Presheaf Topos

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Recall our definition of an elementary topos:

Definition 0.1

A category $\mathcal{E}$ is an elementary topos (or just a topos) if it has the following properties:

- *$\mathcal{E}$ is finitely complete and finitely cocomplete*[1]*,*
- *$\mathcal{E}$ is Cartesian closed,*
- *$\mathcal{E}$ has a subobject classifier.*

Today we will study the presheaf category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ of a small category C. Our main goal is to show that it is a topos.

Note that the size requirement of C is necessary for $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ to be locally small, so $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(-, \Omega) \cong \mathrm{Sub}$ indeed sends any presheaf to a set, so $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is well-powered.

We start we the general theory, then we look at some enlightening examples.

1 The Theory

We always fix a small category C. For a morphism $f : c \to d \in \mathsf{C}$, a presheaf P and an element $x \in Pd$, the element $P(f)(x) \in Pc$ is denoted $x \cdot f$. You can think of a presheaf P as a variable set endowed with *right action* of C (by morphisms), and $x \cdot f$ is f acting on x.

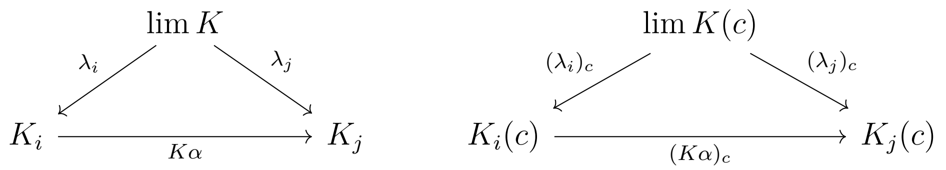
[1] In fact, as mentioned by Ye Lingyuan, the finitely cocompleteness condition can be derived using other axioms. The proof however is quite advanced, so for now let's just say it's an axiom.

1.1 Limits and Colimits

We have the following general theorem.

Proposition 1.1

For small category C *and locally small category* D*, if* D *has all* J*-shaped (co)limit, then so does* D^{C}*. The (co)limit of a diagram* $K : \mathsf{J} \to \mathsf{D}^{\mathsf{C}}$ *can be computed* object-wise*: if* $\lambda : \lim K \to K$ *is a limit cone in* D^{C}*, then given any* $c \in \mathsf{C}$*,* $\lambda^c : \lim K(c) \to K(c)$ *is also a limit cone. The same goes for colimits.*



Proof. We prove the case for limit. Suppose D has all J-shaped limits, then we have an adjunction:

$$\mathsf{D} \underset{\lim}{\overset{\Delta}{\underset{\perp}{\rightleftarrows}}} \mathsf{D}^{\mathsf{J}}$$

expressing the universal property of lim.

Since CAT is Cartesian closed (by the exercise below), by adjoint functor calculus, we have:

$$\mathsf{D}^{\mathsf{C}} \underset{\lim_*}{\overset{\Delta_*}{\underset{\perp}{\rightleftarrows}}} (\mathsf{D}^{\mathsf{J}})^{\mathsf{C}} \quad \simeq \quad (\mathsf{D}^{\mathsf{C}})^{\mathsf{J}}$$

Thus, for any $F : \mathsf{C} \to D$ and $K : \mathsf{J} \to (\mathsf{D}^{\mathsf{C}})$,

$$(\mathsf{D}^{\mathsf{C}})^{\mathsf{J}}(\Delta F, K) \cong \mathsf{D}^{\mathsf{C}}(F, \lim K).$$

And by $(\mathsf{D}^{\mathsf{J}})^{\mathsf{C}} \cong (\mathsf{D}^{\mathsf{C}})^{\mathsf{J}}$, $\lim K : \mathsf{C} \to \mathsf{D}$ maps $c \in \mathsf{C}$ to the limit of $Kc : \mathsf{J} \to \mathsf{D}$. This is exactly the meaning of *object-wise*. □

Exercise 1.2

Show that CAT *is a ccc.*

Since Set is both complete and cocomplete, we have:

Proposition 1.3

Presheaf category $\mathsf{Set}^{\mathsf{C}^{op}}$ *is both complete and cocomplete. Limits and colimits are computed object-wise.*

1.2 Cartesian Closedness

Given two presheaves P, Q, we need to figure out what their exponential Q^P presheaf is. It has to satisfy the adjunction property:

$$\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(R \times P, Q) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(R, Q^P).$$

Consider the case where R is representable, say $R \cong \mathsf{C}(-, c)$. Applying Yoneda lemma, we get:

$$\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c) \times P, Q) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c), Q^P) \cong Q^P(c).$$

So the definition of Q^P is forced by the property above. We *have* to define it that way.

Definition 1.4

Given two presheaves P, Q, their exponential Q^P is defined to be

$$Q^P(c) = \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c) \times P, Q).$$

Proposition 1.5

Q^P is indeed the exponential, i.e. $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(R \times P, Q) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(R, Q^P)$.

Proof. Propositions like this is meant to be proved by the technique of co-end calculus.

$$\begin{aligned}
\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(R, Q^P) &\cong \int_{c \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(Rc, \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c) \times P, Q)) \\
&\cong \int_{c \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}\left(Rc, \int_{d \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(\mathsf{C}(d, c) \times Pd, Qd)\right) \\
&\cong \int_{c \in \mathsf{C}^{\mathrm{op}}, d \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(Rc, \mathsf{Set}(\mathsf{C}(d, c) \times Pd, Qd)) \\
&\cong \int_{c \in \mathsf{C}^{\mathrm{op}}, d \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(Rc \times \mathsf{C}(d, c), \mathsf{Set}(Pd, Qd)) \\
&\cong \int_{d \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}\left(\int^{c \in \mathsf{C}} Rc \times \mathsf{C}(d, c), \mathsf{Set}(Pd, Qd)\right) \\
&\cong \int_{d \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(Rd, \mathsf{Set}(Pd, Qd)) \\
&\cong \int_{d \in \mathsf{C}^{\mathrm{op}}} \mathsf{Set}(Rd \times Pd, Qd) \\
&\cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(R \times P, Q).
\end{aligned}$$

□

I'm not sure what's the intuition behind Q^P. Any suggestion is welcome!

1.3 Subobject Classifier

Lemma 1.6

In a category C *with all pullbacks, a morphism* $f : c \to d$ *is monic iff the the pullback of* f *along itself is* $1_c : c \to c$*, i.e. the following diagram is a pullback square.*

$$\begin{array}{ccc} c & \xrightarrow{1_c} & c \\ {\scriptstyle 1_c}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ c & \underset{f}{\rightarrowtail} & d \end{array}$$

Exercise 1.7

Prove the lemma above.

Suppose one has a mono of presheaf $m : P \rightarrowtail Q$, then by the lemma, the following square is a pullback.

$$\begin{array}{ccc} P & \xrightarrow{1_P} & P \\ {\scriptstyle 1_P}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ P & \underset{m}{\rightarrowtail} & Q \end{array}$$

thus for any $c \in \mathsf{C}$, the following square is a pullback in Set.

$$\begin{array}{ccc} Pc & \xrightarrow{1} & Pc \\ {\scriptstyle 1}\downarrow & \lrcorner & \downarrow{\scriptstyle m_c} \\ Pc & \underset{m_c}{\longrightarrow} & Qc \end{array}$$

again by the lemma, this implies that m_c is monic in Set, thus injective.

Conversely, suppose every component of a presheaf morphism $m : P \to Q$ is injective, then it easily follows that m itself is monic. In conclusion, the property of being monic can be decided object-wise.

A subpresheaf of a representable presheaf $S \rightarrowtail \mathsf{C}(-, c)$ has the following characterization:

Definition 1.8

A sieve S *on an object* c *is a set of morphisms into* c *which is downward closed: if* $f : d \to c \in S$*, then for any* $g : e \to d$*,* $fg : e \to c \in S$*.*

Suppose we have such a sieve S. How do we get an actual subpresheaf of $\mathsf{C}(-, c)$? Well, $S(d)$ is just morphisms in S whose source is d. Conversely, when $S \rightarrowtail \mathsf{C}(-, c)$ is a subpresheaf, then the sieve is just $\coprod_{d \in \mathsf{C}} S(d)$.

A morphism $f : c \to d$ can be Yoneda embedded to $f_* : \mathsf{C}(-, c) \to \mathsf{C}(-, d)$. Applying subobject, we obtains a function $(f_*)^* : \mathrm{Sub}(\mathsf{C}(-, d)) \to \mathrm{Sub}(\mathsf{C}(-, c))$, sending every sieve on d to a sieve on c by pullback:

$$\begin{array}{ccc} f^*S & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathsf{C}(-,c) & \xrightarrow[f_*]{} & \mathsf{C}(-,d) \end{array}$$

so $k : b \to c \in f^*S$ iff $fk : b \to d \in S$. This deserves some emphasis:

$$f^*S = \{k : \bullet \to c \mid fk \in S\}.$$

For example, when C is a poset, then a sieve on c is just a downward closed subset of $\mathsf{C}_{\leq c}$. When $c \leq d$ and S in a sieve on d, then its pullback along $c \leq d$ is just $S \cap \mathsf{C}_{\leq c}$.

Any object c has a *maximal sieve* t_c, consisting of all the morphisms targetting at c. Since a sieve is required to be downward closed, a sieve S on c is maximal iff the identity morphism 1_c is in S. Althought this fact is easy, it will be frequently used, so I'll re-state it as a lemma.

Lemma 1.9

A sieve S on c is maximal iff $1_c \in S$.

Another frequently used lemma is:

Lemma 1.10

Suppose we have a morphism $f : c \to d$ and a sieve S on d, then $f \in S$ iff $f^*S = t_c$. So the pullback of a sieve along a morphism that is already in the sieve will always be maximal, and vice versa.

Exercise 1.11

Prove the lemma above.

Now suppose $\mathsf{Set}^{\mathsf{C}^{op}}$ has a subobject classifier Ω, then we should have a natural isomorphism:

$$\mathrm{Sub}(P) \cong \mathsf{Set}^{\mathsf{C}^{op}}(P, \Omega).$$

Let P be a representable presheaf $\mathsf{C}(-, c)$, we get:

$$\mathrm{Sub}(\mathsf{C}(-, c)) \cong \mathsf{Set}^{\mathsf{C}^{op}}(\mathsf{C}(-, c), \Omega) \cong \Omega(c).$$

We see that the definition of Ω is also forced by representable presheaves.

Proposition 1.12

The presheaf Ω *defined above is indeed the subobject classifier, and the component of* $\top : 1 \to \Omega$ *at* c, $\top_c : 1 \to \Omega(c)$, *picks out the maximal sieve* t_c.

So the notation of t_c is somewhat justified.

Proof. Suppose we have a subpresheaf $Q \rightarrowtail P$. Define the characteristic morphism $\chi_Q : P \to \Omega$ to be:

$$(\chi_Q)_c : P(c) \to \Omega(c), x \mapsto \{f : d \to c \mid x \cdot f \in Q(d), d \in \mathsf{C}\}.$$

One can easily check that $(\chi_Q)_c(x)$ is always downward closed, thus a sieve on c. We have to check that the pullback of χ_Q and $\top$ is Q. Fix an object c.

$$\begin{array}{ccc} Qc & \longrightarrow & 1 \\ \downarrow & & \downarrow \top \\ Pc & \xrightarrow[(\chi_Q)_c]{} & \Omega(c) \end{array}$$

First of all it's a commutative square. To see this, take any $x \in Qc$ and chase the diagram:

$$\begin{array}{ccccccc} x & \in & Qc & \longrightarrow & 1 & \ni & \bullet \\ & & \downarrow & & \downarrow \top & & \\ x & \in & Pc & \xrightarrow[(\chi_Q)_c]{} & \Omega(c) & \ni & t_c \end{array}$$

Since $x \in Qc$, $x \cdot 1_c = x \in Qc$, so $1_c \in (\chi_Q)_c(x)$, meaning $(\chi_Q)_c(x) = t_c$.

To see that it's a pullback, let's compute the actual pullback of $(\chi_Q)_c$ and $\top_c$:

$$Pc \times_{\Omega(c)} 1 \cong \{x \in Pc \mid (\chi_Q)_c(x) = t_c\}.$$

But again, if $(\chi_Q)_c(x) = t_c$, then $1_c \in (\chi_Q)_c(x)$, so $x \cdot 1_c = x \in Q(c)$. So the diagram is indeed a pullback.

Conversely, suppose $\phi : P \to \Omega$ is any morphism, we form the subpresheaf $[\phi] \rightarrowtail P$ by pulling back $\top$ along ϕ, so

$$x \in [\phi](c) \Leftrightarrow \phi_c(x) = t_c.$$

The characteristic morphism of $[\phi]$, however, maps $x \in Pc$ to:

$$(\chi_{[\phi]})_c(x) = \{f : d \to c \mid x \cdot f \in [\phi](d), d \in \mathsf{C}\}.$$

But according to the definition of $[\phi]$,

$$x \cdot f \in [\phi](d) \Leftrightarrow \phi_d(x \cdot f) = t_d.$$

And by the naturality of ϕ,

$$\begin{array}{ccccccc} x & \in & Pc & \xrightarrow{\phi_c} & \Omega(c) & \ni & \phi_c(x) \\ & & {\scriptstyle Pf}\downarrow & & \downarrow & & \\ x \cdot f & \in & Pd & \xrightarrow[\phi_d]{} & \Omega(d) & \ni & \phi_d(x \cdot f) = f^*\phi_c(x) \end{array}$$

$$\phi_d(x \cdot f) = t_d \Leftrightarrow f^*\phi_c(x) = t_d \Leftrightarrow f \in \phi_c(x).$$

Putting everything together we get:

$$f \in (\chi_{[\phi]})_c(x) \Leftrightarrow f \in \phi_c(x).$$

meaning $\chi_\phi = \phi$. So the construction $(Q \rightarrowtail P) \mapsto \chi_Q$ and $\phi \mapsto [\phi]$ are mutually inversed, thus $\mathrm{Sub}(P) \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(P, \Omega)$. ☐

We have shown that

Theorem 1.13

For any small category C, $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ *is a topos. Moreover it's both complete and cocomplete.*

2 The Examples

Depending on the nature of C, $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ may have different understandings:

- If C looks like a certain algebraic structure whose morphisms compose in an algebraic way, then a presheaf of C can be seen as a (variable) set with a right C-action. Primary example: when C has only one object, it can be seen as a monoid M, and the presheaf category is just the category of right M-sets.

- If C looks like a model of time, then a presheaf can be seen as a set through time. Primary example: $\mathsf{C} = \omega^{\mathrm{op}}$, then a presheaf category is a mathematical universe whose time looks like ω.

- If C looks like a category of spaces, the object are some simple spaces and morphisms are *continuous maps* in some sense, then a presheaf is a *generalized space*. Primary example: the category of simplicial sets $\mathsf{Set}^{\Delta^{\mathrm{op}}}$.

- There are probably something more that I don't know. Again, any suggestion is welcome.

We have thoroughly study Set^{ω} before. Let's turn our attention to the first and third intuitions.

2.1 Monoid Actions

Fix a monoid M, regarded as a small category, then the presheaf category $\mathsf{Set}^{M^{\mathrm{op}}}$ is the category of right M-sets. Indeed, each presheaf is a contravariant functor

$$X : M^{\mathrm{op}} \to \mathsf{Set}$$

which picks out a set X, and turn every element (morphism) in M into an endofunction of X. Functoriality condition becomes the action condition.

Let's compute the subobject classifier Ω. It's the set of sieves of the unique object in M, i.e. the set of *right ideals* of M.

For example, let $M = (\mathbb{N}, 0, +)$, then every natural number n gives you a right ideal $\mathbb{N}_{\geq n}$.

However when M is a *group* G, the only two right ideals of a group is the maximal one and the empty one, because whenever an ideal $S \subseteq G$ is non-empty, say $g \in S$, then $gg^{-1} = 1 \in S$, then for any $h \in G$, $1h = h \in S$, so $S = G$.

So when G is a group, the subobject classifier Ω of $\mathsf{Set}^{G^{\mathrm{op}}}$, the "truth value" object, has only two elements (as a G-set), and G acts on it trivially, Ω is the binary coproduct $1 + 1$. This amounts to say that $\mathsf{Set}^{G^{\mathrm{op}}}$ is a *Boolean* topos, its internal logic has LEM, while for most monoids M, the internal logic of $\mathsf{Set}^{M^{\mathrm{op}}}$ is intuitionistic.

There's another notion of *two-valuedness*, which says the terminal object 1 has only two subobjects: the initial object and 1 itself. Although two-valuedness and Booleannes may appear similar, they are unrelated. For example, the topos considered above $\mathsf{Set}^{(\mathbb{N},0,+)}$ is two-valued but not Boolean. We will encounter some examples of Boolean yet not two-valued toposes after introducing sheaf topoese.

2.2 Generalized Spaces

There's an important theorem I haven't proved yet, called *coYoneda lemma*, which says presheaf category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is the free cocompletion of C. The proof requires a technique of Kan extension, but I'm afraid that the note might become overwhelming, so let's talk about it later. However we will be using at least the intuition of coYoneda lemma here.

Definition 2.1

The simplex category Δ *is defined as follows.*

- *Objects: finite ordinals.* $[n]$ *means the ordinal* $n + 1$.

- *Morphisms: order-preserving functions.*

The intuition is that $[n]$ is sort of an n-dimensional triangle. For example, $[0]$ is a point, $[1]$ is a line, $[2]$ is an actual triangle, $[3]$ is a tetrahedron and so on. Topologists call these things *simplices*. The morphisms are continuous mappings in the sense that they map vertices to vertices and they preserve edges (order).

A presheaf $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is called a *simplicial set*. The intuition is that X is like a space, and $X[n]$ should be seen as the ways of laying out the simplex $[n]$ inside X. The Yoneda embedding

$$y : \Delta \to \mathsf{Set}^{\Delta^{\mathrm{op}}}$$

turns a *model space* $[n]$ into a *generalized space* $\Delta(-, [n])$. The Yoneda lemma says:

$$X[n] \cong \mathsf{Set}^{\Delta^{\mathrm{op}}}(\Delta(-, [n]), X)$$

which, under our interpretation, simply says that the way of mapping $[n]$ into X is the same, no matter whether you treat $[n]$ as a model space or a generalized space.

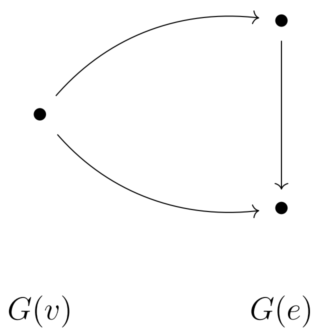
CoYoneda lemma further justifies this interpretation: a simplicial set X is canonically some simplices put together (coprodut), then stick together along certain rules (coequalizer).

Another good example is the category of directed graph. Let $\rightrightarrows$ be the category with two objects v, e and two non-identity morphisms $v \rightrightarrows e$. Then a presheaf G consists of:

- Two sets $G(e), G(v)$,
- Two functions $G(e) \rightrightarrows G(v)$.

G is exactly a *directed graph*: $G(e)$ is its set of edges and $G(v)$ is its set of vertices. The two functions $G(e) \rightrightarrows G(v)$ gives you the source and the target of any edge in G.

By Yoneda embedding, the category $\rightrightarrows$ can be embedding into the category of directed graphs. The graph $y(v)$ has only one vertex and no edge. The graph $y(e)$ has two vertices and one edge. So we might draw the whole diagram like:



By coYoneda lemma, they are exactly the *building blocks* of any directed graph.

Perhaps the most interesting thing about this presheaf category is its subobject classifier. Explore it yourself!

Exercise 2.2 (Understanding the Subobject *Classifier*)

1. *How many sieves does v have?*

2. *How many sieves does e have?*

3. *What's the two pullback functions?*

4. *Draw the directed graph of* Ω.

5. *Take any graph you like, then take any subgraph. Compute its characteristic function. How does* Ω classify *the subgraph?*