The Category of Sets

Prepared by CanaanZhou ;)

March 16

1 Why Topos?

There are *many*[1] ways to look at topos theory. But from the logical perspective, topos theory and set theory share the same goal: finding a suitable foundation for mathematics. But they have much more different characteristics than commonalities.

1. Set theory is based on first-order logic: The standard ZFC system is a first-order theory. Topos theory is based on *category theory* instead. A topos is just a category with some additional structures.

2. Set theory emphasizes on membership relation $\in$. In ZFC, we have a *global* membership relation. For any two sets x, y, the sentence $x \in y$ makes sense (it might be right or wrong, but it's a valid sentence). In topos theory, we can only study an object (e.g. a set) via its *morphisms* (e.g. functions) to/from other objects.

3. For platonic set theorist (like Kurt Gödel), there is *the* mathematical universe V. The goal of set theory then is to add more suitable axioms (like large cardinals) to ZFC and make it characterize V as precisely as possible. But in topos theory we do the exact opposite. We look for the *minimal* set of axioms (the axioms of *elemantary topos*, as we will see later on) that can give us a *reasonable* mathematical universe (a

[1] Famous saying by Peter Johnstone, the author of *Elephant*: "However you approach it, it is still the same animal."

topos). In other words, we study *all possible* mathematical universes, or you can say, *all possible mathematics*.

4. Any topos has a internal logic, just like we have *the* logic of ZFC in V. But the internal logic of a topos is natively *intuitionistic*, and being classical (i.e. Boolean) is a special property of a topos.

Enough talking. Let's get down to business.

2 Your first category: Set

Definition 2.1

*A **category** C consists of the following data:*

- *A collection of objects. We write $c \in \mathsf{C}$ if c is an object of C.*
- *For every pair of objects $c, d \in \mathsf{C}$, a collection of morphisms from c to d, denoted as $\mathsf{C}(c, d)$. We write $f : c \to d$ if $f \in \mathsf{C}(c, d)$. In this case, c is the **source** of f and d is the **target** of f. We call $\mathsf{C}(c, d)$ a **hom-set**.*
- *If two morphisms are connected like $f : c \to d$, $g : d \to e$, we can **compose** them into one morphism $g \circ f : c \to e$.*

and satisties the following axioms:

- *Morphisms composition is **associative**. If there are three connected morphisms $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, then $(h \circ g) \circ f = h \circ (g \circ f)$, which we simply denote as $h \circ g \circ f$.*
- *For each object $c \in \mathsf{C}$, there's an **identity** morphism $1_c : c \to c$.*
- *Morphism composition also satisties **unit law**: By composing $a \xrightarrow{f} b \xrightarrow{1_b} b$ we get f, and by composing $a \xrightarrow{1_a} a \xrightarrow{f} b$ we still get f.*

Idea 2.2 (The philosophy of category theory)

To study any mathematical object (e.g. a set, a group, a topological space), we should put it in a suitable category. And to study any object in a category, we can only look at its relationship (morphisms) with other objects.

*So unlike in set theory where we look **inside**, in category theory we look **outside**.*

A notion embodying this idea is *isomorphism*.

Definition 2.3 (Isomorphism)
In a category C*, we call a morphism* $f : c \to d$ *an **isomorphism** if there is another morphism* $g : d \to c$ *in reverse direction, such that* $g \circ f = 1_c$ *and* $f \circ g = 1_d$*. If such a isomorphism exists, we say objects* c *and* d *are **isomorphic**, denoted as* $c \cong d$*.*

In category theory we almost never talk about objects being *equal*. We use *isomorphic* instead, because according to the philosophy, there's really no difference between isomorphic objects.[2] This is related to the idea of structuralism (I think). *Structural set theory* deals with sets in this perspective, while traditional ZFC-style set theory is called *material set theory*.

Exercise 2.4
Show that if a morphism $f : c \to d$ *is an isomorphism, than the* g *witnessing that is unique. The morphism* g *is called the inverse of* f*, written as* f^{-1}*.*

The first category (also the first topos) you will learn, and the most important one, is *the category of sets*.

Definition 2.5
The category of sets Set *is:*

- *An object is a set* X*.*
- *A morphism* f *from* X *to* Y *is a function* $f : X \to Y$*.*

Discussion 2.6
Is Set *just* V*?*

Sort of, but you shouldn't think like that. The point is that there's no more hereditary structure to look at. Since every element of a set is now

[2] Note that even though we don't talk about objects being equal, we still talk about *morphisms* being equal. If we try to completely avoid talking about equality, we reach the idea of higher category theory. Maybe an ∞-topos would be an ideal mathematical universe for structuralists, but that's far beyond the scope of this seminar and my knowledge.

indistinguishable from other elements[3], every set now has only one property: how many elements are there in the set.[4]

This is furthur illustrated by the following exercise.

Exercise 2.7

Show that in Set*, an isomorphism is just a bijection.*

3 A different type of categories: small categories

Here are some categories with a different flavor.

- There's an *empty* category that has no objects and no morphisms, as well as a *singleton* category with only one object and one morphism: the identity morphism.
- Every preordered set $(P, \leq)$ (i.e. a set with a binary relation that is reflexive and transitive) can be seen as a category. The objects are the elements of P. For each $p, q \in P$, if $p \leq q$, then $P(p, q)$ is a singleton. Otherwise, $P(p, q) = \varnothing$.[5]
- Every monoid can be seen as a category with a single object. In fact...

Exercise 3.1

Show that by considering a category M with a single object $\bullet$, the category axioms become the monoid axioms for $M(\bullet, \bullet)$.

For example, there are many reasonable ways of turning the set of natural numbers $\mathbb{N}$ into a category. You can think of it as an ordered set $(\mathbb{N}, \leq)$ (an *ordinal* if you will), or as a monoid $(\mathbb{N}, +)$.

[3]This is an example of Lawvere's notion of *cohesion*. The idea is that even though there are different elements in a set, they are totally indistinguishable. It has a fascinating connection with Hegel's dialectic method.

[4]Structural set theory is closer to Cantor's original version of set theory, where the number of elements in a set is called its *kardinale*. Meanwhile, a *cardinal* is a special type of ordinal. There are some very subtle differences.

[5]You may reasonably think of the element in the singleton as the *proof* or *witness* of $p \leq q$. If $P(p, q) = \varnothing$, then there's no proof of $p \leq q$, meaning $p \not\leq q$.

You might've noticed that categories of this type are significantly smaller than Set.

Definition 3.2

*For any category C, if its collection of morphisms is small enough to be a set, we call it a **small** category. A category that is not small is called **large**.*

So for example, $(\mathbb{N}, \leq)$ as a category is a small one, while Set is a large one. However, Set is *locally* small.

Definition 3.3

*For any category C, if all of its **hom-sets** are small enough to be a set, we call it a **locally small** category.*

Discussion 3.4 (Size Issue)

If the collection of objects of Set is not even a set, what are we even talking about? A class?

It's totally fine to think of it like that it you want to do it rigorously. You may also assume that we're working in a meta-theory of ZFC plus an inaccessible cardinal κ, we can take any worldly cardinal λ below that and let Set be V_λ. You can also ignore that completely (like I do) because size issue is barely important here.

Other examples of large categories include the category of groups and homomorphisms Group, the category of topological spaces Top and so on. You may think of them as categories with structured objects and structure-preserving mappings.

Small categories, on the other hand, are often thought of as models of *time*[6] where an object is a certain time, a morphism is the passing of time. Another commonly-used perspective is to think of them as *generalized spaces* where an object is a place, and a morphism $f : c \to d$ is a directed path from c to d.

[6] For example, a Kripke frame in Kripke semantics for intuitionistic logic can be thought of as a model of time.

4 The first example of universal property and diagram chasing

Let's see what we can do in Set just with morphisms.

For a natural number $n \in \mathbb{N}$, let n also denote the n-element set. For a finite set X, let $|X| \in \mathbb{N}$ denote the number of elements in X.

Exercise 4.1 (Counting functions)
Calculate the following.

- *What's* $|\mathsf{Set}(2,3)|$*?*
- *More generally, for* $m, n \in \mathbb{N}$*, what's* $|\mathsf{Set}(m,n)|$*?*

Use your conclusion to show that, for any set X*,*

- $|\mathsf{Set}(\varnothing, X)| = 1$*;*
- $|\mathsf{Set}(X, 1)| = 1$*;*
- $|\mathsf{Set}(1, X)| = X$*.*

Definition 4.2 (Initial and Terminal)
Fix a category C*. We call an object* 0 *a* ***initial*** *object if for any object* $c \in \mathsf{C}$*, there's a unique morphism* $0 \to c$*.*

Dually*, we call an object* 1 *a* ***terminal*** *object if for any object* $c \in \mathsf{C}$*, there's a unique morphism* $c \to 1$*.*

Initial and teriminal objects are *universal* among all the objects in C in opposite ways.

Proposition 4.3 (Universal property implies uniqueness)
If C *has a initial object* 0*, then it's unique up to unique isomorphism.*

Proof. The proof technique is called *diagram chasing*.

Suppose there are two initial objects $0, 0'$ in C. Since 0 is initial, there's a unique morphism $f : 0 \to 0'$. Since $0'$ is also initial, there's a unique morphism $g : 0' \to 0$.

Again, since 0 is initial, there's a unique morphism $0 \to 0$. But now we have two such morphisms: $1_0, g \circ f$, so they have to be equal. In the same way we can show that $1_{0'} = f \circ g$, so $0 \cong 0'$.

Now suppose there are two isomorphisms $f, f' : 0 \to 0'$. Since 0 is initial, there's only one morphism $0 \to 0'$, so f, f' have to be equal. □

We can do the same thing for terminal objects. However, there's a useful metatheorem in category theory.

Definition 4.4 (Opposite category)

For any category C*, we can fix the objects but* ***reverse*** *all the morphisms to obtain another category* C^{op}*. This is called the opposite (or dual) category of* C*.*

Idea 4.5 (Dual principle)

Any definition or theorem in category theory has a dual version. We simply reverse all the morphisms.

Exercise 4.6

Show the uniqueness of terminal objects in the following ways:

- *Prove it directly by diagram chasing.*
- *Show that the notion of initial and termianl objects are dual: a initial object in* C *is a terminal object in* C^{op} *and vice versa. Then apply the dual principle.*

Note that initial and termianl objects may or may not exist in a category. But when they do, they're unique up to unique isomorphism.

Exercise 4.7

Show that in Set*,* $\varnothing$ *is initial while* 1 *is terminal, so the notation makes sense.*

Exercise 4.8

Show that in a preordered set $(P, \leq)$*, a initial is a minimum while a terminal is a maximum (if they exist).*

Sometimes, initial and terminal might even coincide. That special object is called a *zero* object.

Example 4.9 (Zero object)

Consider the category $\mathsf{Vect}_{\mathbb{R}}$*. Its objects are real vector spaces, and its morphisms are linear functions. In this category, the initial and the terminal are both zero-dimension space* 1*: a singleton with* $\mathbb{R}$ *acting trivially on it.*

5 Monomorphism and Epimorphism

Now we look at the categorical way of expressing *injectiveness* and *surjectiveness*.

From now on, fix a category C.

Definition 5.1 (Monomorphism)
*A morphism $f : c \to d$ is a **monomorphism**, denoted as $f : c \rightarrowtail d$, if for any pair of morphisms $h, k : b \rightrightarrows c$, whenever $f \circ h = f \circ k$, we have $h = k$.*

Monomorphism is often abbreviated as mono. The adjective form is monic.

Exercise 5.2 (Mono means injection)
Show that Set, a monomorphism is exactly an injection.

Hint: for one direction, let $b = 1$.

Definition 5.3 (Epimorphism)
*A morphism $f : c \to d$ is a **epimorphism**, denoted as $f : c \twoheadrightarrow d$, if for any pair of morphisms $h, k : d \rightrightarrows e$, whenever $h \circ f = k \circ f$, we have $h = k$.*

Epimorphism is often abbreviated as epi. The adjective form is epic. It can be easily seen that monomorphism and epimorphism are dual definitions: a monomorphism in C is exactly an epimorphism in C^{op} and vice versa.

Exercise 5.4 (Epi means surjection)
Show that in Set, an epimorphism is exactly a surjection.

There's a stronger notion of mono and epi. It's inspired by the following definition. In Set, whenever there are two functions

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

such that $g \circ f = 1_X$, then f has to be injective and g has to be surjective.

Definition 5.5 (Splitting)
*A morphism $f : c \to d$ is a **split monomorphism** if there's another morphism $g : d \to c$ such that $g \circ f = 1_c$. We call g the **retraction** of f.*

*Dually, a morphism $g : d \to c$ is a **split epimorphism** if there's another morphism $f : c \to d$ such that $g \circ f = 1_d$. We call f the **section** of g.*

If a morphism is a split mono/epi, we simply say that it splits.

Exercise 5.6

Show that in any category C*, split monic/epic implies monic/epic.*

Exercise 5.7

Show that in Set*, every monomorphism whose sourse is not* $\varnothing$ *splits.*

Hint: you have to use law of excluded middle to get an element from a non-empty set.

Definition 5.8 (The Axiom of Choice)

If every epimorphism splits in a category C*, we say* C *satisfies the* ***axiom of choice****, abbreviated as AC.*

In particular, Set satisfies AC.

Exercise 5.9 (More on AC for set theory lovers)

Here's AC in its usual form: for every inhabited[7] *family of sets* $\{X_i\}_{i \in I}$*, there's a* ***choice function*** $f : I \to \coprod_{i \in I} X_i$ *such that for every* $i \in I$*,* $f(i) \in X_i$*, where* $\coprod$ *means disjoint union.*

Show that these two forms of AC are equivalent under ZF*.*

Here's a fun exercise of diagram chasing. Note that in Set, every function whose source is the singleton 1 is injective. This can be generalized to any category.

Exercise 5.10

Fix a category C *with a terminal* 1*. Take any morphism* $f : 1 \to c$ *out of* 1*. Show that* f *is a split monomorphism, therefore is monic.*

Hint: by the universal property of 1*, there's a unique morphism* $c \to 1$*.*

6 Product and coproduct

Now we deal with *Cartesian product* operation categorically.

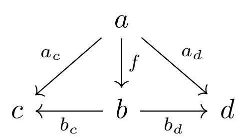
Definition 6.1 (Product)

Fix a category C *and two objects* $c, d \in \mathsf{C}$*. We look at the following category* $\int \mathsf{C}(-, c) \times \mathsf{C}(-, d)$[8]*:*

[7] "Inhabited" is just a constructive way of saying "non-empty".

[8] The notation is standard. It refers to a construction called category of elements, but you don't have to know that.

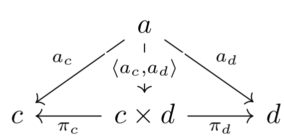
- *Objects: an object is a triple* (a, a_c, a_d)*, where* $a \in \mathsf{C}$ *is an object of* C*,* $a_c : a \to c, a_d : a \to d$ *are two morphisms.*
- *Morphisms: a morphism* $f : (a, a_c, a_d) \to (b, b_c, b_d)$ *is a morphism* $f : a \to b$ *such that the following diagram* ***commutes****, meaning* $a_c = b_c \circ f$ *and* $a_d = b_d \circ f$*.*



The ***product*** $(c \times d, \pi_c, \pi_d)$ *of* c *and* d *is defined as the terminal object of this category.*

Here's an equivalent definition that you will see on most category theory textbook.

Definition 6.2 (Product again)
Fix an category C*. The* ***product*** *of* $c, d \in \mathsf{C}$ *is an object* $c \times d$ *equipped with two morphisms* $\pi_c : c \times d \to c, \pi_d : c \times d \to d$*, called* ***projections****, such that: for every object* a *with two morphisms* $a_c : a \to c$ *and* $a_d : a \to d$*, there exists a* ***unique*** *morphism* $a \to c \times d$ *such that the following diagram commutes.*



Once you unpack everything, these definitions are clearly equivalent.

Exercise 6.3 (Product deserves its name)
Show that in Set*,* $X \times Y$ *really is (isomorphic to) the cartesian product of* X *and* Y*. So* Set *has all binary product.*

Exercise 6.4 (Product also deserves its notation)
Show that in Set*, for any finite sets* X, Y*,* $|X \times Y| = |X| \times |Y|$*.*

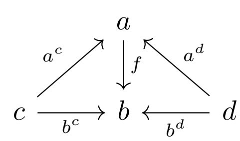
Exercise 6.5 (Product in poset is meet)
In a poset $(P, \leq)$*, the* ***meet*** $p \wedge q$ *of* $p, q \in P$ *is the greatest element* r *such that* $r \leq p$ *and* $r \leq q$*. Show that meet is exactly product in a poset.*

Fix a set X. Consider the poset $(\mathcal{P}X, \subseteq)$, the power set of X ordered by inclusion. Take $A, B \in \mathcal{P}X$. What's their product in $\mathcal{P}X$? What's their product in Set*? What's the difference?*

By reversing the morphisms, we get the dual version of product: coproduct. In category theory we often use *co-something* to represent the dual of that thing.

Definition 6.6 (Coproduct)
Fix a category C and two objects $c, d \in \mathsf{C}$. We look at the following category $\int \mathsf{C}(c, -) \times \mathsf{C}(d, -)$:

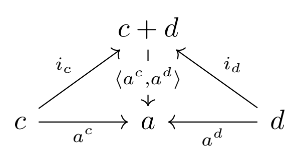
- *Objects: an object is a triple (a, a^c, a^d)*[9] *where $a \in \mathsf{C}$, $a^c : c \to a, a^d : d \to a$.*
- *Morphisms: a morphism $f : (a, a^c, a^d) \to (b, b^c, b^d)$ is a morphism $f : a \to b$ such that the following diagram commutes.*



The ***coproduct*** *$(c + d, i_c, i_d)$ is defined as the initial object of this category.*

Equivalently…

Definition 6.7 (Coproduct again)
Fix a category C. The ***coproduct*** *of $c, d \in \mathsf{C}$ is an object $c + d$ equipped with two morphisms $i_c : c \to c + d, i_d : d \to c + d$, called* ***embeddings****, such that: for every object a with two morphisms $a^c : c \to a$ and $a^d : d \to a$, there exists a* ***unique*** *morphism $c + d \to a$ such that the following diagram commutes.*



[9] This is **not** a standard notation, I made it up myself. In fact I don't think there's a standard notation for this.

Exercise 6.8 (Coproduct in Set)
Show that in Set*,* $X + Y$ *is just the disjoint union of* X *and* Y*.*

Also show that, for any finite sets X, Y*,* $|X + Y| = |X| + |Y|$*.*

Just like product, in a poset $(P, \leq)$, the coproduct of $p, q \in P$ is their *join* $p \vee q$, which is their binary supremum.

The interplay between product, coproduct, initial, and terminal. Can be quite complicated. In Set, for any inhabited set X, we have the following:

- $X \times 0 \cong 0$.
- $X + 0 \cong X$.
- $X \times 1 \cong X$.

However, that's not always the case.

Example 6.9
Consider the category $\mathsf{Vect}_{\mathbb{R}}^{\text{fin}}$ *of* ***finite****-dimension vector spaces and linear functions. In this category, even product and coproduct coincide. They are both* ***direct sum****. So* $V \times 1 \cong V + 1 \cong V$*.*

One may also observe that, in Set, embeddings are always injective. This is also not always true in any category.

Exercise 6.10
Show the following claims.

- *Projections are not always epic.*

Hint: consider $X \times \varnothing$ *in* Set*.*

- *Embeddings are not always monic.*

Hint: this is the dual of the previous claim.

7 Furthur reading

- *Sets for Mathematics* - F. William Lawvere, Robert Rosebrugh, Chapter 1-4
- *Category Theory in Context* - Emily Riehl, Chapter 1, 3