Equivalence of Categories, Limit and Colimit

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1 Equivalence of Categories

Equivalence is the notion expressing two categories being *the same*.

Definition 1.1 (Equivalence of Categories)

An equivalence *between two categories* C, D *consists of:*

- *A pair of functors in opposite directions* $F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}$*.*
- *Two natural transformations* $\eta : 1_{\mathsf{C}} \cong GF, \epsilon : FG \cong 1_{\mathsf{D}}$*.*

If there exists an equivalence between C, D*, then* C, D *are* equivalent*, denoted as* $\mathsf{C} \simeq \mathsf{D}$*.*

We also say F is an equivalence, or F witnesses the equivalence between C and D.

Here's a toy model of equivalence. Let C be the singleton category 1, D be the preorder with as many objects as you like, but all of them are isomorphic. $F : \mathsf{C} \to \mathsf{D}$ picks out an object in D, while $G : \mathsf{D} \to \mathsf{C}$ sends everything to the unique object. You can figure out η and ϵ yourself.

Non-trivial examples of equivalences will gradually appear later. One of the most important example in topos theory is the equivalence of category of *sheaves* and category of *étale bundle* on a given topological space X:

$$\mathrm{Sh}(X) \simeq \mathrm{Et}(X)$$

meaning the concept of sheaf and étale bundle are actually describing the same thing, even though they seem drastically different.

We introduce a simple criterion for equivalence.

Definition 1.2

A functor $F : \mathsf{C} \to \mathsf{D}$ *is* essentially surjective *if for every* $d \in \mathsf{D}$*, there exists* $c \in \mathsf{C}$ *such that* $F(c) \cong d$*.*

Here, "*essentially*" means that we don't require strict equality $F(c) = d$, we only need isomorphism.

Exercise 1.3

Show that if $F : \mathsf{C} \to \mathsf{D}$ *is an equivalence, then* F *is fully faithful and essentially surjective.*

Proposition 1.4

Assuming the axiom of choice (for class probably), if $F : \mathsf{C} \to \mathsf{D}$ *is fully faithful and essentially surjective, then* F *is an equivalence.*

Proof.

- Stage 1. Construct the inverse G.

Given any $d \in \mathsf{D}$. Since F is essentially surjective, there exists $c \in \mathsf{C}$ such that $F(c) \cong d$, but we need AC to pick out such a c for every $d \in \mathsf{D}$. Let $G(d)$ be this c.

Given any $g : d \to d' \in \mathsf{D}$. Since F is fully faithful, $\mathsf{C}(G(d), G(d')) \cong \mathsf{D}(F(G(d)), F(G(d')))$. But according to the definition of G on objects, $F(G(d)) \cong d$ and $F(G(d')) \cong d'$. Again, pick out two such isomorphisms $\epsilon_d : F(G(d)) \cong d, \epsilon_{d'} : F(G(d')) \cong d'$, a simple diagram chase shows that $\mathsf{D}(F(G(d)), F(G(d'))) \cong \mathsf{D}(d, d')$. Let $G(g) : G(d) \to G(d')$ be the image of g under

$$D(d, d') \cong \mathsf{D}(F(G(d)), F(G(d'))) \cong \mathsf{C}(G(d), G(d')).$$

One can easily verify the functoriality of G.

- Stage 2. Construct $\eta : 1_{\mathsf{C}} \cong GF$ and $\epsilon : FG \cong 1_{\mathsf{D}}$.

We have constructed ϵ already, so let's focus on η.

Let $c \in \mathsf{C}$. We need to construct an isomorphism $\eta_c : c \to GF(c)$. But since F is fully faithful, $\mathsf{C}(c, GF(c)) \cong \mathsf{D}(F(c), FGF(c)) \ni \epsilon^{-1}_{F(c)}$. Let $\eta_c : c \to GF(c)$ be the image of $\epsilon^{-1}_{F(c)}$ under this bijection. Again, since F is fully faithful, η_c is an isomorphism.

- Stage 3. Verify the naturality of η and ϵ.

First we consider ϵ. We've mentioned that $\mathsf{D}(FG(d), FG(d')) \cong D(d, d')$. This is done as follows.

$$\begin{array}{ccc} FG(d) & \xrightarrow{FG(g)} & FG(d') \\ {\scriptstyle \epsilon_d^{-1}}\uparrow & & \downarrow{\scriptstyle \epsilon_{d'}} \\ d & \xrightarrow[g]{} & d' \end{array}$$

Reverse ϵ_d^{-1} to ϵ_d and we're done.

Now we prove the naturality of η. Given $f : c \to c'$, we need to show that the diagram on the left commutes.

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \eta_{c'}} \\ GF(c) & \xrightarrow[GF(f)]{} & GF(c') \end{array} \qquad \begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ {\scriptstyle F(\eta_c)=\epsilon_{F(c)}^{-1}}\downarrow & & \downarrow{\scriptstyle F(\eta_{c'})=\epsilon_{F(c')}^{-1}} \\ FGF(c) & \xrightarrow[FGF(f)]{} & FGF(c') \end{array}$$

But since F is fully faithful, we can apply F to that diagram and show that the result diagram (on the right) commutes. This is simply because ϵ is natural. □

The proof may seem long, but this theorem is really just a fancy version of "a bijective function is invertible". Let's summarize the result again, just to emphasize.

Theorem 1.5

Assuming AC. A functor $F : \mathsf{C} \to \mathsf{D}$ *is an equivalence iff* F *is fully faithful and essentially surjective.*

Whenever a functor $F : \mathsf{C} \to \mathsf{D}$ is fully faithful, we can regard it as an inclusion functor of full subcategory. We can ask if F is essentially surjective or not. If it is, then it witnesses $\mathsf{C} \simeq \mathsf{D}$. If not, well, it really is (in some sense) the inclusion functor of a proper subcategory.

Discussion 1.6 (Equivalence vs. Isomorphism)

What's the difference between equivalence *and* isomorphism *of categories?*

The notion of isomorphism between categories is just isomorphism in CAT. Let's spell out the definition. For any pair of categories C, D, $\mathsf{C} \cong \mathsf{D}$ if there's a pair of functors $F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}$, such that:

$$1_{\mathsf{C}} = GF, FG = 1_{\mathsf{D}}.$$

But talking about *equality* between functors is against the philosophy of category theory. It's more *natural*[1], and as it turned out, more useful to talk about functors being naturally isomorphic.

[1] No pun intended, but the notion of natural transformation really is natural.

The key difference is that objects may be isomorphic but not equal. Sometimes we wish isomorphic objects are always equal, this property is called being *skeletal*.

Definition 1.7 (Skeletal)

A category C *is* skeletal *if for any* $c, d \in \mathsf{C}$, $c \cong d$ *implies* $c = d$.

Proposition 1.8

If $\mathsf{C} \simeq \mathsf{D}$ *and* C, D *are both skeletal, then* $\mathsf{C} \cong \mathsf{D}$.

Proof. Suppose (F, G, η, ϵ) witness $\mathsf{C} \simeq \mathsf{D}$. Each component of $\eta : 1_{\mathsf{C}} \cong GF$ is an isomorphism $\eta_c : c \cong GFc$. But since C is skeletal, $c = GFc$. Thus $1_{\mathsf{C}} = GF$. The same goes for $FG = 1_{\mathsf{D}}$. □

Given any category C, if we can pick out *one* object from any isomorphism class and take the full subcategory spanned by them, then it's like we have forced C to be skeletal. To be precise:

Proposition 1.9 (Skeleton)

Assuming AC. Every category C *is equivalent to a skeletal category* $\mathrm{sk}(\mathsf{C})$*, unique up to isomorphism. Actually, if* $\mathsf{C} \simeq \mathsf{D}$*, then* $\mathrm{sk}(\mathsf{C}) \cong \mathrm{sk}(\mathsf{D})$.

Proof. AC allows us to pick out a single object from any isomorphism class of objects in C. Take the full subcategory $\mathrm{sk}(\mathsf{C})$ spanned by them. The canonical inclusion functor $\mathrm{sk}(\mathsf{C}) \to \mathsf{C}$ is fully faithful (since it's the full subcategory) and essentially surjective (by definition). Clearly $\mathrm{sk}(\mathsf{C})$ is skeletal.

If $\mathsf{C} \simeq \mathsf{D}$, then compose the equivalence:

$$\mathrm{sk}(\mathsf{C}) \simeq \mathsf{C} \simeq \mathsf{D} \simeq \mathrm{sk}(\mathsf{D})$$

we get $\mathrm{sk}(\mathsf{C}) \simeq \mathrm{sk}(\mathsf{C})$, which implies $\mathrm{sk}(\mathsf{C}) \cong \mathrm{sk}(\mathsf{C})$. □

The choice involved in the proof can often be explicitly constructed. In this case, the construction of skeleton often serves to make the category more *concrete*.

For example, in linear algebra, every n-dimensional real vector space V is isomorphic to $\mathbb{R}^n$. Let's define Mat to be the category whose objects are standart Euclidean spaces $\mathbb{R}^n$, morphisms are linear functions between them, which can be written down as *matrices*. Then $\mathsf{Mat} \cong \mathrm{sk}(\mathsf{Vect}_{\mathbb{R}}^{\mathrm{fin}})$. The functor $\mathsf{Vect}_{\mathbb{R}}^{\mathrm{fin}} \to \mathsf{Mat}$ converts any abstract linear algebra argument to concrete matrices computaion.

In set theory, every well-ordered set X has a *order type* $o(X)$, which is an *ordinal*. Every order-preserving function between well-ordered sets $f : X \to Y$ induces an order-preserving function between their order types $o(f) : o(X) \to o(Y)$. This functor o is an equivalence between the category of well-ordered sets WO and the category of ordinals Ord, and $\mathsf{Ord} \cong \mathrm{sk}(\mathsf{WO})$.

2 Limit and Colimit

Limit and colimit are a kind of *universal construction*. We give the definition first, then we discuss the intuition.

2.1 The definition

Definition 2.1 (Diagram)

Fix a small category J*, a category* C*, a functor* $K : \mathsf{J} \to \mathsf{C}$*. The triple* $(\mathsf{C}, \mathsf{J}, K)$*, or just* K*, is a* J*-shaped* diagram *in* C*.* J *is called the index category of* K*.*

For example, a pair of sets (X, Y) can be seen as a J-shaped diagram in Set, where J is the discrete category with two objects. A sequence

$$K(0) \to K(1) \to K(2) \to \cdots$$

is a $(\mathbb{N}, \leq)$-shaped diagram in Set.

Definition 2.2 (Constant functor)

For any category J, C*, any object* $c \in \mathsf{C}$*, we have a* constant functor $\Delta c : \mathsf{J} \to \mathsf{C}$*, sending every object in* J *to* c*, every morphism to* 1_c*.*

Definition 2.3 (Cone, Cocone)

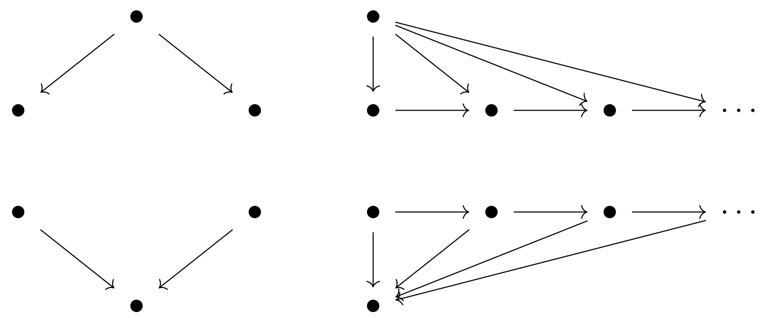
Fix a diagram $K : \mathsf{J} \to \mathsf{C}$*.*

- *A* cone *over* K *consists of an object* $c \in \mathsf{C}$ *(the* summit*), and a natural transformation* $\lambda : \Delta c \to K$ *(the* legs*).*
- *Dually, a* cocone *under* K *consists of an object* $c \in \mathsf{C}$*, (the* nadir*), and a natural transformation* $\lambda : K \to \Delta c$ *(the* legs*).*

Here are some pictures of cones and cocones.

Cones:

Cocones:



Note that since we require that all the legs form a natural transformation, every triangle on the right commutes.

Definition 2.4 (Morphism between cones and cocones)
Fix a diagram $K : \mathsf{J} \to \mathsf{C}$ *and two cones* $\lambda : \Delta c \to K, \mu : \Delta d \to K$. *A morphism* $f : (c, \lambda) \to (d, \mu)$ *is a morphism* $f : c \to d \in \mathsf{C}$ *such that it commutes with every leg. Morphisms between cocones are dual.*

Let $\mathrm{Cone}(-, K) := \mathsf{C}^{\mathsf{J}}(\Delta -, K)$, $\mathrm{Cone}(K, -) := \mathsf{C}^{\mathsf{J}}(K, \Delta -)$. Intuitively, $\mathrm{Cone}(c, K)$ is the set of all the cones over K with summit c, $\mathrm{Cone}(K, c)$ is similar.

Let $\int \mathrm{Cone}(-, K)$ be the category of all cones over K, and $\int \mathrm{Cone}(K, -)$ be the category of all cocones under K. The notation stands for category of elements but you don't have to know that.

Finally we can define limit and colimit.

Definition 2.5 (Limit, Colimit as Terminal, Initial)

- *The limit cone over* K *is the terminal object of* $\int \mathrm{Cone}(-, K)$. *The summit is called the limit of* K, *denoted as* $\lim K$.

- *Dually, the colimit cone under* K *is the initial object of* $\int \mathrm{Cone}(K, -)$. *The nadir is called the colimit, of* K, *denoted as* $\operatorname{colim} K$.

An equivalent, and probably *better* definition is this.

Definition 2.6 (Limit, Colimit as Representing Objects)

- *A diagram* $K : \mathsf{J} \to \mathsf{C}$ *has limit iff* $\mathrm{Cone}(-, K)$ *is representable. The representing object is* $\lim K$.

- *Dually, it has colimit iff* $\mathrm{Cone}(K, -)$ *is representable. The representing object is* $\operatorname{colim} K$.

Personally I think the terminal-initial definition is easier to imagine, but the representing-objects definition proves to be more useful. The equivalence of these two definitions a special case of Proposition 2.4.8 in [Rie16]. You may also verify it yourself.

Intuitively, the limit cone $\lambda : \lim K \to K$ is like the *best* cone, because it can do the job of any other cone. If I have a limit cone $\lambda : \lim K \to K$ in my hand, you may give me any cone $\mu : d \to K$ whatsoever, but in my eyes your cone is just a morphism $d \to \lim K$ to my cone.[2] The same goes for colimit. That's why people call them *universal* constructions.[3]

Fun fact: category theory community haven't quite agreed on the terminology yet. Some may call limit *inverse limit* or *projective limit*, they may also call colimit *directed limit* or *inductive limit*.

For certain index category J, (co)limits of J-shaped diagrams have fixed names. Three of them are particularly important.

Fix a category C.

Let 2 be the discrete category with two objects.

Definition 2.7 ((Co)product)

A (co)product in C is the (co)limit of a 2-shaped diagram.

Let $\rightrightarrows$ be the category that looks just like that.

Definition 2.8 ((Co)equalizer)

A (co)equalizer in C is the (co)limit of a $\rightrightarrows$-shaped diagram.

Let $\to\leftarrow$ be the category that looks like $\bullet \to \bullet \leftarrow \bullet$, $\leftarrow\to$ be $(\to\leftarrow)^{\mathrm{op}}$.

Definition 2.9 (Pullback, Pushout)

- *A pullback in C is the limit of a $\to\leftarrow$-shaped diagram.*
- *A pushout in C is the colimit of a $\leftarrow\to$-shaped diagram.*

Terminal and initial are special cases of (co)limit.

Exercise 2.10

Fix a category C with initial 0 and terminal 1. Suppose J is the empty *category, so there is exactly one functor $K : \mathsf{J} \to \mathsf{C}$. Show that $\lim K \cong 1, \operatorname{colim} K \cong 0$.*

[2] This intuition fits both definitions. Pause a second and think about this.

[3] In fact, one might *define* the universal property of an object $c \in \mathsf{C}$ as certain Set-valued functor it represents. Universal objects are *very common* in mathematics. In fact, I believe that behind every construction there's some universal property that truly captures the spirit of that construction.

Exercise 2.11 (When J has initial or terminal)

- *Suppose* J *has an initial* j. *Show that for any diagram* $K : \mathsf{J} \to \mathsf{C}$*,* $\lim K \cong K(j)$.
- *Dually, suppose* J *has a terminal* k. *Show that for any diagram* $K : \mathsf{J} \to \mathsf{C}$*,* $\operatorname{colim} K \cong K(k)$.

Hint: for limit case, the initiality of j *gives you the limit cone.*

This is why we never talk about stuff like ω-shaped limit, they are trivial, while ω-shaped colimit can be highly interesting.

To understand what (co)limit does, we first focus on (co)limits in Set.

2.2 Limit in Set

Idea 2.12 (How do you know what's the (co)limit of a diagram?)

There are two ideas in general.

1. *You* guess*. But you can guess in a smart way. In this section we will see that in many cases, the universal property of certain object on* global elements *determines that object completely.*
2. *You can construct the (co)limit of a complicated diagram by putting together (co)limits of some smaller diagrams. We will learn that in the next section.*

A 2-shaped diagram in Set is just a pair of sets X, Y. The product of them is an *universal* object $X \times Y$ equipped two projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$.

To see what $X \times Y$ is, let's apply the category theory philosophy. We *probe* it by global elements. That is, we ask the question *what is a global element* $e : 1 \to X \times Y$.

One such element gives rise to a pair of elements $\pi_X(e) : 1 \to X$ and $\pi_Y(e) : 1 \to Y$. Moreover, given any pair of such element $(x : 1 \to X, y : 1 \to Y)$, there is a unique element $\langle x, y\rangle : 1 \to X \times Y$ such that $\pi_X\langle x, y\rangle = x$ and $\pi_Y\langle x, y\rangle = y$.

Since we are in the category Set, we may identify a global element $x : 1 \to X$ with an *actual* element $x \in X$. So the set $X \times Y$ contains exactly elements of this form $\langle x \in X, y \in Y\rangle$. That is, if $X \times Y$ exists, it has to be the *Cartesian product* of X and Y. Indeed, one may easily verify that the Cartesian satisfies the universal property.

Let's investigate equalizer with this idea. A $\rightrightarrows$-shaped diagram in Set is a parallel pair of functions $f, g : X \rightrightarrows Y$, suppose E is the equalizer with $e : E \to X$, such that $fe = ge$ (by definition of cone).

A global element in E is precisely a global element $x : 1 \to X$, such that $fx = gx$. Thus E has to be the set $\{x \in X \mid f(x) = g(x)\}$. It's easy to check that this set indeed satisfies the universal property of equalizer.

Exercise 2.13 (Pullback in Set)

Study pullback with global element:

- *Given a diagram* $X \xrightarrow{f} Z \xleftarrow{g} Y \in \mathsf{Set}$*. Show that the pullback of this diagram* $X \times_Z Y$ *is the set* $\{(x, y) \in X \times Y \mid f(x) = g(x)\}$*.*

- *Suppose* Y *is a subset of* Z *and* g *is the inclusion function of subset. Show that* $X \times_Z Y$ *is* $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\}$*. This is called the pullback of subset* Y *along* f*. Conclude that pullback preserves monomorphism.*

- *Suppose even further that* Y *is a singleton set* $\{z\}$*, where* $z \in Z$*. Conclude that* $X \times_Z Y$ *is* $f^{-1}(z)$*. This is called the* fiber *of* f *at* z*. This is the reason why pullback is also called* fiber product

Let's reflect on this idea for a second.

Idea 2.14 (What we have done)

Our idea can be expressed like this.

- *In* Set*, every object* X *(a set) is made up of all the global elements* $\mathsf{Set}(1, X)$ *in it.*

- *If the limit* $\lim K$ *of a cone* K *were to exist, then by universal property, a global element of* $\lim K$ *is just a cone* $\mu : 1 \to K$*.*

- *So* $\lim K \cong \mathrm{Cone}(1, K)$ *is the only possibility.*

In fact, one may use global element to show that Set has *all* limit.

Definition 2.15 (Completeness)

- *Fix a small category* J*. A category* C *is* J*-(co)complete, if for* any J*-shaped diagram* $K : \mathsf{J} \to \mathsf{C}$, $\lim K$ *(*$\mathrm{colim}\, K$*) exists.*

- *A category* C *is (co)complete, if it's* J*-(co)complete for* any *small category* J*.*

- *A category* C *is finitely (co)complete if it's* J*-(co)complete for any* finite[4] J*.*

[4] A small category is finite if its set of morphisms is finite.

Now, suppose $K : \mathsf{J} \to \mathsf{Set}$ is diagram in Set. If $\lim K$ exists, then it represents $\mathrm{Cone}(-, K)$. Idea 2.13 can be expressed via the following isomorphism.

$$\lim K \cong \mathsf{Set}(1, \lim K) \cong \mathrm{Cone}(1, K).$$

Just like before, we now *define* $\lim K$ to be $\mathrm{Cone}(1, K)$.

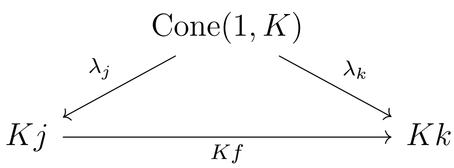
Lemma 2.16 (Reality Check)

$\mathrm{Cone}(1, K)$ *has the required universal property of limit.*

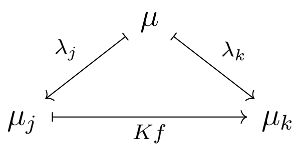
Proof. We define the cone $\lambda : \mathrm{Cone}(1, K) \to K$. For each $j \in \mathsf{J}$, $\lambda_j : \mathrm{Cone}(1, K) \to Kj$ sends a cone $\mu : 1 \to K$ to its component at j, $\mu_j \in K_j$.

Claim 1. λ is a cone.

For each $f : j \to k \to \mathsf{J}$, we need to show that the following triangle commutes.



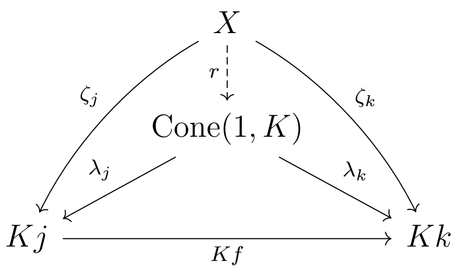
That is to say, for each cone $\mu : 1 \to K$, we have:



This is simply because $\mu : 1 \to K$ is a cone. So λ is a cone.

Claim 2. λ is the limit cone.

Consider a cone $\zeta : X \to K$. We need to show that ζ factors uniquely through λ along a function $f : X \to \mathrm{Cone}(1, K)$.



Suppose $x \in X$. If $r(x) : 1 \to K$ exists, it has to be such a cone that $r(x)_j = \zeta_j$. We take it as the *definition* of $r(x)$. It other words,

> $\mathrm{Cone}(1, K)$ has all the possible cones over K. $\zeta : X \to K$ is X-many cones put together. The function r picks out the cone $r(x)$ for every $x \in X$.

Clearly r is unique. □

Corollary 2.17

Set *is complete.*

In fact, Set is also cocomplete, but the method above doesn't work for colimit. We now introduce a more general method of proving (co)completeness.

2.3 Limit in Set again

Let's compute some limit in Set and find some patterns.

Example 2.18 (Arbitrary Product)

Suppose we have a family of sets $\{A_j\}_{j\in J}$*, indexed by* J*.* J *can be* any *set, even* empty*.*

Regard J *as a discrete category, then* $\{A_j\}$ *is a* J*-shaped diagram in* Set*. The limit of it is the cartesian product:*

$$\prod_{j\in J} A_j = \{(a_j \in A_j)_{j\in J}\}.$$

Note that when $J = \varnothing$, a J-shaped diagram is an empty diagram, then $\prod_{j\in J} A_j$ is the terminal object 1.

We will often use terminologies like these:

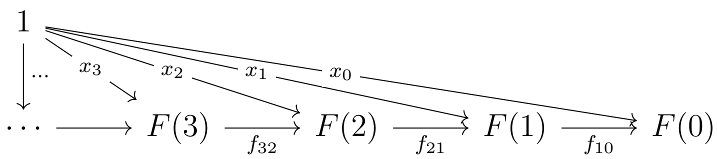
- "C has *binary product*" means C has all 2-shaped limits.
- "C has *finite product*" means C has all J-shaped limits for every finite set J.
- "C has *arbitrary product*" means C has all J-shaped limits for every set J.

Exercise 2.19 (Finite product from binary and terminal)

Show that C *has finite product iff* C *has binary product and terminal.*

Example 2.20

Elements of the limit of a diagram $F : \omega^{\text{op}} \to$ Set *are cones:*



The data of such a cone is given by a tuple of elements $(x_n \in F(n))_{n\in\omega}$ *making each triangle commutes. Thus, we see that*

$$\lim F = \left\{ (x_n)_{n\in\omega} \in \prod_{n\in\omega} F(n) \mid f_{n,n-1}(x_n) = x_{n-1} \right\}.$$

Example 2.21

The pullback of $B \xrightarrow{f} A \xrightarrow{g} C$ *is*

$$B \times_A C = \{(b,c) \in B \times C \mid f(b) = g(c)\}.$$

A pattern has occurred. To construct the limit of a diagram in Set, it seems like you need to take the (probably infinite) *product* of every set in the diagram, then take the subset of the elements that satisfy the cone condition. The latter is done via *equalizer*.

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ B & \xrightarrow[f]{} & A \end{array}$$

How to construct the pullback?

$$\begin{array}{ccc} B \times C & \longrightarrow & A \\ \downarrow & \searrow & \downarrow g \\ B & \xrightarrow[f]{} & C \end{array}$$

First you take the product to form a (non-commutative) square,

$$\begin{array}{ccc} B \times_A C \subseteq B \times C & \longrightarrow & C \\ \downarrow & \searrow & \downarrow g \\ B & \xrightarrow[f]{} & A \end{array}$$

then you take the subset so that the diagram commutes,

$$B \times_A C \rightarrowtail B \times C \underset{(b,c)\mapsto g(c)}{\overset{(b,c)\mapsto f(b)}{\rightrightarrows}} A$$

which is done by taking equalizer.

In fact, every limit in Set can be constructed in this way.

Theorem 2.22 (Limit via product and equalizer)

For any diagram $K : \mathsf{J} \to \mathsf{Set}$*, there is an equalizer diagram:*

$$\lim_{\mathsf{J}} K \longrightarrow \prod_{j \in \mathsf{J}} Kj \underset{d}{\overset{c}{\rightrightarrows}} \prod_{f \in \operatorname{mor} \mathsf{J}} K(\operatorname{cod} f)$$

where for any $(\mu_j)_{j \in \mathsf{J}} \in \prod_{j \in \mathsf{J}} K_j$, $(c(\mu_j))_f = K(f)(\mu_{\operatorname{dom} f})$ *and* $(d(\mu_j))_f = \mu_{\operatorname{cod} f}$.

Proof. By the characterization of equalizer,

$$(\mu_j) \in \lim_{\mathsf{J}} K \Leftrightarrow \forall f : k \to l \in \mathsf{J}. K(f)(\mu_k) = \mu_l.$$

Identifying an element of a set (for example $\mu_k \in Kk$) as a global element ($\mu_k : 1 \to Kk$), $(\mu_j) \in \lim_{\mathsf{J}} K$ simply expresses that μ is a cone $\mu : 1 \to K$. So $\lim_{\mathsf{J}} K = \operatorname{Cone}(1, K)$ is the equalizer of the diagram. □

3 Towards Completeness

3.1 (Co)products and (co)equalizers

Now we aim to generalize the previous theorem to any category. The idea is to use *Yoneda embedding.*

Definition 3.1

For any class of diagrams $K : \mathsf{J} \to \mathsf{C}$ *in* C*, a functor* $F : \mathsf{C} \to \mathsf{D}$

- preserves *these limits, if for any such diagram* K *with a limit cone* $\lambda : \lim K \to K$*,* $F\lambda : F \lim K \to FK$ *is also a limit cone, so* $F \lim K \cong \lim FK$*;*
- reflects *these limits, if for any such diagram* K *and any cone* $\lambda : c \to K$ *in* C*, whenever* $F\lambda : Fc \to FK$ *is a limit cone in* D*,* λ *is already a limit cone in* C*;*
- *more rarely,* creates *these limits, if for any such diagram* K*, the* mere existence *of a limit cone* $\mu : d \to FK$ *in* D *implies that there is a limit cone* $\lambda : c \to K$ *in* C*, and* F *reflects these limits.*

Again, everything is dual for colimit.

The following exercise shows just how strong a condition "F creates limit" is.

Exercise 3.2

If $F : \mathsf{C} \to \mathsf{D}$ *creates limits for some class of diagrams in* C *and* D *has limits for those diagrams, then* C *admits those limits and* F *preserves them.*

To me, it's mind-blowing to just think about constructing limits in a category C by studying a functor $F : \mathsf{C} \to \mathsf{D}$ mapping *out* of C.

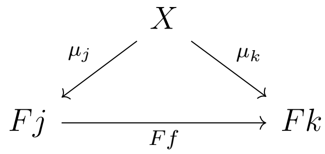
Exercise 3.3

Show that a fully faithful functor $F : \mathsf{C} \to \mathsf{D}$ *reflects every limit and colimit that exists.*

Now we study the property of Hom-functors. Fix a category C, an object X, and a diagram $F : \mathsf{J} \to \mathsf{C}$. We have a J-shaped diagram in Set:

$$\mathsf{C}(X, F-) := \mathsf{J} \xrightarrow{F} \mathsf{C} \xrightarrow{\mathsf{C}(X,-)} \mathsf{Set}.$$

We know that $\lim \mathsf{C}(X, F-) \cong \mathrm{Cone}(1, \mathsf{C}(X, F-))$. An element $\mu \in \mathrm{Cone}(1, \mathsf{C}(X, F-))$ is just a family of functions $\mu_j : X \to Fj$, subjected to some compatibility conditions. Every non-identity morphism $f : j \to k \in \mathsf{J}$ imposes that the following triangle commutes.



So μ is precisely a cone $\mathrm{Cone}(X, F)$.

$$\lim_{\mathsf{J}} \mathsf{C}(X, F-) \cong \mathrm{Cone}(X, F) \cong \mathsf{C}(X, \lim F).$$

We have proved the following theorem.

Theorem 3.4

For any diagram $F : \mathsf{J} \to \mathsf{C}$ whose limit exists, there is a natural isomorphism

$$\mathsf{C}(X, \lim_{\mathsf{J}} F) \cong \lim_{\mathsf{J}} \mathsf{C}(X, F-).$$

In other words, covariant Hom-functors preserve every limit that exists.

Dualizing the argument, by considering the diagram

$$\mathsf{C}(F-, X) := \mathsf{J}^{\mathrm{op}} \xrightarrow{F} \mathsf{C}^{\mathrm{op}} \xrightarrow{\mathsf{C}(-,X)} \mathsf{Set}.$$

We get $\lim_{\mathsf{J}^{\mathrm{op}}} \mathsf{C}(F-, X) \cong \mathrm{Cone}(F, X) \cong \mathsf{C}(\mathrm{colim}\, F, X)$.

Theorem 3.5

For any diagram $F : \mathsf{J} \to \mathsf{C}$ whose colimit exists, there is a natural isomorphism

$$\mathsf{C}(\mathrm{colim}_{\mathsf{J}}\, F, X) \cong \lim_{\mathsf{J}^{\mathrm{op}}} \mathsf{C}(F-, X).$$

In other words, contravariant Hom-functors send every colimit that exists to limit in Set.

We can finally prove the final completeness theorem.

Theorem 3.6

The colimit of any diagram $F : \mathsf{J} \to \mathsf{C}$ may be expressed as a coequalizer of a pair of maps between coproducts.

$$\textstyle\coprod_{f \in \mathrm{mor}\,\mathsf{J}} F(\mathrm{dom}\, f) \underset{c}{\overset{d}{\rightrightarrows}} \coprod_{j \in \mathsf{J}} Fj \longrightarrow \mathrm{colim}_{\mathsf{J}}\, F$$

Dually, the limit of any diagram may be expressed as an equalizer of a pair of maps between products.

Proof. We prove the case for colimit. Suppose coequalizers and coproducts exist in C. Consider a diagram $F : \mathsf{J} \to \mathsf{C}$. Construct the following diagram.

$$C \longleftarrow \coprod_{j\in\mathsf{J}} Fj \underset{c}{\overset{d}{\leftleftarrows}} \coprod_{f\in\operatorname{mor}\mathsf{J}} F(\operatorname{dom} f)$$

where $d = \langle i_k : Fk \to \coprod_{j\in\mathsf{J}} Fj\rangle_{f:k\to l}$, $c = \langle Fk \xrightarrow{Ff} Fl \xrightarrow{i_l} \coprod_{j\in\mathsf{J}} F_j\rangle_{f:k\to l}$. Our aim is to show that C defines a colimit of F.

Apply Yoneda embedding to the diagram. For any object $X \in \mathsf{C}$, we get the equalizer diagram (by the properties of Hom-functors on (co)limit):

$$\mathsf{C}(C, X) \longrightarrow \prod_{j\in\mathsf{J}^{\mathrm{op}}}(Fj, X) \underset{c}{\overset{d}{\rightrightarrows}} \prod_{f\in\operatorname{mor}\mathsf{J}^{\mathrm{op}}}(F(\operatorname{cod} f), X)$$

where

$$(c\langle \varphi_j : Fj \to X\rangle_{j\in\mathsf{J}^{\mathrm{op}}})_{f:k\to l\in\mathsf{J}} = \varphi_k : Fk \to X,$$
$$(d\langle \varphi_j : Fj \to X\rangle_{j\in\mathsf{J}^{\mathrm{op}}})_{f:k\to l\in\mathsf{J}} = Fk \xrightarrow{Ff} Fl \xrightarrow{\varphi_l} X.$$

This equalizer defines the limit of the diagram:

$$\lim_{\mathsf{J}^{\mathrm{op}}} \mathsf{C}(F-, X) \cong \mathsf{C}(C, X).$$

These isomorphisms are natural in X, so $\mathsf{C}(C, -)$ is the limit of the J^{op}-shaped diagram of $\mathsf{C}(Fj, -)$. Since Yoneda embedding is fully faithful, it reflects limits, so $C \cong \lim_{\mathsf{J}^{\mathrm{op}}} Fj \cong \operatorname{colim}_{\mathsf{J}} Fj$. □

Note that the proof makes use of (mor J)-sized product.

Corollary 3.7

For any category C*,*

- *if* C *has equalizers and arbitrary products, then* C *is complete;*
- *if* C *has equalizers and finite products , then* C *is finitely complete.*
- *In particular, if* C *has equalizers, terminal, and binary products, then* C *is finitely complete.*

Colimit is similar.

Now let's use these theorems to prove the cocompleteness of Set.

3.2 Colimit in Set

It's easy to check that in Set, the coproduct of X, Y is the disjoint union $X + Y$. The coproduct of a family of sets $\{X_j\}_{j \in J}$ is the disjoint union of everything $\coprod_{j \in J} X_j$.

The coequalizer of $f, g : X \rightrightarrows Y$ is $Y/\sim$, where $\sim$ is the following equivalence relation:

$$y_1 \sim y_2 \Leftrightarrow \exists x \in X. f(x) = y_1 \wedge g(x) = y_2.$$

The picture becomes intuitive when you let both f, g be injective. In this case, the coequalizer is two copies of Y *glued* along f, g. Here's a simple yet illustrative example.

Example 3.8 (Glueing the interval)

Working in the category Top *of topological spaces and continuous functions. Let* $*$ *be the singleton,* $[0, 1]$ *be the unit interval in* $\mathbb{R}$*. There are two functions* $0, 1 : * \rightrightarrows [0, 1]$*, picking out* 0 *and* 1 *respectively. The coequalizer of them can be thought of as a* circle S^1.

Thus, colimit in Set (or other similar categories) can be thought of as putting a bunch of sets together (taking the coproduct), and then glueing them along certain rules (taking the coequalizer).

Corollary 3.9

Set *is cocomplete.*

The first axiom of topos states that a topos has to be finitely complete and finitely cocomplete. It does not require arbitrary (co)completeness. For example, the category of finite sets FinSet is a topos, but it's neither complete nor cocomplete.

References

[Rie16] Emily Riehl. *Category theory in context.* Aurora: Modern Math Originals. Dover, 2016.