Universal Property and Adjunctions

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April 13

Our goal today is to learn the concept of adjunction. In order to that, we need to develop a better understanding of universal property[1] first.

1 Universal Property

Recall that Yoneda lemma says for any category C, we have an isomorphism

$$Fc \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c), F)$$

natural in both $c \in \mathsf{C}$ and $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. Explicitly,

- Given any $\alpha : \mathsf{C}(-, c) \to F$, we have $\alpha_c : \mathsf{C}(c, c) \to Fc$, thus $\alpha_c(1_c) \in Fc$.
- Given any $u \in Fc$, there's only one $\alpha : \mathsf{C}(-, c) \to F$ such that $\alpha_c(1_c) = u \in Fc$.

Now suppose we have a presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, we want to know if it is *representable*, that is, if there exists a $c \in \mathsf{C}$ and a *natural isomorphism* $\alpha : \mathsf{C}(-, c) \cong F$.

Here's the rough idea. We consider every natural transformation $\alpha : \mathsf{C}(-, c) \to F$. Each one may be seen as an *approximation* of F with a Hom-functor. We look for the best approximation. The following definition is just this idea made precise.

Definition 1.1

Suppose we have a functor $F : \mathsf{C} \to \mathsf{D}$ and an object $d \in \mathsf{D}$, the category of F over *d, F/d, is defined as follows.*

- *Objects: an object consists of an object $a \in \mathsf{C}$ and a morphism $\alpha : Fa \to d \in \mathsf{D}$.*

[1] Xu Yiqi reminded me that there's a precise definition of universal property. Credit to him! <3

- *Morphisms: a morphism $f : (a, h) \to (b, k) \in F/d$ is a morphism $f : a \to b$ such that the following triangle commutes.*

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle k} & \\ & d & \end{array}$$

In most cases, $F : \mathsf{C} \to \mathsf{D}$ is the inclusion functor of subcategory. So the *terminal object* of F/d is the best approximation of d in C.

Example 1.2

Take your favourite irrational number, for example π. Let $i : \mathbb{Q} \to \mathbb{R}$ denote the canonical inclusion, regarded as a functor. The category i/π is equivalent (in this case even isomorphic) to $\mathbb{Q}_{\leq \pi}$, containing every rational number below π. It has no terminal object (maximum). However, let $j : \mathbb{Z} \to \mathbb{R}$ denote the inclusion from integers to $\mathbb{R}$. The category $j/\pi \cong \mathbb{Z}_{\leq \pi}$ has a terminal object: the number 3. This is the best approximation of π with integers from below.

Example 1.3 (Slice Category)

When $\mathsf{C} = \mathsf{D}$ and F is the identity functor, F/d is simply denoted as C/d (or D/d of course), called slice category*. It's easy to show that it always has a terminal object $1_d : d \to d$. Here's my favourite example of slice category.*

- *Take the three-element set $C = \{r, g, b\}$.*[2] *The category Set/C is the category of C-colored sets. The morphisms are color-preserving functions.*

For our purpose, we need to compute y/F, where $y : \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is Yoneda embedding $\mathrm{y}(c) := \mathsf{C}(-, c)$.

- Objects: an object $c \in \mathsf{C}$ and a natural transformation $\alpha : \mathsf{C}(-, c) \to F$.

- Morphisms: $f : (c, \alpha) \to (d, \beta)$ is a morphism $f : c \to d$ such that the following triangle commutes.

$$\begin{array}{ccc} \mathsf{C}(-, c) & \xrightarrow{f_*} & \mathsf{C}(-, d) \\ & {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle \beta} & \\ & F & \end{array}$$

However by Yoneda lemma, every $\alpha : \mathsf{C}(-, c) \to F$ can be seen as a $\alpha \in Fc$, and the above triangle translates to $Ff : Fd \to Fc$ maps $\beta \in Fd$ to α.

[2] C stands for *colors* and r, g, b stands for *red, green, blue* respectively.

Exercise 1.4

Show that under the Yoneda isomorphism $Fc \cong \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-,c), F)$*, for any* $\alpha : \mathsf{C}(-,c) \to F$ *and* $\beta : \mathsf{C}(-,d) \to F$*,* $\beta \circ f_* = \alpha$ *(i.e. the above triangle commutes) is equivalent to* $(Ff)(\beta) = \alpha$*.*

So y/F is equivalent to the following category.

Definition 1.5 (Category of Elements: Contravariant Case)

The category of elements $\int F$ *for a contravariant* Set*-valued functor* $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ *(a presheaf on* C*) is defined as follows.*

- *Objects: an object* (c, x) *consists of an object* $c \in \mathsf{C}$ *and an element* $x \in Fc$*.*
- *Morphisms: a morphism* $f : (c, x) \to (d, y)$ *is a morphism* $f : c \to d \in \mathsf{C}$ *such that* $(Ff)(y) = x$*.*

Thus for a presheaf F on C, $\mathrm{y}/F \cong \int F$.

Dualizing everything we get:

Definition 1.6 (Category of Elements: Covariant Case)

The category of elements $\int F$ *for a covariant* Set*-valued functor* $F : \mathsf{C} \to \mathsf{Set}$ *is defined as follows.*

- *Objects:* (c, x) *consists of* $c \in \mathsf{C}$ *and* $x \in Fc$*.*
- *Morphisms:* $f : (c, x) \to (d, y)$ *is* $f : c \to d \in \mathsf{C}$ *such that* $(Ff)(x) = y$*.*

Proposition 1.7

- *For* $F : \mathsf{C} \to \mathsf{Set}$*,* $\int F \simeq \mathrm{y}/F$*, where* $\mathrm{y} : \mathsf{C} \to (\mathsf{Set}^{\mathsf{C}})^{\mathrm{op}}$ *is the contravariant Yoneda embedding.*
- *For* $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$*,* $\int F \simeq \mathrm{y}/F$*, where* $\mathrm{y} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}}$ *is the covariant Yoneda embedding.*

Theorem 1.8

Covariant Set*-valued functor* $F : \mathsf{C} \to \mathsf{Set}$ *is representable iff* $\int F$ *has an initial. Dually, contravariant* Set*-valued functor* $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ *is representable iff* $\int F$ *has a terminal.*

Proof. Let's prove the covariant case.

$\Rightarrow$ Suppose F is representable. We may let F be a Hom-functor $\mathsf{C}(c, -)$. Then $\int \mathsf{C}(c, -)$ is just the slice category c/C, so it has an initial $1_c : c \to c$.

$\Leftarrow$ Suppose $\int F$ has an initial (c, u), we show that $u : \mathsf{C}(c, -) \to F$ is an isomorphism.

Fix any $d \in \mathsf{C}$, we need to prove $u_d : \mathsf{C}(c, d) \to Fd$ is a bijection. But given any $v \in Fd$, by the initiality of (c, u), there exists a unique $f : c \to d$ such that $Ff(u) = v \in Fd$, which is exactly $u_d(f) \in F(d)$, by Yoneda. Thus u_d is a bijection for every $d \in \mathsf{C}$, so u is a natural isomorphism. □

The *universal property* of an object $c \in \mathsf{C}$ is characterized by a functor $F : \mathsf{C} \to \mathsf{Set}$ (or $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$) and a natural isomorphism $u : \mathsf{C}(c, -) \cong F$ (or $u : \mathsf{C}(-, c) \cong F$), that is, what functor it represents and how it represents that. By the theorem above, $u : \mathsf{C}(c, -) \cong F$ is just a $u \in Fc$ such that $(c, u) \in \int F$ is initial (or dually, terminal).

2 Adjoint Functors

At the very beginning of nlab page adjoint functor, it says:

> The concept of *adjoint functors* is a key conpect in category theory, if not *the* key concept.

Let's begin by one simple example.

2.1 Free Vector Space

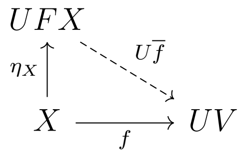
Let $\mathsf{Vect}_{\mathbb{R}}$ be the category of real vector spaces and linear mappings. We have two functors in opposite directions:

- the *free* functor $F : \mathsf{Set} \to \mathsf{Vect}_{\mathbb{R}}$, sending any set X to the space $\bigoplus_{x \in X} \mathbb{R}$,
- the *forgetful* functor $U : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Set}$ sending any space V to its underlying set.

We now study the property of the composition of F and U, the endofunctors $UF : \mathsf{Set} \to \mathsf{Set}$, $FU : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{R}}$.

Given a set X, UFX is the underlying set of $\bigoplus_{x \in X} \mathbb{R}$. The elements are finite $\mathbb{R}$-linear combinations of X. Note that we have a canonical embedding $X \to UFX$, sending each $x \in X$ to $1 \cdot x$, regareded as a really simple linear combination. Let's denote it $\eta_X : X \to UFX$. You can easily check that as X goes through Set, η forms a natural transformation $\eta : 1_{\mathsf{Set}} \to UF$.

The set UFX has a *universal property.* Suppose we have a vector space V and a function $f : X \to UV$. We want to extend f to a $\overline{f} : FX \to V$. By “extend” we mean the following triangle commutes.



which means when retricting $\overline{f}$ along η_X, $\overline{f}$ agrees with f.

The mapping $\overline{f}$ has only one choice. Given $\sum_{i\in I} r_i x_i \in UFX$, where I is some finite set, $r_i \in \mathbb{R}, x_i \in X$, since $\overline{f}$ is linear,

$$\overline{f}\left(\sum_{i\in I} r_i x_i\right) = \sum_{i\in I} r_i \overline{f}(x_i).$$

But since $\overline{f}$ agrees with f when restricting along η_X, for any $i \in I$,

$$\overline{f}(x_i) = f(x_i).$$

Putting everything together we get:

$$\overline{f}\left(\sum_{i\in I} r_i x_i\right) = \sum_{i\in I} r_i f(x_i).$$

We can treat the formula above as a *definition* for $\overline{f}$. Note that for any $i \in I$, $f(x_i) \in UV$. Since I is finite and V is closed under finite linear combination, $\sum_{i\in I} r_i f(x_i) \in UV$.

Exercise 2.1

Show that had we defined $F : \mathsf{Set} \to \mathsf{Vect}_{\mathbb{R}}$ *to be* $F(X) = \prod_{x\in X} \mathbb{R}$, *the* X*-index* direct product *(rather than direct sum) of* $\mathbb{R}$, *an* $f : X \to UV$ *might not admit an extension* $\overline{f} : UFX \to UV$. *Hint: let* V *be* $\bigoplus_{x\in X} \mathbb{R}$.

Everything so far has been taught in first-year linear algebra course: any linear mapping $f : V \to X$ is determined by its value on the basis of V.

Let’s rephrase the above universal property in the language of category of elements: (FX, η_X) is the initial in $\int \mathsf{Set}(X, U-)$, meaning FX represents $\mathsf{Set}(X, U-)$ via η_X:

$$\mathsf{Vect}_{\mathbb{R}}(FX, -) \cong \mathsf{Set}(X, U-)$$

where the image of $1_{FX} : FX \to FX$ under $\mathsf{Vect}_{\mathbb{R}}(FX, FX) \to \mathsf{Set}(X, UFX)$ is η_X.

It can be easily shown that for any fixed V, the above natural isomorphism $\mathsf{Vect}_{\mathbb{R}}(FX, V) \cong \mathsf{Set}(X, UV)$ is natural in X too.

Exercise 2.2

Show the claim above.

For any vector space V, the vector space FUV also has a univeral property. The elements in FUV are linear combinations of elements in V. Here's an example.

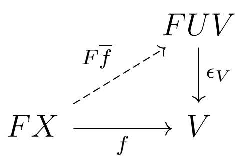
$$4(v) + 5(-7u + 2w) \in FUV.$$

So the base vectors of FUV are vectors in V. It might seem monstrously huge at the first glance, since the dimension of FUV is the cardinality of UV! Here's an easier way to comprehend FUV: vectors in it are like formal linear combinations of vectors in V, *waiting to be evaluated.*

You give the above vector in FUV to your friend and he will happily *evaluate* it to:

$$4(v) + 5(-7u + 2w) = 4v - 35u + 10w \in V.$$

So we have a canonical evaluation linear mapping $\epsilon_V : FUV \to V$. It has a dual universal property: fix a set X and a $f : FX \to V$, then it uniquely determines a $X \to UV$ such that the following triangle commutes.



In other words, (UV, ϵ_V) is the terminal in $\int \mathsf{Vect}_{\mathbb{R}}(F-, V)$, so we have a natural isomorphism for any fixed V:

$$\mathsf{Vect}_{\mathbb{R}}(F-, V) \cong \mathsf{Set}(-, UV)$$

where the preimage of $1_{UV} : UV \to UV$ is ϵ_V.

A mutual functorial universal property is expressed by F and U. They are a pair of *adjoint functors.*

2.2 Adjoint Functors: The Definitions

The notion of adjoint functor, or adjunction pair, has at least three equivalent definitions.

Fix categories C, D and functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$. We now define the relation $F \dashv G$, meaning F is the left adjoint of G, and G is the right adjoint of F.

Definition 2.3 (1.Adjunction via Hom-functor)

$F \dashv G$ iff there is an isomorphism natural in both $c \in \mathsf{C}, d \in \mathsf{D}$:

$$\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd).$$

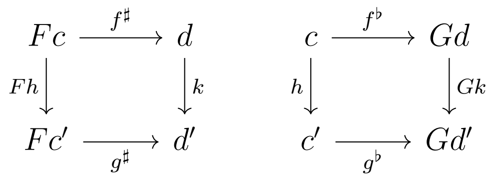
If $f : Fc \to d \in \mathsf{D}$ and $g : c \to Gd \in \mathsf{C}$ corresponds under this isomorphism, they are called the *adjunct* (by nLab) or the *transposition* (by Riehl) of each other. Here I adopt the notation from Riehl and denote $f^\sharp : Fc \to d$ in $\mathsf{D}(Fc, d)$, it corresponds to $f^\flat c \to Gd$.

In the example above we have seen that $\mathsf{Vect}_\mathbb{R}(FX, V) \cong \mathsf{Set}(X, UV)$, thus $F \dashv U$, "free is left adjoint to forgetful".[3]

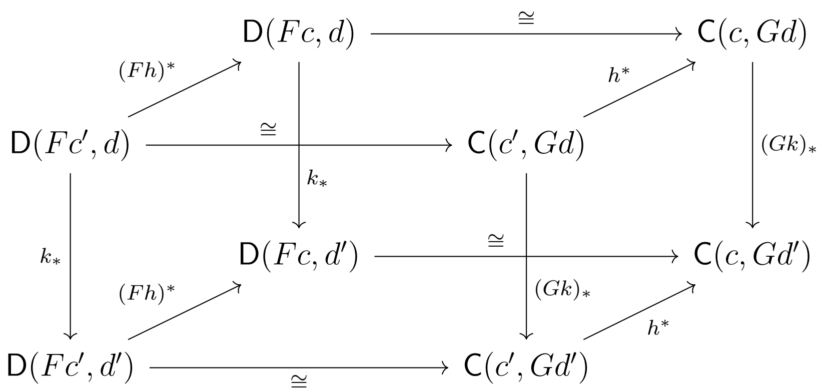
We present a very useful lemma.

Lemma 2.4 (Transposition of Commutative Squares)

Suppose $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$ natural in c, d. Then left square commutes iff right square commutes.



Proof. The following cube commutes.



It might seem scary, but the good thing about diagram chasing is that we can check the whole cube commutes by checking every side commutes. Left and right sides are obvious while the four sides in the middle commutes by the naturality of $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$.

Now suppose the left square commutes, so $f^\sharp \in \mathsf{D}(Fc, d)$ and $g^\sharp \in \mathsf{D}(Fc', d')$ are mapped by k^* and $(Fh)^*$ to the same thing. A simple diagram chasing shows that their

[3] There are countless examples of free-forgetful adjunction in algebra, just think about free group, free Abelian group, free R-module... I don't know if there's an ultimate abstraction of it, but free-forgetful adjunction appears in the theory of algebra over a monad, which is pretty widely applicable. See here.

transposition are mapped to the same thing in $\mathsf{C}(c, Gd')$, hence the right square commutes. The converse is equally easy. □

Definition 2.5 (2.Adjunction via Unit and Counit)

$F \dashv G$ if there is two natural transformations:

$$\eta : 1_{\mathsf{C}} \to GF, \epsilon : FG \to 1_{\mathsf{D}}$$

such that the following two diagram of functors commutes. This condition is called triangle identity*.*

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{1_G} & \downarrow{\scriptstyle G\epsilon} \\ & & G \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{1_F} & \downarrow{\scriptstyle \epsilon F} \\ & & F \end{array}$$

In the example above, the two triangle identities say:

- For any vector space V and any $v \in V$ (perhaps I should say $v \in UV$, you get the point), the formal linear combination $(v) \in UFUV$ is evaluated to v itself.

- For any set X and formal finite linear combination $\sum_i r_i x_i$, the linear combination $\sum_i r_i(x_i) \in FUFX$, where each (x_i) is now regarded as a linear combination itself, is evaluated to $\sum_i r_i x_i$.

Definition 2.6 (3.Adjunction via Universal Morphism)

$F \dashv G$ iff there is a natural transformation $\eta : 1_{\mathsf{C}} \to GF$ such that for any $c \in \mathsf{C}$, (Fc, η_c) is the initial in c/G.

In the example above, for any $X \in \mathsf{Set}$, there is such a universal $\eta_c : X \to UFc$, regarding each $x \in X$ as a formal linear combination $(x) \in UF$.

Theorem 2.7

The three definitions are equivalent.

Proof. We prove $1 \implies 2 \implies 3 \implies 1$.

$1 \implies 2$. Suppose $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$. Let $\eta_c : c \to GFc$ be the image of 1_{Fc}, and $\epsilon : FGd \to d$ be the preimage of 1_{Gd}. The naturality of η comes from transpositions.

$$\begin{array}{ccc} Fc & \longrightarrow & Fc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Ff} \\ Fc' & \longrightarrow & Fc' \end{array} \Longrightarrow \begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ c' & \xrightarrow[\eta_{c'}]{} & GFc' \end{array}$$

ϵ is similar.

Triangle identities also come from transpositions.

$$
\begin{array}{ccc}
FG & \xrightarrow{1_{FG}} & FG \\
{\scriptstyle 1_{FG}}\downarrow & & \downarrow{\scriptstyle \epsilon} \\
FG & \xrightarrow[\epsilon]{} & 1_{\mathsf{D}}
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
G & \xrightarrow{\eta G} & GFG \\
{\scriptstyle 1_G}\downarrow & & \downarrow{\scriptstyle G\epsilon} \\
G & \xrightarrow[1_G]{} & G
\end{array}
$$

$$
\begin{array}{ccc}
F & \xrightarrow{1_F} & F \\
{\scriptstyle F\eta}\downarrow & & \downarrow{\scriptstyle 1_F} \\
FGF & \xrightarrow[\epsilon F]{} & F
\end{array}
\quad \Longleftarrow \quad
\begin{array}{ccc}
1_{\mathsf{C}} & \xrightarrow{\eta} & GF \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle 1_{GF}} \\
GF & \xrightarrow[1_{GF}]{} & GF
\end{array}
$$

$2 \implies 3$. We need to show (Fc, η_c) is initial in c/G, which is $\int \mathsf{C}(c, G-)$. We only need to show that $\mathsf{C}(Fc, -) \cong \mathsf{C}(c, G-)$.

- Given any $f^\sharp : Fc \to d$. Let $f^\flat : c \to Gd$ be $c \xrightarrow{\epsilon_c} GFc \xrightarrow{Gf^\sharp} Gd$.
- Given any $f^\flat : c \to Gd$. Let $f^\sharp : Fc \to d$ be $Fc \xrightarrow{Ff^\flat} FGd \xrightarrow{\epsilon_d} d$.

It's easy to show that they are mutually inverse by the naturality of η and ϵ and triangle identities. The naturality of $\mathsf{C}(Fc, -) \cong \mathsf{C}(c, G-)$ also follows easily.

$3 \implies 1$. According to 3, for every fixed c, $\mathsf{D}(Fc, -) \cong \mathsf{C}(c, G-)$. Fix any $d \in \mathsf{D}$, the fact that $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$ can be proven by a diagram chasing. You need to use η_c to compute $f^\sharp \mapsto f^\flat : \mathsf{D}(Fc, d) \to \mathsf{C}(c, Gd)$. The definition is exactly the same as above. Apply the naturality of η and we're done. $\square$

2.3 Properties of Adjoint Functors

Since the notion of adjoint functor expresses some funtorial universal property, we would expect it to be unique. Indeed,

Proposition 2.8 (Uniqueness)

If $F \dashv G$ and $F \dashv G'$, then $G \cong G'$. Dually, if $F \dashv G$ and $F' \dashv G$, then $F \cong F'$.

Proof. Suppose $F \dashv G$ and $F \dashv G'$, then $\mathsf{C}(-, Gd) \cong \mathsf{D}(F-, d) \cong \mathsf{C}(-, G'd)$, both natural in d, thus $\mathsf{C}(-, Gd) \cong \mathsf{C}(-, G'd)$. By Yoneda, $G \cong G'$. The case for F is dual.

Adjunction pairs can be composed in the following sense.

Proposition 2.9 (Composition)

Suppose we have the following diagram:

$$\mathsf{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{D} \underset{K}{\overset{H}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{E}$$

then $HF \dashv GK$.

Proof. $\mathcal{E}(HFc, e) \cong \mathsf{D}(Fc, Ke) \cong \mathsf{C}(c, GKe)$. □

Proposition 2.10

Given an adjunction $F \dashv G$. Post-composition with F and G defines a pair of adjoint functors

$$\mathsf{C}^{\mathsf{J}} \underset{G_*}{\overset{F_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{D}^{\mathsf{J}}$$

for any small J, and pre-composition with F and G also defines an adjunction

$$\mathsf{E}^{\mathsf{C}} \underset{F^*}{\overset{G^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{E}^{\mathsf{D}}$$

for any locally small E.

Proof. We prove the case for post-composition using definition 2. Let the unit be $\eta : 1_{\mathsf{C}^{\mathsf{J}}} \to (GF)_*$, counit be $\epsilon : (FG)_* \to 1_{\mathsf{D}^{\mathsf{J}}}$. Triangle identity of $F_* \dashv G_*$ comes from triangle identity of $F \dashv G$. □

The following property about the interplay between adjoint functors and (co)limits is super important. We will be using it for like a thousand times in the future.

Theorem 2.11 (LAPC)

Left adjoints preserve colimits.

Proof. Suppose $F \dashv G$, $K : \mathsf{J} \to \mathsf{C}$ is a diagram with colimit $\lambda : K \to \operatorname{colim} K$. The magic step in the list of isomorphisms below is $\Delta Gd \cong G_*(\Delta d) : \mathsf{J} \to \mathsf{C}$.

$$\mathsf{D}(F \operatorname{colim} K, d) \cong \mathsf{C}(\operatorname{colim} K, Gd) \cong \mathsf{C}^{\mathsf{J}}(K, \Delta Gd) \cong \mathsf{C}^{\mathsf{J}}(K, G_*\Delta d) \cong \mathsf{D}^{\mathsf{J}}(FK, \Delta d) \cong \mathsf{D}(\operatorname{colim} FK, d).$$

Apply Yoneda we get $F \operatorname{colim} K \cong \operatorname{colim} FK$. □

Similarly,

Theorem 2.12 (RAPL)

Right adjoints preserve limits.

Now we can define the notion of *cartesian closed category.*

Definition 2.13 (Cartesian Closed Category)
A category C *is a cartesian closed category (abbreviated as ccc), if:*

- C *has finite products (that is, binary products and terminal),*
- *For each* $c \in \mathsf{C}$*, the functor* $c \times - : \mathsf{C} \to \mathsf{C}$ *has a right adjoint* $(-)^c : \mathsf{C} \to \mathsf{C}$.

For example, Set is a ccc:

$$\mathsf{Set}(X \times Y, Z) \cong (X, Z^Y)$$

where Z^Y is just $\mathsf{Set}(Y, Z)$. This process of turning a binary function to a unary higher-order functional is called *Currying* in computer science. We will study ccc in greater depth by introducing the formal language of simply typed λ-calculus next week.