Variants of the Pigeonhole Principle in Fragments of Arithmetic

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Models of Arithmetic

Models of (first order Peano) Arithmetic are structures like $(\mathbb{N}, 0, 1, +, \times, <)$ satisfying certain axioms.

Like groups/rings/fields/vector spaces etc., models of arithmetic are just some abstract mathematical structures.

But from logicians' perspectives, these structures are interesting

- allow formalization of metamathematics, so that we can study metamathematics in mathamtical ways;
- related to axiomatic theory of truth (e.g., works of Halbach, Fujimoto, Parsons);
- connected to other areas in mathematical logic, like computability theory and reverse mathematics;
- ...

Standard and Non-standard Models (of Arithmetic)

$(\mathbb{N}, 0, 1, +, \times, <)$ is called the standard model (of arithmetic). All other models (of arithmetic) are called non-standard models.

Non-standard models could look very wierd. A particularly wierd-looking kind of models are κ-like (κ is a cardinal) models, which are $(M, 0, 1, +, \times, <)$ s.t.

- $|M|$ (the size of M) is κ;
- $|\{b \in M : b < a\}| < \kappa$ for all $a \in M$.

Axiomatizing κ-like Models

Let $\mathcal{M} = (M, 0, 1, +, \times, <)$ be a model. For $a, b \in M$, let

$$[a, b] = \{c \in M : a \leq c \leq b\}.$$

Theorem (Richard Kaye, 1995)

Let $\mathcal{M} = (M, 0, 1, +, \times, <)$ be a model.

(1) *$\mathcal{M}$ is κ like, iff its maximal second order expansion satisfies CARD_2, which states that there is no 1-1 map sending M into $[0, a]$ for any $a \in M$.*

(2) *$\mathcal{M}$ is κ like for some limit cardinal κ, iff its maximal second order expansion satisfies GPHP_2, which states that for any $a \in M$ there is $b \in M$ s.t. there is no 1-1 map sending $[0, b]$ into $[0, a]$.*

More Details about Arithmetic

The usual first-order axiomatization of arithmetic is called PA (for Peano Arithmetic), which includes infinitely many 'axioms'.

Theorem (Paris and Kirby, 1978)

PA *is not finitely axiomatizible, i.e.,* PA *is not equivalent to any finite set of (first-order) 'axioms'.*

The First-Order Part of GPHP_2

Let GPHP be the first-order part of GPHP_2 which consists of infinitely many axioms.

Theorem (Richard Kaye, 1997)

PA *is not equivalent to any finite set of axioms plus* GPHP*.*

To prove the theorem above, Kaye constructs κ-like models with κ being singular. This may seem very interesting, because 'moderately' large infinity is involved in solving some 'elementary' problem of arithmetic.

Fragments

Sometimes, PA and GPHP are too powerful for subtle questions. So we need refinement.

- Σ_n-induction ($I\Sigma_n$) is mathematical induction restricted to Σ_n-definable sets.
- Σ_n-bounding ($B\Sigma_n$): Suppose that $R(x, y)$ is a Σ_n-definable binary relation and $(M, 0, 1, +, \times, <)$ is a model. Then every $a \in M$ corresponds to some $b \in M$, s.t., if for every $c < a$ there is d with $R(c, d)$ holds, then for every $c < a$ there is $d < b$ with $R(c, d)$ holds.
- $\Sigma_n - \mathrm{PHP}$: There is no Σ_n-definable 1-1 map $[0, b+1] \to [0, b]$.
- $\Sigma_n - \mathrm{CARD}$ ($\Sigma_n - \mathrm{GPHP}$) is CARD (resp. GPHP) restricted to Σ_n-definable maps.

Relations between Fragments

Theorem

Over $\mathrm{PA}^- + I\Sigma_1$,

- *(Paris and Kirby, 1978)* $I\Sigma_{n+1}$ *implies* $B\Sigma_{n+1}$ $(I\Sigma_{n+1} \vdash B\Sigma_{n+1})$ *and* $B\Sigma_{n+1} \vdash I\Sigma_n$.
- *(Dimitracopoulos and Paris, 1986)* $B\Sigma_{n+1}$ *is equivalent to* $\Sigma_{n+1} - \mathrm{PHP}$;
- *(Kaye, 1995)* $I\Sigma_n \nvdash \Sigma_{n+1} - \mathrm{CARD}$ *for* $n > 0$.
- *(Kaye, 1997)* $B\Sigma_n + \mathrm{GPHP} \nvdash I\Sigma_n$ *for* $n > 0$.

Quetion (Kaye, 1995)

Does CARD *imply* GPHP *over* PA^- $(+I\Sigma_1)$*?*

This is a question that I am going to address later.

A Fragment of CARD in Reverse Math

$\Sigma_2 - \mathrm{CARD}$ has proved useful in reverse mathematics. E.g.,

Theorem

(1) *(Seetapun and Slaman, 1995)* $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \Sigma_2 - \mathrm{CARD}$.

(2) *(Conidis and Slaman, 2013)* $\mathsf{RCA}_0 + \mathsf{RRT}^2_2 \vdash \Sigma_2 - \mathrm{CARD}$.

So, neither RT^2_2 nor RRT^2_2 is arithmetically conservative over RCA_0.

$\Sigma_n - \mathrm{GPHP}$ vs. $\Sigma_n - \mathrm{CARD}$

Theorem (WW)

$\mathrm{PA}^- + I\Sigma_n + \Sigma_{n+1} - \mathrm{CARD} \nvdash \Sigma_{n+1} - \mathrm{GPHP}.$

$\Sigma_n - \mathrm{GPHP}$ vs. $\Sigma_n - \mathrm{CARD}$: A Lemma

Lemma

Let M be a model of $\mathrm{PA}^- + I\Sigma_n$ for some $n > 0$. Suppose that

- *There exists a Σ_{n+1}^M-injection from M into some $a \in M$;*
- *$N = M[c]$ for some $c < a \in M$;*
- *and N is a Σ_{n+1}-elementary cofinal extension of M.*

Then there also exists a Σ_{n+1}^N-injection from N into a.

$\Sigma_n - \mathrm{GPHP}$ vs. $\Sigma_n - \mathrm{CARD}$: The Model

Let $M \models \mathrm{PA}^- + I\Sigma_n + \neg C\Sigma_{n+1}$ be countable. Fix $a \in M$ and a Σ_{n+1}^M-definable 1-1 map $f\colon M \to [0, a]$. Also fix $(b_k : k \in \mathbb{N})$ cofinal in M.

We build a sequence $(M_k : k \in \mathbb{N})$ s.t.

(1) $M_0 = M$;

(2) $M_{k+1} = M_k[c_k]$ is a Σ_{n+1}-elementary cofinal extension of M_k;

(3) $[0, b_k]^{M_k} = [0, b_k]^{M_{k+1}}$;

(4) For each k, there is a Σ_{n+1}-definable 1-1 map $f_k : M_k \to [0, a]$;

(5) If $\varphi(x, y)$ is a $\Sigma_{n+1}^{M_k}$-formula defining a 1-1 map $M_k \to [0, a]$ then there is $j > k$ s.t. $M_j \models \forall y \neg\varphi(d_j, y)$ for some $d_j \in M_j$.

Let $N = \bigcup_{k\in\mathbb{N}} M_k$.

Then $N \models \mathrm{PA}^- + I\Sigma_n + \Sigma_n - \mathrm{CARD} + \neg\Sigma_n - \mathrm{GPHP}$.

$\Sigma_n - \mathrm{GPHP}$ vs. $\Sigma_n - \mathrm{CARD}$: The Model

Let $M \models \mathrm{PA}^- + I\Sigma_n + \neg C\Sigma_{n+1}$ be countable. Fix $a \in M$ and a Σ_{n+1}^M-definable 1-1 map $f\colon M \to [0, a]$. Also fix $(b_k : k \in \mathbb{N})$ cofinal in M.

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Let $N = \bigcup_{k \in \mathbb{N}} M_k$.

Then $N \models \mathrm{PA}^- + I\Sigma_n + \Sigma_n - \mathrm{CARD} + \neg\Sigma_n - \mathrm{GPHP}$.