridge University Press.

The structure of Borel actions of Polish *locally compact*, i.e., second countable locally compact, topological groups has long been studied in ergodic theory, operator algebras and group representation theory. See, for example, Auslander–Moore [66], Feldman–Hahn–Moore [78], Glimm [61], Kechris [92a], Mackey [57, 62, 89], Moore [82], Ramsay [82, 85], Sinai [89], Varadarajan [63], Vershik–Fedorov [87], Zimmer [84] for a sample of this work. This is closely related to the subject matter of this book. More recently, there has been increasing interest in an extension of the above: studying the structure of Borel actions of *arbitrary*, not necessarily locally compact, Polish groups. Such groups are ubiquitous in mathematics, most often as groups of symmetries of mathematical structures, e.g., the symmetric group S_∞ on a countably infinite set, the group of homeomorphisms $H(X)$ of a compact metrizable space X, the unitary group $U(H)$ of a separable infinite dimensional Hilbert space H, groups of automorphisms of standard measure spaces, diffeomorphism groups, etc. The nonexistence of Haar measure for such groups makes their study quite challenging and requires the development of new methods, one of which is the use of Baire

An example: von Neumann's isomorphism problem

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morphieinvarianten Eigenschaften. Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden[13], vielleicht sogar eine stetig-differentiierbare, oder gar eine mechanische.

[13] Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben.

Halmos-von Neumann Theorem

Theorem (Halmos-von Neumann (1942))

The isomorphism problem for compact groups can be reduced to the equality of countable sets in certain Polish spaces.

Ornstein's Classification Theorem (1970)

Definition

A Bernoulli shift is a quadruple $(X, \mathcal{B}, \mu, T)$ such that
1. $X = \{1, 2, .., n\}^{\mathbb{Z}}$ for some natural number n
2. $\mathcal{B}$ is the Borel σ-algebra on X
3. μ is a product measure given by a probability distribution $(p_1, ..., p_n)$ with $\sum p_i = 1$
4. T shifts the space: for $x = (x_n)_{n \in \mathbb{Z}}$, $(Tx)_n = x_{n-1}$

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Definition

The Kolmogorov-Sinai entropy of a Bernoulli shift is $-\sum_{i=1}^{n} p_i \log p_i$

Definition

Two Bernoulli shifts $(X, \mathcal{B}, \mu, T)$ and$(Y, \mathcal{C}, \nu, S)$ are isomorphic if there is a measure-preserving map Φ from a μ-measure 1 subset of X onto a ν-measure 1 subset of Y such that $\Phi(Tx) = S\Phi(x)$ for μ-a.e. $x \in X$.

Kolmogorov-Sinai entropy

Theorem (Kolmogorov-Sinai, 50s)

If two Bernoulli shifts are isomorphic, then they have the same KS entropy (which is a real number).

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Remark

This theorem was used to disprove a conjecture by von Neumann, asking whether two specific transformations are isomorphic. K-S showed it in the negative by showing their entropies are different.

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Remark

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K-S provided a concrete procedure to associate to each Bernoulli shift with a real number, so that the problem of non-isomorphism is reduced to the problem of non-identity on the real number.

Ornstein's Classification theorem

Question

If there any sort of measure-preserving transformation that's completely classified by its K-S entropy (i.e., the converse holds)?

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Theorem (Ornstein 1970)

Two Bernoulli shifts are isomorphic if and only if they have the same entropy.

A classic case of Borel reduction

Point: for each Bernoulli shift, we associate (in a Borel way) a real number, i.e., its entropy, such that the problem of isomorphism is completely reduced to the problem of identity.

Template

Borel map F : Bernoulli shifts $\to \mathbb{R}$, such that

$$X \cong Y \Leftrightarrow F(X) = F(Y)$$

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We now say the Bernoulli shifts are completely *classified* by their entropy.

What we don't know (or what nobody has bothered to write down yet)

My goal

- To find out what the descriptive set theorists were doing then.
- To identify the "spiritual ancestors" to the more modern results.
- Challenge: bring the classical concerns of the Polish and Russian schools closer to the stuff in the 60's, so that the emergence of IDST seems not so sudden or miracuous.

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- (really provocative, CRAZY stuff) To suggest a plausible non-mathematical development that had non-trivial impact.

Thinking in terms of equivalence relations is a very natural thing to do for early set theorists. For instance, Cantor's first formulation of CH was in terms of the number of equivalence classes.

Cantor's *Beitrage*

Via an inductive procedure (*Inductionsverfahren*), whose presentation we shall not detail here, the proposition is suggested that the number of classes resulting from this principle of divisionis finite, specifically, that it is equal to two [...] We will postpone a thorough examination of this question to a later occasion.

Luzin's Program

Luzin's goals, 1917-1936

To determine the structure of the projective hierarchy. And to determine if the interplay between complexity and regularity continues at higher levels.

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Luzin (1925), *Les propriétés des ensembles projectifs*

One does not know, and one will never know, whether the PCA ($\underset{\sim}{\Sigma}^1_2$) sets are Lebesgue measurable.

Luzin's works on equivalence relations

Kanovei

"Luzin was probably the first in the descriptive theory to turn his attention to the difficulties associated with equivalence relations."

Luzin (1927), *Sur les ensembles analytiques*, XIV. Le Transfini

The study arose out of concerns about the axiom of choice.

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The study arose out of concerns about the axiom of choice.

Nous dirons que nous sommes dans le cas du *choix lebesguien* si nous pouvons tirer de Λ une loi nouvelle λ toute finie qui permette de définir d'une manière précise et sans ambiguïté possible un ensemble de points L contenant un et un seul point dans chacun des ensembles M'.

S'il n'existe pas une telle possibilité, nous dirons que nous sommes dans le cas du *choix zermellien*: c'est le fameux *Principe du Choix arbitraire* appelé souvent *Axiome de M. Zermelo* qui fait l'affirmation catégorique, *dans tous les cas*, de l'existence *réelle* d'un ensemble Z ayant un et un seul point dans chaque M', en donnant aux mots une extension extraordinaire et trop peu kroneckérienne pour qu'on puisse, à mon avis, lui attribuer un sens.

Mais il y a lieu ici de distinguer un cas particulier important, celui où l'ensemble M est *dénombrable*. Dans ce cas, le choix lebesguien est bien imposé, puisque il est *à priori* impossible, à notre avis, d'avoir les ensembles M' *eux-mêmes* numérotés au moyen des entiers positifs sans avoir préalablement une loi qui définit les points choisis, un dans chacun des ensembles M', et numérotés d'une manière analogue.

Luzin (1927), *Sur les ensembles analytiques*, XIV. Le Transfini

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Ces préliminaires terminés, l'Axiome du Partage [2]) *affirme l'existence réelle de la totalité T des systèmes complets distincts qui épuise d'une manière définitive le continu.*

Les arguments qui peuvent être inventés en faveur de cet axiome sont tous d'ordre psychologique, comme dans le cas de l'Axiome du Choix, d'ailleurs, et ne valent pas mieux [3]); la collection fini-de phrases qui constitue la relation R, en nous prêtant la possibilité de définir de systèmes complets particuliers, est elle-même bien loin de nous présenter une décomposition totale du continu.

Luzin (1927), *Sur les ensembles analytiques*, XIV. Le Transfini

Luzin's *partage lebesguien*: the origin of what's known today as "perfectly many equivalence classes"

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Luzin's *partage lebesguien*: the origin of what's known today as "perfectly many equivalence classes"

Il en est tout autrement pour le cas singulier où nous pouvons tirer de R une loi finie λ qui définit un ensemble de points L ouissant des deux propriétés suivantes:

1° xRx' est fausse si les points x et x' ($x \neq x'$) appartiennent à L;

2° Quel que soit le point y pris dans le continu, il existe un point x de L tel que xRy est vraie.

Nous appellerons *partage lebesguien* tout partage qui possède ces deux propriétés. C'est dans ce cas seul que la totalité T existe réellement, étant achevée; elle est donc légitime.

Luzin (1927), *Sur les ensembles analytiques*, XIV. Le Transfini

Strong words from Luzin:

Mais, dans le cas général où nous n'avons plus du partage lebesguien, la totalité T est, à notre avis, tout illégitime: *ce n'est qu'une pure virtualité.*

"Common Motivation" in Luzin's 1927 treatment of equivalence relation

Luzin's claim

Only two types of equivalence relations are "nice":

1. Those with at most countably many equivalence classes.
2. Those with perfectly many equivalence classes.

Much later...

Recall the common motivation: "concretely definable objects are nice"

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Theorem (Silver (1980))

If E is a $\underset{\sim}{\Pi}^1_1$ equivalence relation on a Polish space, then either E has at most $\aleph_0$ equivalence classes or there exits a perfect set of mutually inequivalent elements.

Theorem (Burgess (1978))

If E is a $\underset{\sim}{\Sigma}^1_1$ equivalence relation on a Polish space, then either E has at most $\aleph_1$ equivalence classes or there exits a perfect set of mutually inequivalent elements.

Spiritual ancestors to the dichotomy theorems of Burgess, Silver, and others

Many associate the works of Silver and Burgess with the TVC. But we have seen that the spirit of such dichotomy has been there very early on, consistent with Luzin's program.

Sierpiński and the Axiom of Choice

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- Earlier debates about the axiom of choice tended to proceed on philosophical grounds. As evidenced in the *Cinq Lettres* (Hadamard, 1905).
- Entered Sierpński:
- "a broad ideological program was outlined by Sierpiński ... According to Sierpiński's program, it is most desirable to distinguish between theorems which can be proved without the aid of AC and those which are not provable without the help of this axiom. "

Equivalence Relations: first curiosities

A friend of mine once said...

Studying $\mathbb{R}/\mathbb{Q}$ is how you become a descriptive set theorist.

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Definition

Two reals are Vitali-equivalent iff they are some rational distance apart. The Vitali classes are the equivalence classes for this relation. The set of Vitali classes is denoted $\mathbb{R}/\mathbb{Q}$.

One of the earliest papers in Sierpiński's program

Sierpiński (1917), *Sur quelques problèmes qui impliquent des fonctions non-mesurables*.

Nous dirons qu'un problème implique des fonctions non mesurables si, en admettant ce problème résolu (affirmativement), on en déduit *sans l'axiome du choix* l'existence des fonctions non mesurables. Par exemple le problème d'existence d'un ensemble bien ordonné ayant la puissance du continu implique des fonctions non mesurables.

Le but de cette Note est d'appeler l'attention sur quelques autres problèmes de la théorie des ensembles et certaines questions d'analyse qui impliquent l'existence des fonctions non mesurables.

Goal of Sierpiński 1917

To show that intuitively obvious ("bien démontré", "well-established") principles already imply the existence of non-measurable sets.

Announcing the first result in Borel equivalence relations

Theorem (Sierpiński (1917), *Sur quelques problèmes qui impliquent des fonctions non-mesurables.*)

If the set of countable subsets of the reals has the cardinality as the continuum ($|[\mathbb{R}]^\omega| = |\mathbb{R}|$), then there is a non-measurable set.

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Soit maintenant x un nombre réel donné. Designons par $E(x)$ l'ensemble de tous les nombres $x+r$, r étant un nombre rationnel quelconque: on voit sans peine que ce sera un ensemble dénombrable et que nous aurons toujours $E(x) = E(x')$ pour $x-x'$ rationnel et $E(x) \neq E(x')$ pour $x-x'$ irrationnel.

A tout nombre réel donné x correspondra donc un nombre réel $\varphi(x) = f[E(x)]$, et il suit des propriétés de $E(x)$ et $f(E)$ que nous aurons $\varphi(x) = \varphi(x')$ pour $x-x'$ rationnel et $\varphi(x) \neq \varphi(x')$ pour $x-x'$ irrationnel.

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Or, je dis que toute fonction $\varphi(x)$ jouissant de cette propriété est non mesurable (1).

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Sierpiński's 1917 lemma, in modern language

There is no measurable reduction from Vitali equivalence to Identity.

Sierpiński's 1917 lemma

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Measure-theoretic proof.

Assume $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable and $\varphi(x) = \varphi(y)$ iff $x - y \in \mathbb{Q}$. Then $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = \varphi(x) - \varphi(-x)$ is also measurable.

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Now:

$$A := \{x \mid \psi(x) > 0\}$$

and

$$B := \{x \mid \psi(x) < 0\}$$

are both measurable. And $\psi(x) = 0$ iff $x \in \mathbb{Q}$. So A and B are symmetrical about every rational ($\psi(2r - x) = -\psi(2r + x)$). And they have the same measure in every open intervals with rational endpoints.

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And so A and B each occupies half of each rational interval, contradicting Lebesgue density. $\square$

Preview of Friday

weaker version of Sierpiński's 1917 lemma

There is no Borel reduction from Vitali equivalence to Identity.

Metamathematical proof.

Stay tuned on Friday... □

Introducing the Partition Principle

Sur une proposition qui entraîne l'existence des ensembles non mesurables.

Par

Wacław Sierpiński (Warszawa).

On pourrait regarder comme presque évidente la proposition P suivante:

P. *Une image univoque d'un ensemble ne peut pas avoir une puissance supérieure que cet ensemble lui-même.*

Or, je démontrerai ici ce

Théorème 1. *Il résulte de la proposition P sans l'aide de l'axiome du choix que*

a) $\aleph_1 \leqslant 2^{\aleph_0}$,

b) *on n'a pas* $2^{\aleph_0} = \mathfrak{m} + \mathfrak{n}$, *où* $\mathfrak{m} < 2^{\aleph_0}$ *et* $\mathfrak{n} < 2^{\aleph_0}$,

c) *il existe un ensemble linéaire ne contenant aucun ensemble parfait (non vide),*

d) *il existe un ensemble linéaire non mesurable (au sens de Lebesgue).*

Introducing the Partition Principle

Theorem (Sierpiński (1947), *Sur une proposition qui entraîne l'existence des ensembles non mesurables.*)

If $|\mathbb{R}/\mathbb{Q}| \leq |\mathbb{R}|$, then there is a non-measurable set.

Proof sketch.

An injection $\mathbb{R}/\mathbb{Q} \to \mathbb{R}$ induces a linear order $\prec$ on $\mathbb{R}/\mathbb{Q}$. Then $\{x \in \mathbb{R} \mid [x]_{E_v} \prec [-x]_{E_v}\}$ is non-measurable. This follows from similar reasoning as the measure-theoretic proof above, defining $\psi : x \mapsto -x$ and appealing to density or 0-1 law considerations. $\square$

Other piece of the puzzle

Theorem (Mycielski (1964), *Independent sets in topological algebras*)

$=_{2^\omega} \leq_B E$ *for any Borel equivalence relations* E *with meager classes.*

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Proof.

Stay tuned on Friday... $\square$

More clues from Sierpiński

Theorem (Sierpiński (1954), *Sur une proposition équivalente à l'existence d'un ensemble de nombres réels de puissance* $\aleph_1$)

TFAE:

1. $\aleph_1 \leq |\mathbb{R}|$
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2. *There is a diagonalizer for countable sets of reals: a function $F : [\mathbb{R}]^\omega \to \mathbb{R}$ such that $F(X) \notin X$.*

Note that the existence of such a function is equivalent to the existence of a function $G : \mathbb{R}^\omega \to \mathbb{R}$, such that $G(S) \neq$ any $S(n)$, and if S and S' are permutations of each other, then $G(S) = G(S')$.

Spiritual ancestor to Borel diagonalization theorem

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Sierpiński revealed that the existence of a uniform diagonalizer is a choice principle. It would have been well within the spirit of Sierpiński's program to guess that such a diagonalizer cannot be nice.

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Theorem (Borel diagonalization theorem. Friedman (1981), *On the necessary use of abstract set theory*)

Define the equivalence relation $\sim$ *on* R^ω*:* $S \sim T$ *iff* $rng(S) = rng(T)$*. Then there is no Borel map* $F : \mathbb{R}^\omega \to \mathbb{R}$ *satisfying*

1. $S \sim T \Rightarrow F(S) = F(T)$
2. $\forall n(F(S) \neq S(n))$

That is, there is no (uniform) Borel diagonalizer.

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This is a canonical way to increase complexity, i.e., the Friedman-Stanley jump of $=_\mathbb{R}$. Sierpiński's 1954 theorem was a spiritual ancestor to it.

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Proof.

Stay tuned on Friday... □

Conclusion

Alghouth Sierpiński was not primarily concerned with a general theory of equivalence relations...

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- His program inadvertently demonstrated that equivalence relations can engage fruitfully with concepts like measure and category.
- And in doing so, he provided some of the earliest results in Borel equivalence relations.
- This shows that turning to the complexities of equivalence relations is not a sudden occurrence.
- All the clues were there, in one form or another, in the youth days of DST.

What is Lusitania?

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- The participants included many future leaders of the mathematical world: Sierpiński, Kolmogorov, Aleksandrov, Lyapunov, Keldysh, Novikov, etc
- They called the group Lusitania, a wordplay between the name of (what was once briefly) the largest passenger ship and the name of their advisor.

Luzin's 1930 Paris Lectures

The group's collective work culminated in the 1930 lecture notes *Leçons sur les ensembles analytiques et leurs applications*.

Luzin (1930), *Leçons sur les ensembles analytiques et leurs applications*.

Either later research will one day lead to precise relations between projective sets as well as to the complete solution of questions relating to the measure, category, and cardinality of these sets. In that case, the projective sets will have conquered citizenship in mathematics, in the same way as [the Borel sets].

Luzin (1930), *Leçons sur les ensembles analytiques et leurs applications*.

Or the problems indicated on the projective sets will remain forever unsolved, augmented by a great deal of new problems that are just as natural and just as unattainable. In this case it is clear that the day would have come to reform our ideas on the arithmetic continuum.

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Here, Luzin is proposing a two-pronged attack:

1. Take the projective sets head-on and solve the open problems.
2. Collect enough natural unsolvable problem to self-impose a paradigm shift on how to deal with the continuum.

Both approaches resemble what set theorists are doing today. So why didn't it happen sooner?

The 1936 Luzin Affair

Answer: The Luzin Affair

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- Relevant charges: bullying and plagiarizing from his students, fawning over the French, bourgeois idealism, only publishing second-tier works in the USSR.
- Many of Luzin's students testified against him: Alexandrov, Kolmogorov, Lyusternik, Khinchin, and L. G. Shnirelman.

There are two different theories as to how this happened:

1. The academic and philosopher Ernst Kol'man was behind it (Cooke et al., 2016)
2. It was a collective mutiny by Luzin's students (Neretin, 2017).

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Consequence

Either way, this dealt heavy damages to Luzin's program.

Luzin's defense

"Everyone knows the articles of Comrade Kol'man ... he has said that my theoretical works are saturated through and through with idealism, thta they are all dangerous nonsense."

- Alexandrov described Luzin as "deferring to French mathematics not just with respect, but fawning obsequiously and praising French mathematicians in a way that was literally servile"

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- "He ascribes his own works to Lebesgue, and does it in such a silly manner. ... in this way he gets himself the reputation of a man who ascribes even his own ideas to someone else, and then when it's a matter of his own students, uses that as a screen to expropriate theirs."

Lebesgue's letter

... You will see there that I was already mixed in this by contrasting "my" science, which is bourgeoise and useless, to analysis situs [topology], a proletarian and useful science. Since the former was the science of Luzin; whereas the latter, the science of Aleksandroff. ... Aleksandroff has never cited me anymore since he must now speak badly of me in his struggle against Luzin!

The Wild, Crazy Conjecture

The Luzin affair effectively caused the demise of Luzin's program outlined in his 1930 lectures notes. Coupled with the subsequent dissolution of Lusitania, this effectively sent a next generation of experts with DST training to work in other areas of mathematics.

Loose ends

Some things not touched on today (work-in-progress):

- History of the uniformization theorems
- Discovery of the versatility of Polish spaces

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