Sheaf on a Topological Space: Functor and Étale Space

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The construction of sheaf topos is probably the central construction in topos theory. We begin by looking at sheaf topos on a topological space, and then we move on to more abstract constructions like sheaf topos on a site.

There are two equivalent[1] characterizations of a sheaf on a topological space, both important in their own right. The equivalence itself also has some non-trivial consequences, like the construction of *sheafification*.

We fix some notations. $X, Y, Z \cdots$ will be some topological spaces. $O(X)$ will be the poset of opens of space X, ordered by inclusion. $O(X)$ will always be seen as a small category. $U, V, W \cdots$ will always denote some opens. In a space X, if we have an open U and a family of opens $\{U_i\}_{i \in I}$ indexed by some set I such that $U = \bigcup_{i \in I} U_i$, we say $\{U_i\}_{i \in I}$ is an open cover of U, or $\{U_i\}_{i \in I}$ covers U, or $U \lhd \{U_i\}_{i \in I}$.[2] We will often omit the index set I and just write $\{U_i\}$. We write U_{ij} for $U_i \cap U_j$.

1 Sheaf as a Functor

Definition 1.1

A presheaf P over a topological space X is a presheaf over category $O(X)$, i.e. a contravariant functor $P : O(X)^{\text{op}} \to \mathsf{Set}$.

For a presheaf P and an open U, an element $s \in P(U)$ is called a *section* of P over U.

[1] They are equivalent in a strict categorical sense, as we will see later: there will be two equivalent *categories*.

[2] I invented this $\lhd$ notation myself while learning Grothendieck topology. For a site (C, J), if $S \in J(c)$, I would write $c \lhd S$. For $f : d \to c$ and a sieve S on c, if $f^*(S) \in J(d)$, I would write $f \lhd S$. The idea is that S and $\in$ kinda share some similar calculus. This is by no means standard, but it really simplified my life and I'm proud of it :D

Suppose $V \subseteq U$, then we have a function $P(U) \to P(V)$, called *restriction* along $V \subseteq U$. We let $s \cdot V$ denote the image of $s \in P(U)$ under $P(U) \to P(V)$, called the *restriction* of x on V.[3] For course, when $W \subseteq V \subseteq U$ and $s \in P(U)$, we have $s \cdot V \cdot W = s \cdot W$.

Fix a topological space X, We study the presheaf $C \cong \mathsf{Top}(-, \mathbb{R})$, sending each open U to the set of continuous $\mathbb{R}$-valued functions on U. We have the obvious restriction mappings. Suppose we have an open U, covered by two opens $U = U_0 \cup U_1$. Let $U_{01} := U_0 \cap U_1$. Now suppose we have two *sections* $f_0 \in C(U_0), f_1 \in C(U_1)$, such that they match on the intersection part:

$$f_0 \cdot U_{01} = f_1 \cdot U_{01}.$$

Then we can patch them together uniquely and obtain a section $f \in C(U)$, such that $f \cdot U_i = f_i$ for $i = 0, 1$. This motivates the definition of *sheaf*.

Definition 1.2 (Matching Family)

Fix a topological space X, a presheaf P, an open U and a cover $U \lhd \{U_i\}$. For each pair i, j, we have two mappings $\prod_i P(U_i) \rightrightarrows P(U_{ij})$:

- $\prod_i P(U_i) \xrightarrow{\pi_i} P(U_i) \xrightarrow{- \,\cdot\, U_{ij}} P(U_{ij})$,
- $\prod_i P(U_i) \xrightarrow{\pi_j} P(U_j) \xrightarrow{- \,\cdot\, U_{ij}} P(U_{ij})$,

which gives rise to a pair of functions

$$\prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_{ij}).$$

Let $\mathrm{Match}(\{U_i\}, P)$ *be the equalizer of the above diagram. An element* $\mathrm{Match}(\{U_i\}, P)$ *is called a matching family of P over $\{U_i\}$.*

In other words, a matching family $\{s_i\} \in \mathrm{Match}(\{U_i\}, P)$ consists of a family of sections $\{s_i \in P(U_i)\}_{i \in I}$, such that for each i, j, $s_i \cdot U_{ij} = s_j = U_{ij}$.

We have a canonical function $P(U) \to \mathrm{Match}(\{U_i\}, P)$. Given any section $s \in P(U)$, $\{s \cdot U_i\}$ is obviously a matching family, since for each i, j, $x \cdot U_i \cdot U_{ij} = x \cdot U_{ij} = x \cdot U_j \cdot U_{ij}$.

Definition 1.3

Fix a presheaf P on a space X.

- *P is a* seperated presheaf *if for any open U and any cover $U \lhd \{U_i\}$, the canonical mapping $P(U) \to \mathrm{Match}(\{U_i\}, P)$ is injective.*

[3] A more standard notation is $s|_V$, but this may get annoying as soon as V becomes complicated, or when you try to write down a chain of restrictions, like $(s|_V)|_W$.

- *P is a* sheaf *if for any U and any $U \lhd \{U_i\}$, $P(U) \to \mathrm{Match}(\{U_i\}, P)$ is bijective.*

One can even spell out the definition more explicitly, as Ravi Vakil did in his note *The Rising Sea*:

- *Identity axiom.* If $U \lhd \{U_i\}$ and $s, t \in P(U)$ and $\forall i. s \cdot U_i = t \cdot U_i$, then $s = t$.
- *Gluability axiom.* If $U \lhd \{U_i\}$ and $\{s_i \in P(U_i)\} \in \mathrm{Match}(\{U_i\}, P)$, then there is some $s \in P(U)$ such that $\forall i. s \cdot U_i = s_i$.

The two axioms express the condition of $P(U) \to \mathrm{Match}(\{U_i\}, P)$ being injective/surjective.

The prototypical example of a sheaf is $C \cong \mathsf{Top}(-, \mathbb{R})$. Now we look at some non-examples. Since there are two axioms to meet, there are two flavors of presheaves that fail to be a sheaf:

- Gluability fails, so there exists some matching family $\{s_i \in U_i\}$ that can't be patched together.
- Identity fails, so there exists some matching family $\{s_i \in U_i\}$ with more that one possible patch.

Example 1.4 (Seperated presheaf of bounded functions)
Let U be an open of $\mathbb{R}$. A continuous function $f : U \to \mathbb{R}$ is bounded iff its image is bounded in $\mathbb{R}$. Consider a presheaf $B : O(\mathbb{R})^{\mathrm{op}} \to \mathsf{Set}$, sending each open U to $\{f \in \mathsf{Top}(U, \mathbb{R}) \mid f \text{ is bounded}\}$. B is a seperated presheaf, but not a sheaf.

Proof. It's easy to see that identity axiom is met. For the failure of gluability, consider $1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ which is clearly not bounded, so $1_{\mathbb{R}} \notin B(\mathbb{R})$. However, let $\{U_i\}_{i \in I}$ be a cover of $\mathbb{R}$ such that each U_i has finite length, then each $1_{\mathbb{R}} \cdot U_i$ are all bounded. $\{1_{\mathbb{R}} \cdot U_i\}$ is then a matching family that can't be patched together. □

The spirit of this example is that "bounded" is not a *local* property.

A typical example of a non-seperated presheaf is constant presheaf. We need the following observation.

Lemma 1.5
Suppose $P : O(X)^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf, then $P(\varnothing) = 1$, where 1 means the singleton.

Proof. The empty set $\varnothing$, while being an open set, has a special open cover: the *empty* cover $\{U_i\}_{i \in \varnothing}$. Note that is not each U_i being empty, but the *index set* being empty. (So technically there is no such U_i!)

Trivially, there is a matching family for $\{U_i\}_{i\in\varnothing}$, since to construct a matching family, we need to provide a $s_i \in P(U_i)$ for each i, but there is no i in this case. Although we don't have to do anything, we get such a matching family.

By sheaf axioms, the matching family can be patched uniquely to a section in $P(\varnothing)$, and each section in $P(\varnothing)$ can be obtained this way, so $P(\varnothing)$ has to be a singleton. □

Example 1.6 (Constant presheaf)

For every set A, we have the constant presheaf $\Delta A : O(X)^{\mathrm{op}} \to \mathsf{Set}$. Now let A be a set with at least two elements, then ΔA is not even a seperated presheaf.

This is because $\Delta A(\varnothing) = A$ which is not a singleton.

Definition 1.7

Let $\mathrm{Sep}(X)$ be the full subcategory of $\mathsf{Set}^{O(X)^{\mathrm{op}}}$ spanned by seperated presheaves. Similarly, let $\mathrm{Sh}(X)$ be the full subcategory of $\mathsf{Set}^{O(X)^{\mathrm{op}}}$ spanned by sheaves.

Now we develop a few properties of sheaf.

Proposition 1.8

If F is a sheaf on X, then a subfunctor $S \rightarrowtail F$ is a subsheaf iff for every open U, every $s \in F(U)$, every cover $U \lhd \{U_i\}$, one has $s \in S(U)$ iff $\forall i. s \cdot U_i \in S(U_i)$.

Proof. $\Rightarrow$ is obvious. To prove $\Leftarrow$, we need to show that S is itself a sheaf.

Take any open U, cover $U \lhd \{U_i\}$, matching family $\{s_i \in S(U_i)\}$. Since S is a subfunctor of F, each s_i also belongs to $F(U_i)$. It can be easily seen that $\{s_i \in F(U_i)\}$ is also a matching family of F. (This is essentially due to the naturality of morphism $S \rightarrowtail F$, so the matching condition remains.)

Since F is a sheaf, $\{s_i\}$ can be patched to a $s \in F(U)$. However, for every i, $s \cdot U_i = s_i \in S(U_i)$, so by assumption, $s \in S(U)$. The uniqueness of s is obvious. Thus S is a sheaf. □

Every continuous function $f : X \to Y$ gives rise to a functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ in an obvious way: given any sheaf F on X, we define:

$$f_*F : O(Y)^{\mathrm{op}} \to \mathsf{Set}$$
$$U \mapsto F(f^{-1}(U))$$

The sheaf f_*F is called the *direct image* of F under f. Let's check the sheaf condition. Given any $U \lhd \{U_i\}$ in $O(Y)$ and a matching family $\{s_i \in f_*F(U_i) = F(f^{-1}(U_i))\}$. Since

$f^{-1} : O(Y) \to O(X)$ preserves arbitrary union, $f^{-1}U \lhd \{f^{-1}U_i\}$. Then the family $\{s_i\}$ can be patched uniquely to a $s \in F(f^{-1}U) = f_*F(U)$.

By definition, $\mathrm{Sh}(X)$ is a full subcategory of $\mathsf{Set}^{O(X)^{\mathrm{op}}}$. We have a fully faithful inclusion functor $i_* : \mathrm{Sh}(X) \to \mathsf{Set}^{O(X)^{\mathrm{op}}}$. The central construction of sheaf theory is to show that i_* has a left adjoint $i^* : \mathsf{Set}^{O(X)^{\mathrm{op}}} \to \mathrm{Sh}(X)$, called *sheafification*. The construction involves a detour to the study of *bundles*.

2 Bundles

3 Basic Definitions

The following paragraphs are taken directly from 2.4 of [MM12]

For any space X, a continuous map $p : Y \to X$ is called a *space over* X, or a *bundle over* X. The category of bundles over X is defined as the slice category Top/X, where X is called the *base space*.

A *cross-section* of a bundle $p : Y \to X$ is a continuous map $s : X \to Y$ with $ps = 1_X$; that is, it's a morphism from $1_X : X \to X$ to $p : Y \to X$ in Top/X.

For any $x \in X$, the inverse image $p^{-1}x$ is called the *fiber* of Y over x. It's convenient to think of a bundle as the indexed family of fibers $p^{-1}x$, one for each point $x \in X$, "glued together" by the topology of Y.

If U is an open subset of the base space X of a bundle $p : Y \to X$, then p restricts to a map $p_U : p^{-1}U \to U$ which is a bundle over U. Moreover, the square diagram

$$\begin{array}{ccc} p^{-1}U & \longrightarrow & Y \\ {\scriptstyle p_U}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ U & \longrightarrow & X \end{array}$$

with horizontal arrows the inclusions, is a pullback diagram in Top. A cross-section s of the bundle p_U, also called a cross-section of the bundle p *over* U, is a continuous map $s : U \to Y$ such that the composite ps is the inclusion $i : U \to X$. Let

$$\Gamma_p U = \{s \mid s : U \to Y, ps = i : U \to X\}$$

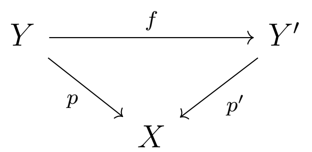
denote the set of all such cross-sections over U.

If $U \subseteq V$, one has a restriction operation $\Gamma_p U \to \Gamma_p V$, so $\Gamma_p(-)$ defines a functor $O(X)^{\mathrm{op}} \to \mathsf{Set}$. Since "being a cross-section" is a *local* property, one may check that Γ_p is in fact a sheaf.

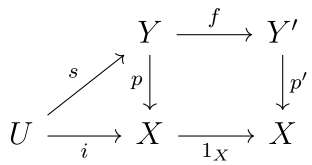
Exercise 3.1

Show that for any bundle $p : Y \to X$*, the presheaf defined above* $\Gamma_p : O(X)^{\mathrm{op}} \to \mathsf{Set}$ *is a sheaf.*

Suppose we have a bundle morphism $p \to p'$ over X, given by $f : Y \to Y'$.



We want to show that it gives rise to a sheaf morphism $\Gamma_p \to \Gamma_{p'}$. Indeed, given any open U, the component $\Gamma_p U \to \Gamma_{p'} U$ is defined as $s \mapsto f \circ s$.



The naturality condition is obvious, so $\Gamma_p U \to \Gamma_{p'} U$ is a sheaf morphism. So Γ is in fact a *functor*:

$$\Gamma : \mathsf{Top}/X \to \mathrm{Sh}(X)$$

4 Étale Bundles

There is a special kind of bundle called *étale bundle*.

Definition 4.1

A bundle $p : E \to X$ *is said to be étale when* p *is a* local homeomorphism *is the following sense: To each* $e \in E$ *there is an open neighbourhood* V *of* e *such that* pV *is open in* X *and* $p|_V$ *is a homeomorphism* $V \to pV$*.*

The category of étale bundles over X is defined as the full subcategory of Top/X spanned by étale bundles, denoted as $\mathrm{Et}(X)$.

Readers with some topology backgrounds might be familiar with the concept of *covering space*:

Definition 4.2

A bundle $p : E \to X$ is a covering space *if for each $x \in X$, the fiber $p^{-1}x = E_x$ is discrete, and there is an open neighbourhood U of x such that $p^{-1}U$ is isomorphic to the product bundle $\pi : U \times E_x \to U$.*

Every covering space is a étale space, but not the contrary, as we will see.

Now we develop some basic properties of étale bundles.

Proposition 4.3

For $p : E \to X$ étale, both p and any sections of p are open maps (in that they carry open sets to open sets). Through every point $e \in E$ there is at least one section $s : U \to E$, and the images sU of all sections form a base for the topology of E. If s, t are two sections, the set $W = \{x \mid sx = tx\}$ of points where they are both defined and agree is open in X.

Exercise 4.4

Prove the proposition above.

There's a construction from presheaf to étale space. Given any presheaf P on X and a point $x \in X$, a *germ* at x is an equivalence class of local sections: we take the set of all local sections $\coprod_{x \in U} P(U)$ and define an equivalence relation as follows: $s \in P(U)$ and $t \in P(V)$ are equivalent iff there's a smaller open $W \subseteq U \cap V$ containing x, such that $s \cdot W = t \cdot W$.

For any section s, its corresponding germ at x is denoted as s_x. We will often say "suppose we have a germ s_x", meaning s_x is a germ at x that comes from some $s \in P(U)$.

The set of germs at x is called the *stalk* of P at x, denoted as P_x.

Equivalently P_x can be defined as a colimit:

$$P_x := \mathrm{colim}_{x \in U} P(U).$$

One can easily check that given a point $x \in X$, then any presheaf morphism $h : P \to Q$ gives rise to a function $P_x \to Q_x$. This is essentially due to the functoriality of colimit. It follows that $P \mapsto P_x$ is a functor $\mathsf{Set}^{O(X)^{\mathrm{op}}} \to \mathsf{Set}$.

To construct the bundle, we take the whole disjoint union ΛP:

$$\Lambda P = \coprod_{x \in X} P_x$$

and define $p : \Lambda P \to X$ as the function sending each germ s_x to x.

Each $s \in PU$ determines a function $\dot{s}$ by:

$$\dot{s} : U \to \Lambda P, \quad \dot{s}x = s_x, \quad x \in U.$$

Topologize ΛP by taking as a base of open sets all the image sets $\dot{s}(U) \subset \Lambda P$; thus an open set in ΛP is a union of images of the sections $\dot{s}$.

Now we check the desired properties.

Proposition 4.5

$p : \Lambda P \to X$ is continuous.

Proof. Given any open $U \subseteq X$, then

$$p^{-1}(U) = \{s_x \mid \exists V \subseteq U.s \in P(V), x \in V\} = \bigcup_{V \subseteq U} \bigcup_{s \in P(V)} \dot{s}(V).$$

Hence $p^{-1}U$ is open. □

Lemma 4.6

Suppose $s, t \in P(U)$, then $\{x \in U \mid s_x = t_x\}$ is open.

Proof. For any $x \in X$, $s_x = t_x$ iff there exists an open neighbourhood $V_x \subseteq U$ of x such that $s \cdot V_x = t \cdot V_x$, so $\{x \in U \mid s_x = t_x\}$ is exactly the union of these V_xs, hence open. □

Proposition 4.7

For any local section $s \in PU$, $\dot{s} : U \to \Lambda P$ is continuous.

Proof. We only need to check that the inverse image of any basic open $\dot{t}(V)$ is open, where $t \in P(V)$. But by definition,

$$\dot{s}^{-1}(\dot{t}(V)) = \{x \in U \cap V \mid s_x = t_x\}.$$

By the lemma above, $\dot{s}^{-1}(\dot{t}(V))$ is open. □

Definition 4.8

$p : \Lambda P \to X$ is a étale bundle.

Proof. Given any $s_x \in \Lambda P$, it must come from some $s \in P(U)$ where $x \in U$, then $s_x \in \dot{s}(U)$, and $p|_{\dot{s}(U)} : \dot{s}(U) \to U$ has a inverse $\dot{s}$, so it's a homeomorphism. □

Given a presheaf morphism $f : P \to Q$, we have an obvious mapping $\Lambda f : \Lambda P \to \Lambda Q$, mapping any germ s_x to $f_x(s_x)$.

Proposition 4.9

Λf is continuous.

Proof. Given any $\dot{s}(U) \subseteq \Lambda Q$, we need to show that $(\Lambda f)^{-1}(\dot{s}(U))$ is open.

$$t_x \in (\Lambda f)^{-1}(\dot{s}(U)) \Leftrightarrow f_x(t_x) \in \dot{s}(U) \Leftrightarrow f_x(t_x) = s_x.$$

Thus,

$$(\Lambda f)^{-1}(\dot{s}(U)) = \bigcup_{V \subseteq U} \bigcup_{t \in P(V)} \dot{t}\left(\{x \in V \mid (f_V(t))_x = s_x\}\right)$$

which is an open by Lemma 4.6. □

So Λ is a functor:

$$\Lambda : \mathsf{Set}^{O(X)^{\mathrm{op}}} \to \mathrm{Et}(X)$$

Our journey leads to an adjunction.

Theorem 4.10 (Presheaf-Bundle Adjunction)

For any space X there is a pair of adjoint functors

$$\Lambda : \mathsf{Set}^{O(X)^{\mathrm{op}}} \rightleftarrows \mathsf{Top}/X : \Gamma, \Lambda \dashv \Gamma$$

.

Next time we will start from this adjunction and show that it restricts to an *adjoint equivalence*:

$$\mathrm{Sh}(X) \simeq \mathrm{Et}(X).$$

The sheafification functor is defined to be $\mathsf{Set}^{O(X)^{\mathrm{op}}} \xrightarrow{\Lambda} \mathrm{Et}(X) \xrightarrow{\Gamma} \mathrm{Sh}(X)$. We will show that it's indeed left adjoint to $i_* : \mathrm{Sh}(X) \to \mathsf{Set}^{O(X)^{\mathrm{op}}}$ and it's left exact. We will use this adjunction pair to show that $\mathrm{Sh}(X)$ is a topos.

References

[MM12] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Universitext. Springer New York, 2012.