Abstraction Principles and Size of Reality

Bokai Yao

Peking University

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Fregean Abstraction in Set Theory

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Two prominent instances: Frege's Basic Law V and Hume's Principle.

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Within a suitable framework, some fragment of mathematics can be recovered from abstraction principles.

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- Gödel-Bernays set theory (GB) quantifiers over classes, but only the first-order definable ones.
- Kelley-Morse set theory (KM) quantifiers over “all” classes.

Basic Law V in set theory

Definition

A model M of set theory satisfies Basic Law V if M defines a map $X \mapsto \varepsilon X$ from classes to the first-order objects such that for any classes X and Y,

$$\varepsilon X = \varepsilon Y \leftrightarrow \forall x (x \in X \leftrightarrow x \in Y).$$

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Proof.

The class $R = \{x \mid \exists X (x = \varepsilon X \wedge x \notin X)\}$ can have no extension satisfying BLV. □

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Hume's Principle follows from Von Neumann's Limitation of Size:

(**Limitation of Size**) All proper classes are equinumerous.

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It is more natural to consider set-theoretic abstraction principles in the context of *absolute generality*.

Set Theory with Absolute Generality

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What does the resulting set theory look like?

ZFU_R

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Definition

The language of urelement set theory contains $\mathscr{A}$ as a unary predicate for urelements. ZU is Zermelo set theory modified to allow a proper class of urelements plus $\forall x(\mathscr{A}(x) \to \forall y(y \notin x))$.

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Definition

$ZFU_R = ZU + \text{Replacement}$.
$ZFCU_R = ZFU_R + AC$.
$ZF(C) = ZF(C)U_R + \forall x \neg \mathscr{A}(x)$.

ZFU_R

Let A be a *set* of urelements.

$V_0(A) = A$;

$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup V_\alpha(A)$;

$V_\gamma(A) = \bigcup_{\alpha<\gamma} V_\alpha(A)$, where γ is a limit;

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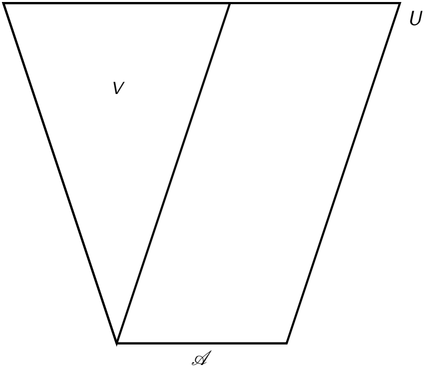
Let $\mathscr{A}$ be the class of urelements (not necessarily a set). Then we have the whole universe U as

$$U = \bigcup_{A \subseteq \mathscr{A}} V(A).$$

The universe with urelements



Abstraction in ZFU_R

Basic Law V in $\mathrm{ZFU_R}$

Theorem (Hamkins)

BLV holds in a model U of ZFU_R if the urelements in U form a set.

Basic Law V in ZFU_R

Proof Sketch.

In the metatheory fix a particular enumeration $\psi_0, \ldots, \psi_n, \ldots$ of the formulas and this enumeration will be the standard part of a definable enumeration in U.

Thus, given a definable class X of U, we let $\varphi(X, \varepsilon X)$ be the second-order assertion

> "εX is an ordered pair $\langle \ulcorner \psi_n \urcorner, u \rangle$, where ψ_n is a Σ_k formula for some k and u is the set of parameters p with minimal rank such that there is a Σ_k truth predicate T with $\forall x (x \in X \leftrightarrow T(\ulcorner \psi_n \urcorner, \langle x, p \rangle))$, and no preceding formula ψ_i has this property."

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The map $X \mapsto \varepsilon X$ then fulfills Basic Law V. □

The definability of $X \mapsto \varepsilon X$ amounts to a solution to the Julius Caesar problem.

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(**Collection**) $\forall x \in w \exists y \varphi(x, y, u) \to \exists v \forall x \in w \exists y \in v\ \varphi(x, y, u)$.

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Theorem

No model U of $\mathrm{ZFU_R}$ + Plenitude satisfies BLV.

Failure of Hume's Principle in ZFU_R

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Point: Natural axioms of urelement set theory cannot save Hume's Principle.

Full Hume's Principle

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A failure of Hume's Principle with GWO

Theorem

Assume the consistency of an inaccessible cardinal. There is a model of KMU + GWO in which the full HP fails.

A failure of Hume's principle with GWO

Proof sketch

Let V be a model of ZFC + an inaccessible cardinal κ. In V, let $\bar{\mathscr{A}} = \{0\} \times Ord$ and define

$$V[\![Ord]\!] = \bar{\mathscr{A}} \cup \{\bar{x} \in V \colon \exists x(\bar{x} = \langle 1, x\rangle \wedge x \subseteq V[\![Ord]\!])\}.$$

For every $\bar{x}, \bar{y} \in V[\![Ord]\!]$, define $\bar{x} \mathrel{\bar{\in}} \bar{y}$ as $\exists y(\bar{y} = \langle 1, y\rangle \wedge \bar{x} \in y)$.

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Let U denote the model $\langle V[\![Ord]\!],\ \bar{\mathscr{A}},\ \bar{\in}\rangle$. $U \models \mathrm{ZFCU_R}$ + Plenitude. Moreover, $V^U \cong V$ so U also contains (a copy of) κ as an inaccessible cardinal.

A failure of second-order Hume's principle with GWO

Proof sketch

In U, let $\lambda = \aleph_{\kappa^+}$ and A be a set of urelements of size λ. Define

$$U_\kappa(A) = \bigcup_{B \in P_\kappa(A)} V_\kappa(B).$$

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$$U_\kappa(A) = \bigcup_{B \in P_\kappa(A)} V_\kappa(B).$$

$U_\kappa(A) \models \text{KMU} + \text{GWO}$.

If $X \mapsto \#X$ defines a cardinality-assignment function in $U_\kappa(A)$, then one can observe that $\#X \in V$ for every class X. So the image of this map has size κ. But there are at least κ^+-many cardinalities for the proper classes of $U_\kappa(A)$—contradiction. □

Philosophical Remarks

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Hot take: In the context of absolute generality, abstraction principles are far from being analytic.

Thank you!