# Co-end Calculus: An Ultra Crash Course

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The technique of co-end calculus is very powerful and very well blackboxed. We will introduce some of the most basic facts without proving them.

## 1 Definition

Fix a binary functor $F : \mathsf{C}^{\text{op}} \times \mathsf{C} \to \mathsf{D}$ that is contravariant in the first parameter and covariant in the second. You can think of it as a diagram in $\mathsf{D}$ equipped with both left and right action of $\mathsf{C}$.

**Definition 1.1**

*The* end *of $F$ is defined as the following limit.*

$$\int_{c\in\mathsf{C}} Fc \rightarrowtail \prod_{c\in\mathsf{C}} F(c,c) \rightrightarrows \prod_{f:c\to d\in\mathsf{C}} F(c,d)$$

*where the component of the parallel morphisms on a morphism $f : c \to d$ is*

- $\prod_{c\in\mathsf{C}} F(c,c) \xrightarrow{\pi_c} F(c,c) \xrightarrow{F(c,f)} F(c,d)$,
- $\prod_{c\in\mathsf{C}} F(c,c) \xrightarrow{\pi_d} F(d,d) \xrightarrow{F(f,d)} F(c,d)$.

*Similarly, the* coend *of $F$ is defined as the following colimit.*

$$\coprod_{f:c\to d\in\mathsf{C}} F(d,c) \rightrightarrows \coprod_{c\in\mathsf{C}} F(c,c) \twoheadrightarrow \int^{c\in\mathsf{C}} F(c,c).$$

*where the component of the parallel morphisms on a morphism $f : c \to d$ is*

- $F(d,c) \xrightarrow{F(d,f)} F(d,d) \xrightarrow{i_d} \coprod_{c\in\mathsf{C}} F(c,c)$,
- $F(d,c) \xrightarrow{F(f,c)} F(c,c) \xrightarrow{i_c} \coprod_{c\in\mathsf{C}} F(c,c)$.

The definition is summarized by nlab as:

> The *end* of the functor picks out the universal subobject on which the left and right action coincides. Dually, the *coend* of $F$ is the universal quotient of $\coprod_{c\in\mathsf{C}} F(c,c)$ that forces the two actions of $F$ on that object to be equal.

## 2 Da Rules

Since a coend is a special colimit and an end is a special limit, we immediately have:

**Theorem 2.1 (Hom functor commutes with integrals)**

*For any functor* $K : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{D}$,

$$\mathsf{D}\left(\int^{c\in\mathsf{C}} F(c,c), d\right) \cong \int_{c\in\mathsf{C}^{\mathrm{op}}} \mathsf{D}(F(c,c),d),$$

$$\mathsf{D}\left(d, \int_{c\in\mathsf{C}} F(c,c)\right) \cong \int_{c\in\mathsf{C}} \mathsf{D}(d, F(c,c)).$$

**Theorem 2.2 (Fubini Theorem)**

*Given a functor* $F : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \times \mathsf{E}^{\mathrm{op}} \times \mathsf{E} \to \mathsf{D}$, *we have:*

$$\int_{c\in\mathsf{C},e\in\mathsf{E}} F(c,c,e,e) \cong \int_{c\in\mathsf{C}}\int_{e\in\mathsf{E}} F(c,c,e,e) \cong \int_{e\in\mathsf{E}}\int_{c\in\mathsf{C}} F(c,c,e,e),$$
$$\int^{c\in\mathsf{C},e\in\mathsf{E}} F(c,c,e,e) \cong \int^{c\in\mathsf{C}}\int^{e\in\mathsf{E}} F(c,c,e,e) \cong \int^{e\in\mathsf{E}}\int^{c\in\mathsf{C}} F(c,c,e,e).$$

**Definition 2.3 (Tensor, Cotensor)**

*Suppose* $\mathsf{C}$ *has all coproduct, then* tensor *functor* $\otimes : \mathsf{Set} \times \mathsf{C} \to \mathsf{C}$ *is defined as*

$$X \otimes c := \coprod_{x\in X} c.$$

*Dually, suppose* $\mathsf{C}$ *has all product, then* cotensor *functor* $\pitchfork : \mathsf{Set}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$ *is defined as*

$$X \pitchfork c := \prod_{x\in X} c.$$

Suppose $\mathsf{C}$ has both product and coproduct. Immediately we have:

$$\mathsf{C}(X\otimes c, c') \cong \mathsf{Set}(X, \mathsf{C}(c,c')) \cong \mathsf{C}(c, X \pitchfork c').$$

**Theorem 2.4 (Natural Transformation as end)**

*Given functor* $F, G : \mathsf{C} \to \mathsf{D}$ *where* $\mathsf{C}$ *is small and* $\mathsf{D}$ *is locally small, then we have:*

$$\mathsf{D}^{\mathsf{C}}(F,G) \cong \int_{c\in\mathsf{C}} \mathsf{D}(Fc, Gc).$$

**Theorem 2.5 (ninja Yoneda Lemma)**

*For every functor* $K : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$*, we have:*

$$K \cong \int^{c \in \mathsf{C}} Kc \times \mathsf{C}(-, c) \cong \int_{c \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(\mathsf{C}(c, -), Kc).$$

*Dually, for every functor* $H : \mathsf{C} \to \mathsf{Set}$*, we have:*

$$H \cong \int^{c \in \mathsf{C}^{\mathrm{op}}} Hc \times \mathsf{C}(c, -) \cong \int_{c \in \mathsf{C}} \mathsf{Set}(\mathsf{C}(-, c), Hc).$$

*Proof.* Let's prove the case for $K$ to showcase the power of co-end calculus. *Yoneda lemma* is easy:

$$\int_{c \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(\mathsf{C}(c, d), Kc) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, d), K) \cong Kc.$$

while *coYoneda lemma* is proved as follows:

$$\begin{aligned} \mathsf{Set}\left(\int^{c \in \mathsf{C}} Kc \times \mathsf{C}(d, c), X\right) &\cong \int_{c \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(Kc \times \mathsf{C}(d, c), X) \\ &\cong \int_{c \in \mathsf{C}} \mathsf{Set}(\mathsf{C}(d, c), \mathsf{Set}(Kc, X)) \\ &\cong \mathsf{Set}^{\mathsf{C}}(\mathsf{C}(d, -), \mathsf{Set}(K-, X)) \\ &\cong \mathsf{Set}(Kd, X). \end{aligned}$$

thus by Yoneda,

$$\int^{c \in \mathsf{C}} Kc \times \mathsf{C}(d, c) \cong Kd.$$

□

This formula is also called *Yoneda reduction*. It follows that any presheaf $K$ is a canonical colimit of representable presheaves.