Functors, Variable Sets, and The Yoneda Lemma

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1 Functors

Functor is probably *the* most important concept in category theory. Intuitively you may think of a functor as a morphism between *categories*.

Definition 1.1 (Functor)
A functor $F : \mathsf{C} \to \mathsf{D}$ *assigns each object* $c \in \mathsf{C}$ *to an object* $F(c) \in \mathsf{D}$*, assigns each morphism* $f : c \to c' \in \mathsf{C}$ *to a morphism* $F(f) : F(c) \to F(c') \in \mathsf{D}$*, such that:*

- $F(1_c) = 1_{F(c)}$*, so* F *preserves identity.*
- *For* $a \xrightarrow{f} b \xrightarrow{g} c \in \mathsf{C}$*,* $F(g \circ f) = F(g) \circ F(f) \in \mathsf{D}$*, so* F *preserves composition.*

Let Cat denote the category of *small* categories and functors, and let CAT denote the category of *locally small* categories and functors.

For a functor $F : \mathsf{C} \to \mathsf{D}$, we can look at the size of the source and the target of it. Roughly there are three flavors of functors:

1. Both of C, D are small. This kind of functors usually functions as a *morphism*.
2. Both of C, D are large. This kind usually are used for *comparison* different structures.
3. C is small but D is large. Here we may look at F as a *variable* object of D indexed by C.

We rarely see a functor from a large category to a small one[1].

1.1 From small to small

This part is easy. I'll let you figure it out yourself.

Exercise 1.2 (Monoids)

- *Suppose* M, N *are monoids, regarded as categories. Show that a functor* $F : M \to N$ *is exactly a homomorphism.*

[1] You may say hey, I got one: for any category C, there's a unique functor $\mathsf{C} \to 1$, where 1 is the category with one object and one identity morphism. Well, I challenge you to come up with a non-constant functor $\mathsf{Set} \to 2$, where 2 looks like $\bullet \to \bullet$.

- *Given a category* C *and an object* $c \in \mathsf{C}$*, show that* $\mathsf{C}(c, c)$ *is a monoid.*
- *Given a functor* $F : \mathsf{C} \to \mathsf{D}$ *and an object* $c \in \mathsf{C}$*, show that* $F : \mathsf{C}(c, c) \to \mathsf{D}(F(c), F(c))$ *is a monoid homomorphism.*

Exercise 1.3 (Preordered sets)
Suppose P, Q *are preoreder sets. A functor* $F : P \to Q$ *is exactly an order-preserving function.*

1.2 From large to large

Examples of this kind are endless. Let's pick a few.

Example 1.4 (Free vertor space)
Let $\mathsf{Vect}_{\mathbb{R}}$ *be the category of* $\mathbb{R}$*-vector spaces and linear functions. There's a canonical* ***forgetful*** *functor* $U : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Set}$*, sending a vector space* V *to its underlying set.*

There's another functor in the other direction. ***Free vector space*** *functor* $F : \mathsf{Set} \to \mathsf{Vect}_{\mathbb{R}}$ *sends a set* X *to the vector space* $\mathbb{R}^{\oplus X}$*, so elements of* X *form a* ***basis*** *for* $F(X)$*. You may say that* $F(X)$ *is the* X*-dimensional space.*

These two functors are ***adjoint*** *to each other in some sense. We will talk about it next time (hopefully).*

Example 1.5 (List)
Let $\mathrm{List} : \mathsf{Set} \to \mathsf{Set}$ *be such a functor. It sends a set* A *to* $\mathrm{List}(A)$*, which contains all the finite* ***lists*** *built from* A*. You may say that* A *is like an alphabet and* $\mathrm{List}(A)$ *contains all possible words built from it. In particular, we allow empty list to be in* $\mathrm{List}(A)$*.*

For monoid lovers, $\mathrm{List}(A)$ *is exactly the underlying set of the* ***free monoid*** *of* A*, so it's a composition of the free monoid functor* $F : \mathsf{Set} \to \mathsf{Monoid}$ *and the forgetful functor* $U : \mathsf{Monoid} \to \mathsf{Set}$*. From this perspective,* List *is actually a* ***monad****.*

We will gradually see more examples of this kind, but for now let's move on.

1.3 From small to large

Fix a small category, say, $(\mathbb{N}, \leq)$. A functor $X : \mathbb{N} \to \mathsf{Set}$ can be seen a set through time. For example, X may represent the set of particles in the universe at each second from now on[2].

Here, time is discrete and has a starting point (a *genesis* if you will), but it will never end. Let's call this kind of functor X a *variable set*.

But here's the question: what should be a *morphism* between variable sets?

Definition 1.6 (Natural Transformation: Morphisms between Functors)
Fix two category C, D *and two functors* $F, G : \mathsf{C} \to \mathsf{D}$*. A* ***natural transformation*** $\alpha : F \to G$ *is a collection of morphisms in* D*. Each* $c \in \mathsf{C}$ *gives rise to a morphism* $\alpha_c : F(c) \to G(c)$*, such*

[2] You may argue that due to the theory of relativity, we can't quite measure the time in the whole universe simultaneously, and I will say okay.

that for each morphism $f : c \to c'$, the following diagram commutes.

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ {\scriptstyle\alpha_c}\downarrow & & \downarrow{\scriptstyle\alpha_{c'}} \\ G(c) & \xrightarrow[G(f)]{} & G(c') \end{array}$$

Let's look at our example of variable sets. Take two variable sets $X, Y : \mathbb{N} \to \mathsf{Set}$, to say that $f : X \to Y$ is a natural transformation, is just to say that the following diagram commutes.

$$\begin{array}{ccccccc} X(0) & \longrightarrow & X(1) & \longrightarrow & X(2) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y(0) & \longrightarrow & Y(1) & \longrightarrow & Y(2) & \longrightarrow & \cdots \end{array}$$

So at any given moment $n \in \mathbb{N}$, f gives you an actual function $f_n : X(n) \to Y(n)$. But each f_n has to be related with one another so that the diagram will commute, it can't get too wild.

For example, let's say $x_0 \in X(0)$, then it gives rise to a whole sequence $\{x_n \in X(n)\}_{n \in \mathbb{N}}$. We may depict it as follows.

$$\begin{array}{ccccccccc} x & : & x_0 & \longmapsto & x_1 & \longmapsto & x_2 & \longmapsto & \cdots \\ \in & & \in & & \in & & \in & & \\ X & : & X(0) & \longrightarrow & X(1) & \longrightarrow & X(2) & \longrightarrow & \cdots \end{array}$$

Then, by the naturality of f, $f_1(x_1)$ is *forced* to be $(f_0(x_0))_1$. So naturality condition sort of means that the function f is travelling through time as well.

We can describe this phenomenon more precisely. Let's say at time n, there's an element $x_n \in X(n)$ which does not belong to the image of $X(n-1) \to X(n)$. This element seems to have *no history*, it magically appears out of nowhere. So the function f_n has the freedom to map x_n to wherever it wants. However, once $f_n(x_n)$ is fixed, every $f_m(x_m)$ later on is also fixed. It has to be $(f_n(x_n))_m$.

Definition 1.7 (Functor Category)
Any two category C, D gives rise to a category D^{C}:

- *Objects: functors $F : \mathsf{C} \to \mathsf{D}$,*
- *Morphisms: natural transformations $\alpha : F \to G$ for $F, G : \mathsf{C} \to \mathsf{D}$.*

This is called the ***functor category*** *from C to D.*

For example, functor category $\mathsf{Set}^{\mathbb{N}}$ is the category of variable sets where time has the shape of $\mathbb{N}$.

Example 1.8 (Natural transformation elsewhere)
We have talked about the functor List : $\mathsf{Set} \to \mathsf{Set}$ *being a* ***monad****. Roughly it means that there's two natural transformation $\eta : 1_{\mathsf{Set}} \to \mathrm{List}, \mu : \mathrm{List} \circ \mathrm{List} \to \mathrm{List}$. We often call them* ***embedding*** *and* ***concatenating****. Can you guess what they are?*

2 The study of $\mathsf{Set}^{\mathbb{N}}$ and Yoneda Lemma

2.1 Global Elements

The category $\mathsf{Set}^{\mathbb{N}}$ is also a topos. In fact, for any small category C, $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$[3] is always a topos.

Definition 2.1 (Presheaf)
*For any category C, a functor $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is called a **presheaf**[4] on C. The category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is called the **presheaf category** of C.*

For example, $\mathsf{Set}^{\mathbb{N}}$ is the presheaf category of $\mathbb{N}^{\mathrm{op}}$.

Now, being a topos means that $\mathsf{Set}^{\mathbb{N}}$ has some rich structures. For example, initial and terminal object.

Exercise 2.2
*Show that the **empty** functor $\varnothing : \mathbb{N} \to \mathsf{Set}$, sending each $n \in \mathbb{N}$ to $\varnothing$, is the initial of $\mathsf{Set}^{\mathbb{N}}$, while the **singleton** functor $1 : \mathbb{N} \to \mathsf{Set}$, sending each $n \in \mathbb{N}$ to the singleton 1, is the terminal.*

In Set, each element in a set X corresponds precisely to a function $1 \to X$, where 1 is the terminal. So simply by looking at $\mathsf{Set}(1, X)$, everything about X can be recovered. Let's try to do that in $\mathsf{Set}^{\mathbb{N}}$.

Definition 2.3 (Global element)
*Fix a category C with a terminal 1. A **global element** x of an object c is just a morphism $x : 1 \to c$.*

Now, fix a variable set $X : \mathbb{N} \to \mathsf{Set}$. What's a global element of X? It's a morphism (natural transformation) $x : 1 \to X$.

- First of all, $X(0)$ must be inhabited, because x_0 must send the unique element in $1(0)$ to $X(0)$.
- But since $X(0)$ is inhabited, everything after $X(0)$ must also be inhabited, because x_0 has to have a *future*.
- Furthermore, by the naturality of x, when we fix x_0, each x_n is also automatically fixed.

In conclusion, the global elements of X corresponds precisely to $X(0)$. This element x appears at every moment, hence it's *global*. And this correspondence is actually a *natural isomorphism*, which is an isomorphism in the functor category. This is an example of the first profound theorem in category theory, called *Yoneda Lemma*.

[3]For some reason we usually talk about $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ instead of $\mathsf{Set}^{\mathsf{C}}$. You will find the reason after learning about *co Yoneda Lemma*.

[4]I promise you the meaning of this terminology will become clear later. Don't worry about it for now.

2.2 Towards Yoneda

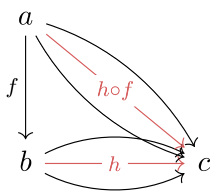
Sometimes we will see functors that looks like $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. In this case, F is *contravariant* at C. On the other hand, a functor $F : \mathsf{C} \to \mathsf{D}$ is *covariant* at C.

This following concept captures the philosophical core of category theory.

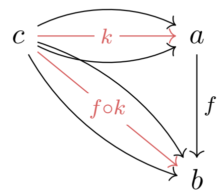
Definition 2.4 (Hom-functor)
Fix a category C *and an object* $c \in \mathsf{C}$. *We have two functors given by* c*:*

- $\mathsf{C}(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, *the contravariant hom-functor,*
- $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$, *the covariant hom-functor.*

Let's study the contravariant case first. Suppose we have a morphism $f : a \to b \in \mathsf{C}$, we should be able to apply $\mathsf{C}(-, c)$ to f and obtain a function $\mathsf{C}(b, c) \to \mathsf{C}(a, c)$. Indeed, this function is called *pre-composition*, denoted as f^*. Given any $h : b \to c$, $f^*(h) := h \circ f : a \to c$. Here's a picture.



The covariant case is essentially dual to the contravariant case. Again suppose we have a morphism $f : a \to b \in \mathsf{C}$. The function $f_* : \mathsf{C}(c, a) \to \mathsf{C}(c, b)$ is called *post-composition*. Given a $k : c \to a$, $f_*(k) := f \circ k : c \to b$.



Proposition 2.5 (First glance of the Yoneda Lemma)
The correspondence $\mathsf{Set}^{\mathbb{N}}(1, X) \cong X(0)$ *is natural in* X.

Proof. For each variable set X, let α_X be the function $\mathsf{Set}^{\mathbb{N}}(1, X) \to X(0)$, sending each global element $x \in \mathsf{Set}^{\mathbb{N}}(1, X)$ to $x_0 \in X(0)$. Let β_X be the function $X(0) \to \mathsf{Set}^{\mathbb{N}}(1, X)$, sending each $x_0 \in X(0)$ to it's *destiny* $x : 1 \to X$. α and β are clearly inverse[5]. We need to show that α and β are natural.

To say α is natural is to say that, for each $f : X \to Y$, the following diagram commutes.

$$\begin{array}{ccc} \mathsf{Set}^{\mathbb{N}}(1, X) & \xrightarrow{\alpha_X} & X(0) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathsf{Set}^{\mathbb{N}}(1, Y) & \xrightarrow[\alpha_Y]{} & Y(0) \end{array}$$

[5] I say clearly despite this inverse *is* the Yoneda Lemma.

Suppose $x \in \mathsf{Set}^{\mathbb{N}}(1, X)$, let's chase the diagram.

$$\begin{array}{ccccccc} x & \in & \mathsf{Set}^{\mathbb{N}}(1,X) & \xrightarrow{\alpha_X} & X(0) & \ni & x_0 \\ & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_0} & & \\ f \circ x & \in & \mathsf{Set}^{\mathbb{N}}(1,Y) & \xrightarrow[\alpha_Y]{} & Y(0) & \ni & f_0(x_0) = (f \circ x)_0 \end{array}$$

Where the last equality $f_0(x_0) = (f \circ x)_0$ is given by composing two natural transformations, thus α is natural.

Since we know that α and β are *inverse*, the naturality of β can be easily deduced. □

Just like $X(0)$ containing all the elements that exist from the beginning, for any $n \in \mathbb{N}$, $X(n)$ contains all the elements that exist at the time n, and will continue to exists forever.

But note that a hom-functor $\mathbb{N}(n, -) : \mathbb{N} \to \mathsf{Set}$ is an abstract "element" that appears at the time n, so the natural transformation $\mathbb{N}(n, -) \to X$ is exactly $X(n)$. **This is the Yoneda Lemma.** Let's look at the generalized case.

Theorem 2.6 (Yoneda Lemma, the Fundamental Theorem of Category Theory)
For any category C*, any object* $c \in \mathsf{C}$*, and any presheaf* $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$*, there's an isomorphism*

$$\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c), F) \cong F(c)$$

that is natural in both F *and* c*.*

Proof. The proof has three stages: construction of the bijection, proof of naturality in F, and proof of naturality in c. Here we only construct the bijection since the left is easy.

Let $\Phi : \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c), F) \to F(c)$ be such a function: for any $\alpha : \mathsf{C}(-, c) \to F$, we first take $\alpha_c : \mathsf{C}(c, c) \to F(c)$, then take $\alpha_c(1_c) \in F(c)$. Let $\Phi(\alpha) = \alpha_c(1_c)$.

Conversely, we construct a function $\Psi : F(c) \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c), F)$. Given $x \in F(c)$, we need to construct a function natural transformation $\Psi(x) : \mathsf{C}(-, c) \to F$, which means, given $d \in \mathsf{C}$ and $f : d \to c \in \mathsf{C}$, we need to construct $\Psi(x)_d(f) \in F(d)$, and $\Psi(x)$ has to be natural.

First let $\Psi(x)_c(1_c)$ be x. And then, by naturality of $\Psi(x)$, every $\Psi(x)_d(f)$ is now *fixed*, since the following diagram must commute.

$$\begin{array}{ccccccc} 1_c & \in & \mathsf{C}(c,c) & \xrightarrow{\Psi(x)_c} & F(c) & \ni & x \\ & & {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle F(f)} & & \\ f & \in & \mathsf{C}(d,c) & \xrightarrow[\Psi(x)_d]{} & F(d) & \ni & F(f)(x) = \Psi(x)_d(f) \end{array}$$

We are forced to let $\Phi(x)_d(f)$ be $F(f)(x)$. By functoriality of F, $\Phi(x)$ is clearly natural.

It's easy to see that Φ and Ψ are inverse. Given $x \in F(c)$, $\Phi(\Psi(x)) = \Psi(x)_c(1_c) = x$. Given $\alpha : \mathsf{C}(-, c) \to F$, α is the *only* natural transformation $\mathsf{C}(-, c) \to F$ such that it maps $1_c \in \mathsf{C}(c, c)$ to $\alpha_c(1_c)$, but so is $\Psi(\Phi(\alpha))$, hence $\alpha = \Psi(\Phi(\alpha))$. □

This theorem has several important corollaries that are also called Yoneda Lemma.

Corollary 2.7
For any category C*, any two objects* $c, d \in \mathsf{C}$*,*

$$\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-, c), \mathsf{C}(-, d)) \cong \mathsf{C}(c, d)$$

where the $\Rightarrow$ direction is given by post composition.

Proof. Let F be $\mathsf{C}(-,d)$ in Yoneda lemma. □

Definition 2.8 (Yoneda Embedding)
For any category C, there is a functor $\mathrm{y} : \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$, where $\mathrm{y}(c) := \mathsf{C}(-,c)$, called ***Yoneda embedding****.*

The previous corollary $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(\mathsf{C}(-,c) \cong \mathsf{C}(-,d)) \cong \mathsf{C}(c,d)$ is essentially saying that Yoneda embedding is *fully faithful*.

Definition 2.9 (Properties of a functor: full and faithful)
Given a functor $F : \mathsf{C} \to \mathsf{D}$.

- *F is* ***faithful*** *if F is injective on each hom-set.*
- *F is* ***full*** *if F is surjective on each hom-set.*
- *F is* ***fully faithful*** *if F is both full and faithful.*
- *If F is an inclusion of subcategory, then C is the* ***full subcategory*** *of D.*

Corollary 2.10
Yoneda embedding is fully faithful.

The cool thing about this version of Yoneda lemma is that, since every presheaf category is a topos, every small category can now be fully and faithfully embedded into a *topos*.[6]

Definition 2.11 (Representable Functors)
A functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is representable if there exists an object $c \in \mathsf{C}$ with $F \cong \mathsf{C}(-,c)$. In this case, F is ***represented*** *by c.*

Corollary 2.12
If a functor is representable, then the object representing it is unique up to isomorphism.

Example 2.13
Proposition 2.5 and Yoneda Lemma shows that the terminal functor $1 \in \mathsf{Set}^{\mathbb{N}}$ is represented by the object $0 \in \mathbb{N}$.

Exercise 2.14
Consider the category $\mathsf{Set}^{\mathbb{Z}}$, where $\mathbb{Z}$ stands for $(\mathbb{Z}, \leq)$ as a category. The terminal object 1 sends every $n \in \mathbb{Z}$ to the singleton $1 \in \mathsf{Set}$. Show that 1 is not representable.

This is immediate once you have the picture of hom-functors in you mind.

Yoneda lemma has a more profound and geometric interpretation by Emily Riehl, who generalized Yoneda lemma to ∞-category and *dependent* case where Yoneda lemma appears as *direct path induction*.

[6]Reader who is familiar with homological algebra may realize that this is quite like the Freyd-Mitchell embedding theorem, which says that every small Abelian category can be embedded into a category of modules by a fully faithful and *exact* functor. I would like to think that Fredy-Mitchell embedding is like an upgraded version of Yoneda embedding that is more suitable for algebra.

2.3 A few more words on subcategory

Suppose C is a subcategory of D and $F : \mathsf{C} \to D$ is the inclusion functor, so clearly F is faithful.

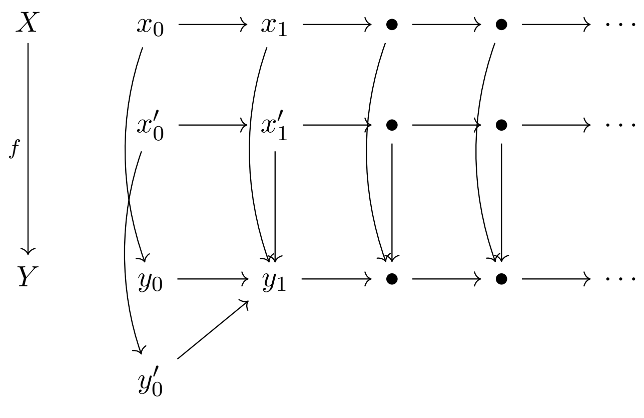
- If F is not full, then we may think of objects in C as certain objects in D with additional structures. For example, the forgetful functor $U : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Set}$ can be seens as an embedding from $\mathsf{Vect}_{\mathbb{R}}$ to Set. But "being a vector space" is not only a property of a set. It requires more *structures* on the set. So $\mathsf{Vect}_{\mathbb{R}}$ in some sense has *less* morphisms than Set, since it only contains the morphisms that preserve the additional structures.
- If F is full, then for any object $d \in \mathsf{D}$, "being an object of C" is a *mere property*. For example, the inclusion from Ab, the category of Abelian groups, to Group, the category of groups, is fully faithful. For any group G, "being Abelian" is a mere property of G. It doesn't require any additional structures.

3 Some topos-theoretic properties of $\mathsf{Set}^{\mathbb{N}}$

Even though we have not yet defined the notion of elementary topos, we can explore some topos-theoretic properties just with examples Set and $\mathsf{Set}^{\mathbb{N}}$ out of curiosity.

Proposition 3.1
$\mathsf{Set}^{\mathbb{N}}$ *doesn't satisfy the axiom of choice.*

Proof. Consider $X, Y : \mathbb{N} \to \mathsf{Set}$, where X is the constant functor on two-element set 2, and $Y(0) = 2$ and for any other $n > 0$, $Y(n) = 1$. The following picture defines an epimorphism $f : X \to Y$.



Since f is pointwise surjective, it's easy to see that f is epic.

Now suppose f splits, so there is a morphism $g : Y \to X$ such that $f \circ g = 1_Y$. On 0, this condition forces $g_0(y_0) = x_0$ and $g_0(y'_0) = x'_0$. Apply the naturality to $0 \to 1 \in \mathbb{N}$ we get the following commutative square.

$$\begin{array}{ccc} X(0) & \longrightarrow & X(1) \\ {\scriptstyle g_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ Y(0) & \longrightarrow & Y(1) \end{array}$$

Chase the diagram for both y_0 and y_0', we get:

$$\begin{array}{ccc} X(0) & \longrightarrow & X(1) \\ {\scriptstyle g_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ Y(0) & \longrightarrow & Y(1) \end{array} \qquad \begin{array}{ccc} x_0 & \longmapsto & x_1 \\ \uparrow & & \uparrow \\ y_0 & \longmapsto & y_1 \end{array} \qquad \begin{array}{ccc} x_0' & \longmapsto & x_1' \\ \uparrow & & \uparrow \\ y_0' & \longmapsto & y_1 \end{array}$$

So y_1 is mapped to both x_1 and x_1', which is absurd. □

Now we introduce the notion of *well-pointedness.*

Definition 3.2 (Well-pointedness)

A category with a terminal 1 *is* ***well-pointed*** *if every morphism is determined by its value on global elements in the following sense:*

- *If* $f, g : a \to b$ *are two morphisms such that for all global elements* $x : 1 \to a$ *we have* $f \circ x = g \circ x$, *then* $f = g$.

Clearly Set is well-pointed.

Exercise 3.3

Suppose C *is a category with a terminal* 1. *Show that the following statements are equivalent.*

- *Category* C *is well-pointed.*
- *The* ***global section*** *functor* $\mathcal{E}(1, -) : \mathcal{E} \to \mathsf{Set}$ *is faithful.*

Exercise 3.4

Show that $\mathsf{Set}^{\mathbb{N}}$ *is not well-pointed.*

Exercise 3.5

Show that in a well-pointed category C, *if an object* c *has only one global element, then* $c \cong 1$.

For a topos $\mathcal{E}$, the closer to being well-pointed it is, the more *extensional* it is. In a topos that is not well-pointed, there might be some objects that are far from being the terminal but has only one global element. For example, in $\mathsf{Set}^{\mathbb{N}}$, any variable set X such that $X(0) = 1$ has only one global element. But if it contains elements that only appear *later*, then it's not the terminal.