# Nonstandard Analysis and Combinatorial Number Theory

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**Day Four:** **Hard Applications to Combinatorics**

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OUTLINE:

1. Multiple Levels of Infinities and Ramsey's Theorem
2. Multidimensional van der Waerden's Theorem
3. Szemerédi's Theorem

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There have been many recent applications of nonstandard analysis to Ramsey type problems in combinatorial number theory. One of the characteristics of these new applications is the use of multiple levels of infinities. We will first construct nonstandard universes with multiple levels of infinities and then solve some combinatorial problems in these nonstandard universes.

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Our first goal in this subsection is to construct a sequence of nonstandard universes and two types of correspondent elementary embeddings satisfying some nice properties.

## Proposition (4.1)

*There exists a sequence of nonstandard universes*

$$\mathcal{V}_0 = \mathcal{V} \prec \mathcal{V}_1 \prec \mathcal{V}_2 \prec \cdots \mathcal{V}_n \prec \cdots$$

*and elementary embeddings*

$$i_{m,n} : \mathcal{V}_n \to \mathcal{V}_{n+1}$$

*for all* $0 \leq m \leq n$ *in* $\mathbb{N}$ *such that*

## Proposition (4.1)

1. $\mathbb{N}_0 := \mathbb{N}$ *and* $\mathbb{N}_{n+1} := i_{n,n}(\mathbb{N}_n) \supseteq i_{n,n}[\mathbb{N}_n] = \mathbb{N}_n$ *is an end-extension of* $\mathbb{N}_n$*, i.e., every number in* $\mathbb{N}_{n+1} \setminus \mathbb{N}_n$ *is greater than any number in* $\mathbb{N}_n$*, for* $n = 0, 1, \ldots$*;*
2. $i_{m,n}[\mathbb{N}_k \setminus \mathbb{N}_{k-1}] \subseteq \mathbb{N}_{k+1} \setminus \mathbb{N}_k$ *for* $k = m+1, m+2, \ldots, n$*;*
3. $i_{m,n}(x) = x$ *for every* $x \in \mathbb{N}_m$ *and* $i_{m,n} \upharpoonright \mathcal{V}_k = i_{m,k}$ *for* $m \le k \le n$*;*
4. $i_{m,n} \upharpoonright \mathcal{V}_k : (\mathcal{V}_k; \mathbb{R}_{k-l+1}, \mathbb{R}_{k-l}) \to (\mathcal{V}_{k+1}; \mathbb{R}_{k-l+2}, \mathbb{R}_{k+1-l})$ *is an elementary embedding where* $(\mathcal{V}_k; \mathbb{R}_{k-l+1}, \mathbb{R}_{k-l})$ *and* $(\mathcal{V}_{k+1}; \mathbb{R}_{k-l+2}, \mathbb{R}_{k+1-1})$ *represent the models* $\mathcal{V}_k$ *and* $\mathcal{V}_{k+1}$ *augmented by unary relations* $\mathbb{R}_{k+1-l}, \mathbb{R}_{k-l} \notin \mathcal{V}_k$ *and* $\mathbb{R}_{k-l+2}, \mathbb{R}_{k+1-l} \notin \mathcal{V}_k$*, respectively, for* $m \le k \le n$ *and* $2 \le l \le k - m$*;*

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Recall that the ultrafilter $\mathcal{F}$ is fixed after Definition 1.6. Let $\mathcal{V}_0 := \mathcal{V}$, $\mathcal{F}_0 := \mathcal{F}$, $\mathcal{V}_1 := {}^*\mathcal{V}$ be the ultrapower of $\mathcal{V}_0$ modulo $\mathcal{F}_0$, and $i_{0,0} := {}^*$ be the elementary embedding from $\mathcal{V}_0$ to $\mathcal{V}_1$ constructed in Definition 1.21. Note that $\mathcal{F}_0 \in \mathcal{V}_0$.

Let $\mathcal{F}_1 := i_{0,0}(\mathcal{F}_0) \in \mathcal{V}_1$. By the transfer principle we have that $\mathcal{F}_1$ satisfies Parts 1 – 4 of Definition 1.6 for any $A, B \in \mathcal{V}_1$ with $X = \mathbb{N}_1 := i_{0,0}(\mathbb{N}_0)$ and co-finite is replaced by co-hyperfinite in $\mathcal{V}_1$. We call $\mathcal{F}_1$ a $\mathcal{V}_1$–internal non-principal ultrafilter on $\mathbb{N}_1$. Notice that $i_{0,0}(\mathscr{P}(\mathbb{N}_0)) = \mathcal{V}_1 \cap \mathscr{P}(\mathbb{N}_1)$ and

$$\begin{aligned} i_{0,0}(\mathscr{P}_{<\mathbb{N}_0}(\mathbb{N}_0)) &= \mathcal{V}_1 \cap \mathscr{P}_{<\mathbb{N}_1}(\mathbb{N}_1) \\ &:= \{A \subseteq \mathbb{N}_1 \mid A \in \mathcal{V}_1 \wedge \exists N \in \mathbb{N}_1 (A \subseteq [N])\}. \end{aligned}$$

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If an $\in$–formula $\varphi$ is coded by a finite sequence of numbers in $\mathbb{N}_0$, then $i_{0,0}(\varphi) = \varphi$.

Without loss of generality we can identify $i_{0,0}[\mathcal{V}_0]$ with $\mathcal{V}_0$ so that $\mathcal{V}_0$ is an elementary submodel of $\mathcal{V}_1$.

Let $\mathcal{F}_0' := \mathcal{F}_0$ and $\mathbb{N}_0' := \mathbb{N}_0$. We use $'$ to indicate the different location where $\mathcal{F}_0$ and $\mathbb{N}_0$ are used. To form an ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$, we obtain an elementary extension

$$\mathcal{V}_2 := (V_1^{\mathbb{N}_0'}/\mathcal{F}_0', {}^*\!\in) = \mathcal{V}_1^{\mathbb{N}_0'}/\mathcal{F}_0' = \left(\mathcal{V}_0^{\mathbb{N}_0}/\mathcal{F}_0\right)^{\mathbb{N}_0'}/\mathcal{F}_0' \qquad (1)$$

and associated elementary embedding $i_{0,1} : \mathcal{V}_1 \to \mathcal{V}_2$ as we did in Definition 1.8 and Corollary 1.10.

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By applying Mostowski collapsing map again we can assume that ${}^*\!\in$ is the real membership relation $\in$ and $\mathbb{N}_1 \subseteq \mathbb{N}_2 := i_{0,1}(\mathbb{N}_1)$. Note that $\mathbb{N}_1$ and $i_{0,1}[\mathbb{N}_1]$ are not the same even after Mostowski collapsing. Let's call $\mathcal{V}_1^{\mathbb{N}_0'}/\mathcal{F}_0'$ the external ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$.

If $\mathbb{N}_1$ had been identified with $i_{0,1}[\mathbb{N}_1]$, then $\mathbb{N}_2$ won't be an end-extension of $\mathbb{N}_1$. Therefore, we should look at $\mathcal{V}_2$ from a different angle.

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## Definition (4.2)

*The $\mathcal{V}_1$-internal ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_1$ is the model with the base set* $\mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1 := \{[f]_{\mathcal{F}_1} \mid f \in \mathcal{V}_1^{\mathbb{N}_1}$ *and* $f \in \mathcal{V}_1\}$, *where*

$$f \sim_{\mathcal{F}_1} g \text{ iff } \{n \in \mathbb{N}_1 \mid f(n) = g(n)\} \in \mathcal{F}_1 \text{ and}$$

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*and the membership relation* $\in_2$ *defined by*

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*The map* $i_{1,1} : \mathcal{V}_1 \to (\mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1)/\mathcal{F}_1$ *with* $i_{1,1}(c) = [\phi_c]_{\mathcal{F}_1}$ *is the elementary embedding from* $\mathcal{V}_1$ *to* $(\mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1)/\mathcal{F}_1$ *associated with the* $\mathcal{V}_1$*-internal ultrapower of* $\mathcal{V}_1$ *modulo the* $\mathcal{V}_1$*-internal ultrafilter* $\mathcal{F}_1$.

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By applying Mostowski collapsing map again we can assume that $\in_2$ is $\in$. An element $a \in \mathcal{V}_2$ is called $\mathcal{V}_2$-internal. An element $a \in \mathcal{V}_2$ is called $\mathcal{V}_1$-internal if $a \in i_{1,1}[\mathcal{V}_1]$.

Note that the $\mathcal{V}_1$-internal ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_1$ is really the same as the external ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$. Indeed, we can make two-step ultrapower process in two different order.

In the external ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$ we view the ultrapower modulo $\mathcal{F}_0$ to get $\mathcal{V}_1$ first and the ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$ the second. If we view the two-step ultrapower process by taking the ultrapower modulo $\mathcal{F}_0'$ first, $\mathbb{N}_0$ and $\mathcal{F}_0$ in $\mathcal{V}_0$ become $\mathbb{N}_1$ and $\mathcal{F}_1$, respectively, and $\mathcal{V}_0^{\mathbb{N}_0}$ because the collection $\mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1$ of all $\mathcal{V}_1$-internal functions from $\mathbb{N}_1$ to $\mathcal{V}_1$. Hence, the process of taking ultrapower of $\mathcal{V}_0$ modulo $\mathcal{F}_0$ is lifted into $\mathcal{V}_1$ to become the $\mathcal{V}_1$-internal ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_1$ to complete the second step.

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Note that the $\mathcal{V}_1$-internal ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_1$ is really the same as the external ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$. Indeed, we can make two-step ultrapower process in two different order.

In the external ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$ we view the ultrapower modulo $\mathcal{F}_0$ to get $\mathcal{V}_1$ first and the ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_0'$ the second. If we view the two-step ultrapower process by taking the ultrapower modulo $\mathcal{F}_0'$ first, $\mathbb{N}_0$ and $\mathcal{F}_0$ in $\mathcal{V}_0$ become $\mathbb{N}_1$ and $\mathcal{F}_1$, respectively, and $\mathcal{V}_0^{\mathbb{N}_0}$ because the collection $\mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1$ of all $\mathcal{V}_1$-internal functions from $\mathbb{N}_1$ to $\mathcal{V}_1$. Hence, the process of taking ultrapower of $\mathcal{V}_0$ modulo $\mathcal{F}_0$ is lifted into $\mathcal{V}_1$ to become the $\mathcal{V}_1$-internal ultrapower of $\mathcal{V}_1$ modulo $\mathcal{F}_1$ to complete the second step.

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Symbolically, we have

$$\mathcal{V}_2 = \left(\mathcal{V}_0^{\mathbb{N}_0}/\mathcal{F}_0\right)^{\mathbb{N}_0'}/\mathcal{F}_0' = (\mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1)/\mathcal{F}_1 \tag{2}$$
$$= ((\mathcal{V}_0^{\mathbb{N}_0'}/\mathcal{F}_0')^{\mathbb{N}_0^{\mathbb{N}_0'}/\mathcal{F}_0} \cap (\mathcal{V}_0^{\mathbb{N}_0'}/\mathcal{F}_0'))/(\mathcal{F}_0^{\mathbb{N}_0'}/\mathcal{F}_0').$$

Roughly speaking, (2) shows that one can change the order of ultrapower of $\mathcal{V}_0$ construction steps first modulo $\mathcal{F}_0$ and then modulo $\mathcal{F}_0'$ to the order that first modulo $\mathcal{F}_0'$ and then modulo $\mathcal{F}_1 = i_{0,0}(\mathcal{F}_0)$.

By applying the transfer principle to the statement that every bounded function from $\mathbb{N}_0$ to $\mathbb{N}_0$ is equivalent, modulo $\mathcal{F}_0$, to a constant function, we have that every bounded $\mathcal{V}_1$-internal function from $\mathbb{N}_1$ to $\mathbb{N}_1$ is equivalent, modulo $\mathcal{F}_1$, to a constant function.

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So, if $[f]_{\mathcal{F}_1} \in \mathbb{N}_2$ and $f(n) \leq m \in \mathbb{N}_1$ for every $n \in \mathbb{N}_1$, then $f$ is equivalent, modulo $\mathcal{F}_1$, to $[\phi_c]_{\mathcal{F}_1}$ for some $c \in \mathbb{N}_1$, which implies $[f]_{\mathcal{F}_1} \in \mathbb{N}_1$.

Thus, $\mathbb{N}_2 := i_{1,1}(\mathbb{N}_1) \supseteq i_{1,1}[\mathbb{N}_1] = \mathbb{N}_1$ is an end-extension of $\mathbb{N}_1$. Note that $i_{0,1} \restriction \mathbb{N}_0 = i_{1,1} \restriction \mathbb{N}_0 = i_{0,0}$. If $\mathcal{V}_2$ is considered as the external ultrapower of $\mathcal{V}_1$, then $\mathbb{N}_1$ can be identified as $\mathbb{N}_0^{\mathbb{N}_0'}/\mathcal{F}_0'$.

It is easy to check that the elementary embeddings $i_{0,0}, i_{0,1}, i_{1,1}$ satisfy Proposition 4.1 except Part 4, which is irrelevant.

In fact, $\mathcal{V}_2$ can be viewed as one-step ultrapower of $\mathcal{V}_0$ modulo the tensor product of $\mathcal{F}_0$ and $\mathcal{F}_0'$ where

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This indicates that $\mathcal{V}_2$ is countably saturated and elements in $\mathcal{V}_2$ can be represented by the equivalence class, modulo $\mathcal{F}_0 \otimes \mathcal{F}_0'$, of functions $f : \mathbb{N}_0 \times \mathbb{N}_0' \to \mathcal{V}_0$.

Now consider a three-step ultrapower construction. Let $\mathcal{F}_0'' := \mathcal{F}_0$, $\mathbb{N}_0'' := \mathbb{N}_0$, and $\mathcal{F}_2 := i_{1,1}(\mathcal{F}_1) \in \mathcal{V}_2$. Then

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The ultrapower in (3) results in the associated elementary embedding $i_{0,2} : \mathcal{V}_2 \to \mathcal{V}_3$. The ultrapower in (4) results in the associated elementary embedding $i_{1,2} : \mathcal{V}_2 \to \mathcal{V}_3$. And the ultrapower in (5) results in the associated elementary embedding $i_{2,2} : \mathcal{V}_2 \to \mathcal{V}_3$.

After applying Mostowski collapsing map we can again assume that $\mathbb{N}_3 := i_{2,2}(\mathbb{N}_2) \supseteq \mathbb{N}_2 = i_{2,2}[\mathbb{N}_2]$ and $\mathbb{N}_3$ is an end-extension of $\mathbb{N}_2$. We can also assume that $\mathcal{V}_2 \subseteq \mathcal{V}_3$ via $i_{2,2}$. It is also easy to check that $i_{0,2} \restriction \mathcal{V}_1 = i_{0,1}$ and $i_{0,2} \restriction \mathcal{V}_0 = i_{0,0}$. Similarly, we have $i_{1,2} \restriction \mathcal{V}_1 = i_{1,1}$. Note that Part 4 in Proposition 4.1 follows from the fact that $(\mathcal{V}_3; \mathbb{R}_2, \mathbb{R}_1)$ is the ultrapower of $(\mathcal{V}_2; \mathbb{R}_1, \mathbb{R}_0)$ modulo $\mathcal{F}'_0$. Hence, $i_{0,2}$ is an elementary embedding from $(\mathcal{V}_2; \mathbb{R}_1, \mathbb{R}_0)$ to $(\mathcal{V}_3; \mathbb{R}_2, \mathbb{R}_1)$.

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In general, we can use the same idea to iterate the ultrapower construction. Given $0 \leq m \leq n$, if we iterate the ultrapower construction $m$ times internally followed by iterating ultrapower construction $n - m$ times within $\mathcal{V}_m$ "externally" we obtain the elementary embedding $i_{m,n} : \mathcal{V}_n \to \mathcal{V}_{n+1}$. These $i_{m,n}$'s satisfy the four parts in Proposition 4.1.

The second gaol of this subsection is to present a probably the simplest proof of Ramsey's Theorem as a testing case for working within a nonstandard universe such as $\mathcal{V}_n$. In the remaining part of this subsection let $[X]^k_* := \{S \subseteq X \mid |S| = k\}$ for any set $X$ and $k \in \mathbb{N}_0$. A coloring of a set $Y$ with $r$ colors is a function $c : Y \to [r]$. A set $Z \subseteq Y$ is monochromatic (with respect to $c$) if $c \upharpoonright Z$ is a constant function.

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**Theorem (4.3, Ramsey's Theorem)**

*Let $k, r \in \mathbb{N}_0$. If $c : [\mathbb{N}_0]^k_* \to [r]$ is a coloring of $[\mathbb{N}_0]^k_*$ with at most $r$ colors, then there exists an infinite set $\mathbb{H} \subseteq \mathbb{N}_0$ such that $[\mathbb{H}]^k_*$ is monochromatic.*

*Proof*: Work within $\mathcal{V}_k$. Let $x_1 = [Id_{\mathbb{N}_0}]_{\mathcal{F}_0} \in \mathbb{N}_1 \setminus \mathbb{N}_0$ and $x_{j+1} := i_{0,k-1}(x_j)$ for $j = 1, 2, \ldots, k-1$. Then $\overline{x} = \{x_1, x_2, \ldots, x_r\} \in [\mathbb{N}_k]^k_*$. Note that $x_j$ is the equivalence class represented by the identity map $Id_{\mathbb{N}_{j-1}} : \mathbb{N}_{j-1} \to \mathbb{N}_{j-1}$.

For convenience we denote still $c$ for the extension of $c$ from $[\mathbb{N}_j]^k_*$ to $[r]$ in $\mathcal{V}_j$. Let $c(\overline{x}) = c_0$.

We construct a sequence $A = \{a_0 < a_1 < \cdots\} \subseteq \mathbb{N}_0$ inductively such that $c \restriction [A \cup \overline{x}]^k_* \equiv c_0$.

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*Let $k, r \in \mathbb{N}_0$. If $c : [\mathbb{N}_0]^k_* \to [r]$ is a coloring of $[\mathbb{N}_0]^k_*$ with at most $r$ colors, then there exists an infinite set $\mathbb{H} \subseteq \mathbb{N}_0$ such that $[\mathbb{H}]^k_*$ is monochromatic.*

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$$\exists y \in \mathbb{N}_1 \, (y > a_{m-1} \text{ and } c \restriction [A_m \cup \{y\} \cup \{i_{0,k-1}(x_1), \ldots, i_{0,k-1}(x_{k-1})\}]^k_* \equiv c_0)$$

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*Proof of Claim*: Let $\overline{b} = \{b_1 < b_2 < \cdots < b_k\} \in [A_{m+1} \cup \overline{x}]^k_*$. We show that $c(\overline{b}) = c_0$.

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*Given any positive $r, n \in \mathbb{N}_0$, one can find an $N \in \mathbb{N}_0$ such that for every coloring $c : [N]^s \to [r]$ there exists $\vec{a}, d$ such that $HC_{\vec{a},d,n} \subseteq [N]^s$ and $c \upharpoonright HC_{\vec{a},d,n} \equiv c_0$ for some $c_0 \in [r]$.*

The proof of Theorem 4.4 in this subsection is inspired by the proof of the one-dimensional version in Khinchin's book "Three Pearls in Number Theory".

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The proof of Theorem 4.4 in this subsection is inspired by the proof of the one-dimensional version in Khinchin's book "Three Pearls in Number Theory".

The multidimensional van der Waerden's Theorem is also called Gallai's Theorem. Fix a dimension $s$ and let $[n]^s = \{(x_1, x_2, \ldots, x_s) \mid x_j \in [n] \text{ for } j = 1, 2, \ldots, s\}$. A homothetic copy of $[n]^s$ is a set of the form

$$HC_{\vec{a},d,n} := \vec{a} + d[n]^s = \{\vec{a} + d\vec{x} \mid \vec{x} \in [n]^s\}$$

for some $\vec{a} \in \mathbb{N}^s$ and $d \in \mathbb{N}$, $d > 0$. The subscript $n$ in $HC_{\vec{a},d,n}$ will be omitted after it is fixed.

**Theorem (4.4, T. Gallai)**

*Given any positive $r, n \in \mathbb{N}_0$, one can find an $N \in \mathbb{N}_0$ such that for every coloring $c : [N]^s \to [r]$ there exists $\vec{a}, d$ such that $HC_{\vec{a},d,n} \subseteq [N]^s$ and $c \upharpoonright HC_{\vec{a},d,n} \equiv c_0$ for some $c_0 \in [r]$.*

The proof of Theorem 4.4 in this subsection is inspired by the proof of the one-dimensional version in Khinchin's book "Three Pearls in Number Theory".

*Proof*: Fix $n \in \mathbb{N}_0$. Let $\lhd$ be the lexicographical order of $HC_{\vec{a},d}$. For each $0 \leq l < n^s$ let $HC_{\vec{a},d}(l)$ denote the $l$-th element of $HC_{\vec{a},d}$ under $\lhd$. Note that $HC_{\vec{a},d}(0) = \vec{a}$.

Let $\varphi_m(r, N)$ be the following first-order sentence:

$$\forall c : [N]^s \to [r] \ \exists HC_{\vec{a},d} \subseteq [N]^s \ \exists c_0 \in [r] \left( c(HC_{\vec{a},d}(l)) = c_0 \ \text{ for } \ l = 0, 1, \ldots, m \right). \tag{7}$$

It suffices to prove the following claim.

**Claim 1**: Let $0 \leq m < n^s$. For every $r \in \mathbb{N}_0$ there exists an $N \in \mathbb{N}_0$ such that $\varphi_m(r, N)$ is true in $\mathcal{V}_0$.

Note that the claim when $m = n^s - 1$ is Theorem 4.4. It suffices to prove the claim by induction on $m \leq n^s - 1$. Call $HC_{\vec{a},d}$ in (7) monochromatic up to $m$ with respect to $c$.

*Proof*: Fix $n \in \mathbb{N}_0$. Let $\lhd$ be the lexicographical order of $HC_{\vec{a},d}$. For each $0 \leq l < n^s$ let $HC_{\vec{a},d}(l)$ denote the $l$-th element of $HC_{\vec{a},d}$ under $\lhd$. Note that $HC_{\vec{a},d}(0) = \vec{a}$.

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*Proof of Claim 1*: The case for $m = 0$ is trivial.

Assume that the claim is true for $m - 1$. We prove that the claim is true for $m < n^s$.

Given $r \in \mathbb{N}_0$, the task now is to find $N \in \mathbb{N}_0$ such that $\varphi_m(r, N)$ is true in $\mathcal{V}_0$.

Work within $\mathcal{V}_{r+1}$. Choose any $N_r \in \mathbb{N}_{r+1} \setminus \mathbb{N}_r$. It suffices to prove that $\varphi_m(r, 2N_r)$ is true in $\mathcal{V}_r$ by the transfer principle.

Fix $c : [2N_r]^s \to [r]$. It suffices to find a $HC_{\vec{a},d} \subseteq [2N_r]^s$ which is monochromatic up to $m$ with respect to $c$.

Choose any $N_j \in \mathbb{N}_{j+1} \setminus \mathbb{N}_j$ for $j = 0, 1, \ldots, r-1$. Since $\mathbb{N}_{j+1}$ is an end-extension of $\mathbb{N}_j$, the number $r^{(2N_{j-1})^s}$ is infinitely smaller than $N_j$. Note also that $N_j + N_{j-1} + \cdots + N_0 < N_{j+1}$.

For any $\vec{x}, \vec{y} \in [N_r]^s$ we say that $\vec{x}$ and $\vec{y}$ have the same $2N_j$-type if for any $\vec{z} \in [2N_j]^s$ we have $c(\vec{x} + \vec{z}) = c(\vec{y} + \vec{z})$, i.e., the color patterns of $\vec{x} + [2N_j]^s$ and $\vec{y} + [2N_j]^s$ with respect to $c$ are the same.

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Since the first-order sentence

$$(\forall r' \in \mathbb{N}_0)\,(\forall N \in \mathbb{N}_1 \setminus \mathbb{N}_0)\,\varphi_{m-1}(r', N)$$

is true in $(\mathcal{V}_1; \mathbb{N}_0)$, the sentence

$$(\forall r' \in \mathbb{N}_j)\,(\forall N \in \mathbb{N}_{j+1} \setminus \mathbb{N}_j)\,\varphi_{m-1}(r', N)$$

is true in $(\mathcal{V}_{j+1}; \mathbb{N}_j)$ for $j = 1, 2, \ldots, r$ by Part 4 of Proposition 4.1.

In particular, $\varphi_{m-1}(r^{(2N_{j-1})^s}, N_j)$ is true in $\mathcal{V}_{j+1}$ for $j = 1, 2, \ldots, r$.

Since the number of different $2N_{j-1}$-types is at most $r^{(2N_{j-1})^s}$, for any $\vec{b} + [N_j]^s$ we can find $HC_{\vec{a}_j, d_j} \subseteq \vec{b} + [N_j]^s$ such that $HC_{\vec{a}_j, d_j}$ is monochromatic up to $m-1$ with respect to $2N_{j-1}$-types, i.e., $HC_{\vec{a}_j, d_j}(l)$ for $l = 0, 1, \ldots, m-1$ have the same $2N_{j-1}$-type.

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So, we can now find a sequence of homothetic copies of $[n]^s$

$$HC_{\vec{a}_r,d_r}, HC_{\vec{a}_{r-1},d_{r-1}}, \ldots, HC_{\vec{a}_0,d_0}$$

such that

- $HC_{\vec{a}_r,d_r} \subseteq [N_r]^s$ is monochromatic up to $m-1$ with respect to $2N_{r-1}$-types;
- $HC_{\vec{a}_{r-1},d_{r-1}} \subseteq [N_{r-1}]^s$ such that $HC_{\vec{a}_r,d_r}(0) + HC_{\vec{a}_{r-1},d_{r-1}}$ is monochromatic up to $m-1$ with respect to $2N_{r-2}$-types. Note that $HC_{\vec{a}_r,d_r}(l) + HC_{\vec{a}_{r-1},d_{r-1}}(l')$ for $0 \leq l, l' \leq m-1$ have the same $2N_{r-2}$-type;
- $HC_{\vec{a}_{r-2},d_{r-2}} \subseteq [N_{r-2}]^s$ such that $HC_{\vec{a}_r,d_r}(0) + HC_{\vec{a}_{r-1},d_{r-1}}(0) + HC_{\vec{a}_{r-2},d_{r-2}}$ is monochromatic up to $m-1$ with respect to $2N_{r-3}$-types. Note that $HC_{\vec{a}_r,d_r}(l) + HC_{\vec{a}_{r-1},d_{r-1}}(l') + HC_{\vec{a}_{r-2},d_{r-2}}(l'')$ for $0 \leq l, l', l'' \leq m-1$ have the same $2N_{r-3}$-type;

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- $HC_{\vec{a}_{r-2},d_{r-2}} \subseteq [N_{r-2}]^s$ such that $HC_{\vec{a}_r,d_r}(0) + HC_{\vec{a}_{r-1},d_{r-1}}(0) + HC_{\vec{a}_{r-2},d_{r-2}}$ is monochromatic up to $m-1$ with respect to $2N_{r-3}$-types. Note that $HC_{\vec{a}_r,d_r}(I) + HC_{\vec{a}_{r-1},d_{r-1}}(I') + HC_{\vec{a}_{r-2},d_{r-2}}(I'')$ for $0 \le I, I', I'' \le m-1$ have the same $2N_{r-3}$-type;

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- $HC_{\vec{a}_{r-1},d_{r-1}} \subseteq [N_{r-1}]^s$ such that $HC_{\vec{a}_r,d_r}(0) + HC_{\vec{a}_{r-1},d_{r-1}}$ is monochromatic up to $m-1$ with respect to $2N_{r-2}$-types. Note that $HC_{\vec{a}_r,d_r}(l) + HC_{\vec{a}_{r-1},d_{r-1}}(l')$ for $0 \leq l, l' \leq m-1$ have the same $2N_{r-2}$-type;
- $HC_{\vec{a}_{r-2},d_{r-2}} \subseteq [N_{r-2}]^s$ such that $HC_{\vec{a}_r,d_r}(0) + HC_{\vec{a}_{r-1},d_{r-1}}(0) + HC_{\vec{a}_{r-2},d_{r-2}}$ is monochromatic up to $m-1$ with respect to $2N_{r-3}$-types. Note that $HC_{\vec{a}_r,d_r}(l) + HC_{\vec{a}_{r-1},d_{r-1}}(l') + HC_{\vec{a}_{r-2},d_{r-2}}(l'')$ for $0 \leq l, l', l'' \leq m-1$ have the same $2N_{r-3}$-type;

So, we can now find a sequence of homothetic copies of $[n]^s$

$$HC_{\vec{a}_r,d_r}, HC_{\vec{a}_{r-1},d_{r-1}}, \ldots, HC_{\vec{a}_0,d_0}$$

such that

- $HC_{\vec{a}_r,d_r} \subseteq [N_r]^s$ is monochromatic up to $m-1$ with respect to $2N_{r-1}$-types;
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- $HC_{\vec{a}_1,d_1} \subseteq [N_1]^s$ such that $\sum_{j=2}^{r} HC_{\vec{a}_j,d_j}(0) + HC_{\vec{a}_1,d_1}$ is monochromatic up to $m-1$ with respect to $2N_0$-types. Note that $\sum_{j=2}^{r} HC_{\vec{a}_j,d_j}(l_j) + HC_{\vec{a}_1,d_1}(l_1)$ for $0 \leq l_1, l_2, \ldots, l_r \leq m-1$ have the same $2N_0$-type;
- $HC_{\vec{a}_0,d_0} \subseteq [N_0]^s$ such that $\sum_{j=1}^{r} HC_{\vec{a}_j,d_j}(0) + HC_{\vec{a}_0,d_0}$ is monochromatic up to $m-1$ with respect to coloring $c$. Note that $\sum_{j=1}^{r} HC_{\vec{a}_j,d_j}(l_j) + HC_{\vec{a}_0,d_0}(l_0)$ for $0 \leq l_0, l_1, \ldots, l_r \leq m-1$ have the same $c$-value.

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Let $HC_{\vec{a},d} \oplus HC_{\vec{a}',d'} := HC_{\vec{a}+\vec{a}',d+d'}$. Clearly, for any $l < n^s$ we have

$$(HC_{\vec{a},d} \oplus HC_{\vec{a}',d'})(l) = HC_{\vec{a},d}(l) + HC_{\vec{a}',d'}(l).$$

For each $j = 0, 1, \ldots, r$ let

$$\vec{y}_j := HC_{\vec{a}_r,d_r}(0)+\cdots+HC_{\vec{a}_j,d_j}(0)+HC_{\vec{a}_{j-1},d_{j-1}}(m)+\cdots+HC_{\vec{a}_0,d_0}(m).$$

Since there are $r+1$ many $y_j$'s and $r$ colors, there must exist $0 \leq j_1 < j_2 \leq r$ such that $c(\vec{y}_{j_1}) = c(\vec{y}_{j_2})$. Let

$$\begin{aligned} D := {} & HC_{\vec{a}_r,d_r}(0) + \cdots + HC_{\vec{a}_{j_2},d_{j_2}}(0) \\ & + HC_{\vec{a}_{j_2-1},d_{j_2-1}} \oplus \cdots \oplus HC_{\vec{a}_{j_1},d_{j_1}} \\ & + HC_{\vec{a}_{j_1-1},d_{j_1-1}}(m) + \cdots + HC_{\vec{a}_0,d_0}(m). \end{aligned} \tag{8}$$

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**Claim 2**: The homothetic copy $D$ of $[n]^s$ in (8) is monochromatic up to $m-1$ with respect to $c$.

Claim 1 follows from Claim 2 because $D(0)=\vec{y}_{j_1}$ and $D(m)=\vec{y}_{j_2}$ have the same $c$–value and hence, the homothetic copy $D$ of $[n]^s$ is monochromatic up to $m$ with respect to $c$.

*Proof of Claim 2*: By the construction of $HC_{\vec{a}_j,d_j}$ we have that

$$\sum_{j=j_2}^{r} HC_{\vec{a}_j,d_j}(0)+\sum_{j=j_1}^{j_2-1} HC_{\vec{a}_j,d_j}(l)$$

for $0\leq l\leq m-1$ have the same $2N_{j_1-1}$-type. Note that

$$\vec{b}:=\sum_{j=0}^{j_1-1} HC_{\vec{a}_j,d_j}(m)\in[2N_{j_1-1}]^s.$$

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Claim 1 follows from Claim 2 because $D(0) = \vec{y}_{j_1}$ and $D(m) = \vec{y}_{j_2}$ have the same $c$–value and hence, the homothetic copy $D$ of $[n]^s$ is monochromatic up to $m$ with respect to $c$.

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Hence,

$$D(l) := HC_{\vec{a}_r,d_r}(0) + \cdots + HC_{\vec{a}_{j_2},d_{j_2}}(0)$$
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Szemerédi's Theorem is the center of attention in additive combinatorics for many years which has attracted many prominent mathematicians.

**Theorem (4.5, E. Szemerédi, 1975)**

*If $D \subseteq \mathbb{N}$ has a positive upper density, then $D$ contains a $k$–term arithmetic progression for every $k \in \mathbb{N}$.*

Szemerédi's Theorem confirms a conjecture of P. Erdős and P. Turán made in 1936, which implies van der Waerden's Theorem.

Nonstandard versions of Furstenberg's ergodic proof and Gowers's harmonic proof of Szemerédi's Theorem have been tried by T. Tao. In August 2017, Tao gave a series of lectures to explain Szemerédi's original combinatorial proof and hope to simplify it so that the proof can be better understood. He believed that Szemerédi's combinatorial method should have a greater impact on combinatorics.

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During these lectures Tao challenged the audience to produce a nonstandard proof of Szemerédi's Theorem which is noticeably simpler and more transparent than Szemerédi's original proof. However, in his later blog post, Tao commented that "in fact there are now signs that perhaps nonstandard analysis is not the optimal framework in which to place this argument." We disagree. To meet Tao's challenge we showed that with the help of a nonstandard universe with three levels of infinities, Szemerédi's original argument can be made simpler and more transparent.

The main simplification in the following proof comparing to the standard proof of Szemerédi–Tao is that a Tower of Hanoi type induction is replaced by a straightforward induction, which makes Szemerédi's idea more transparent.

To achieve this, $\mathcal{V}_3$ (see Proposition 4.1) is used which supply three levels of infinities, plus various elementary embeddings from $\mathcal{V}_j$ to $\mathcal{V}_{j'}$ for some $0 \leq j < j' \leq 3$.

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In this subsection we will do the following:

1. Assume a weak regularity lemma and derive a nonstandard form of mixing lemma;
2. Prove Theorem 4.5 for $k = 3$;
3. Prove Theorem 4.5 for $k = 4$,
4. Prove Theorem 4.5 for any $k$.

The reason to present the proof for $k = 3$ and $k = 4$ is to show how the level of difficulties arises.

Let's fix some notation. The Greek letters $\alpha, \beta, \gamma, \epsilon$, etc. will represent standard real numbers unless otherwise specified. All unspecified sets mentioned are either standard or $\mathcal{V}_j$-internal for $j = 1, 2,$ or 3. If $m, n \in \mathbb{N}_3$, we write $m \ll n$ if $m \in \mathbb{N}_j$ and $n \in \mathbb{N}_{j'} \setminus \mathbb{N}_{j'-1}$ for some $0 \leq j < j' \leq 3$. For example, $1 \ll n$ means that $n$ is hyperfinite.

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Let's fix some notation. The Greek letters $\alpha, \beta, \gamma, \epsilon$, etc. will represent standard real numbers unless otherwise specified. All unspecified sets mentioned are either standard or $\mathcal{V}_j$-internal for $j = 1, 2,$ or 3. If $m, n \in \mathbb{N}_3$, we write $m \ll n$ if $m \in \mathbb{N}_j$ and $n \in \mathbb{N}_{j'} \setminus \mathbb{N}_{j'-1}$ for some $0 \leq j < j' \leq 3$. For example, $1 \ll n$ means that $n$ is hyperfinite.

In this subsection we will do the following:

1. Assume a weak regularity lemma and derive a nonstandard form of mixing lemma;
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The words "arithmetic progression" will be abbreviated to "a.p." The length of an a.p. $p$, denoted by $|p|$, is the number of the terms in $p$. A finite a.p., often with length $k$, will be denoted by $p, q, r$, etc. and an a.p. of hyperfinite length will be denoted by $P, Q, R$, etc. If $P$ (or $p$) is an a.p., the $l$-th term of $P$ is denoted by $P(l)$ for any $1 \leq l \leq |P|$. By $k$-term a.p. or just $k$-a.p. we mean an a.p. with length $k$. If both $p$ and $q$ are $k$-a.p., let $r := p \oplus q$ be the $k$-a.p. such that $r(l) = p(l) + q(l)$ for $1 \leq l \leq k$.

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## Lemma (4.6, Weak Regularity Lemma)

*Let $U, W$ be finite sets, let $\epsilon > 0$, and for each $w \in W$, let $E_w$ be a subset of $U$. Then there exists a partition $U = U_1 \cup U_2 \cup \cdots \cup U_{n_\epsilon}$ for some $n_\epsilon \in \mathbb{N}_0$, and real numbers $0 \leq c_{u,w} \leq 1$ in $\mathbb{R}_0$ for $u \in [n_\epsilon]$ and $w \in W$ such that for any set $F \subseteq U$, one has*

$$\left| |F \cap E_w| - \sum_{u=1}^{n_\epsilon} c_{u,w} |F \cap U_u| \right| \leq \epsilon |U|$$

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## Definition (4.7)

*Let $0 \leq j < j' \leq 3$. For any two numbers $r, r' \in \mathbb{R}_{j'}$ we write $r \approx_j r'$ if $|r - r'| < 1/n$ for every $n \in \mathbb{N}_j$. If $r \in ns_j(\mathbb{R}_{j'})$ where $ns_j(\mathbb{R}_{j'})$ is the set of all near standard elements when considering $\mathcal{V}_j$ as the "standard" universe, denote $st_j(r)$ for the unique number $\alpha \in \mathbb{R}_j$ such that $r \approx_j \alpha$. For any bounded set $A \subseteq \mathbb{N}_{j'}$ and $n \in \mathbb{N}_{j'}$ denote*

$$\delta_n(A) := \frac{|A|}{n} \in \mathbb{R}_{j'} \text{ and } \mu_n^j(A) := st_j(\delta_n(A)).$$

Notice that $\delta_n$ is a $\mathcal{V}_{j'}$-internal function while $\mu_n^j$ are often external functions but definable in $(\mathcal{V}_{j'}; \mathbb{R}_j)$, i.e.,

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If $A \subseteq \Omega$ and $|\Omega| = H$, then $\mu_H(A)$ coincides with the Loeb measure of $A$ in $\Omega$.

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*Let $0 \leq j < j' \leq 3$ and $A \subseteq \mathbb{N}_{j'}$ with $|A| \in \mathbb{N}_{j'} \setminus \mathbb{N}_j$ the strong upper Banach density $SD^j(A)$ of $A$ in $\mathcal{V}_j$ is defined by*

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The uniformity of $A \in [N]$ when $\mu_N(A) = SD(A)$ will be useful.

**Lemma (4.10)**

*Let $0 \leq j < j' \leq 3$. Given $N, H \in \mathbb{N}_{j'} \setminus \mathbb{N}_j$, $H \leq N/2$, and $C \subseteq [N]$ with $\mu^j_N(C) = SD^j(C) = \alpha \in \mathbb{R}_j$, for each $n \in \mathbb{N}_{j'}$ let*

$$D_{n,H,C} := \left\{ x \in [N-H] \mid |\delta_H(C \cap (x + [H])) - \alpha| < \frac{1}{n} \right\}. \quad (11)$$

*Then there exists a $J \in \mathbb{N}_{j'} \setminus \mathbb{N}_j$ such that $\mu^j_{N-H}(D_{J,H,C}) = 1$.*

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Suppose $0 \leq j < j' \leq 3$, $N \geq H \gg 1$ in $\mathbb{N}_{j'}$, $U \subseteq [N]$, $A \subseteq S \subseteq [N]$, $0 \leq \alpha \leq \eta \leq 1$, and $x \in [N]$. For each $n \in \mathbb{N}_j$ let $\xi(x, \alpha, \eta, A, S, U, H, n)$ be the following internal statement:

$$\begin{aligned} |\delta_H(x+[H]) \cap U) - 1| &< 1/n, \\ |\delta_H((x+[H]) \cap S) - \eta| &< 1/n, \text{ and} \\ |\delta_H((x+[H]) \cap A) - \alpha| &< 1/n. \end{aligned} \tag{12}$$

The statement $\xi(x, \alpha, \eta, A, S, U, H, n)$ infers that the densities of $A, S, U$ in the interval $x+[H]$ go to $\alpha, \eta, 1$, respectively, as $n \to \infty$ in $\mathbb{N}_j$. The statement $\xi$ will be referred a few times later.

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The following lemma is the application of Lemma 4.10 to the sets $U, S, A$ simultaneously.

**Lemma (4.11)**

*Let $0 \leq j < j' \leq 3$. Let $N \in \mathbb{N}_{j'} \setminus \mathbb{N}_j$, $U \subseteq [N]$, and $A \subseteq S \subseteq [N]$ be such that $\mu^j_N(U) = 1$, $\mu^j_N(S) = SD(S) = \eta$, and $\mu^j_N(A) = SD^j_S(A) = \alpha$ for some $\eta, \alpha \in \mathbb{R}_j$. For any $n, h \in \mathbb{N}_{j'}$ let*

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(a) *For each $H \in \mathbb{N}_{j'} \setminus \mathbb{N}_j$ with $H \leq N/2$ there exists a $J \in \mathbb{N}_{j'} \setminus \mathbb{N}_j$ such that $\mu^j_{N-H}(G_{J,H}) = 1$;*

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We often write $st$ for $st_0$, $\mu_n$ for $\mu_n^0$, and $SD$ for $SD^0$. One can derive a so-called mixing lemma from Weak Regularity Lemma.

**Lemma (4.12, Mixing Lemma)**

*Let $N \in \mathbb{N}_{j'} \setminus \mathbb{N}_0$, $A \subseteq S \subseteq [N]$, $1 \ll H \leq N/2$, and $R \subseteq [N-H]$ be an a.p. with $|R| \gg 1$ such that*

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# Szemeredi's Theorem for $k = 3$

**Theorem (4.13, K. F. Roth, 1953)**

*If $U \subseteq \mathbb{N}$ and $SD(U) > 0$, then $U$ contains nontrivial 3-term arithmetic progressions.*

*Proof:* We work within $\mathcal{V}_1$. The elementary embedding $i_{0,0}$ is represented by $^*$ for notational convenience.

Let $\alpha = SD(U)$. Then $\alpha > 0$. Let $P \subseteq \mathbb{N}_1$ be an a.p. with $|P| \gg 1$ and $\mu_{|P|}(^*U \cap P) = \alpha$. Without loss of generality we can assume that $P = [N] \cup \{0\}$. Let $A := {^*U} \cap [N]$. It suffices to find a 3-a.p. in $A$.

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For each $t \in [H]$ let

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Notice that $\mu_H(E_t) = \alpha/2 > 0$ because $p(1) - t$ must be even and the density of $A$ in an a.p. of difference 2 and length $\geq \lfloor N/16 \rfloor$ is also $\alpha$. By (iii) of Mixing Lemma, there is an $I \in [H]$ and $T \subseteq [H]$ with $\mu_H(T) = 1$ such that

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Note that there are at least $\alpha^2 H/2$ many 3-a.p. $q$'s in $\mathcal{Q}_{t_0}$ with $p_0 \oplus q \subseteq A$. □

Let $t_0 \in T$ be such that $2H + 2l + t_0 \in A \cap (2H + 2l + T)$.

Let $p_0 = \{0, H + l, 2H + 2l\}$ and $q_0 \in \mathcal{Q}_{t_0}$ with

$$H + l + q_0(2) \in A \cap (H + l + E_{t_0}).$$

Then $p_0 \oplus q_0$ is an 3–a.p. Clearly,

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# Szemeredi's Theorem for $k = 4$

We again work in $\mathcal{V}_1$. If one wants to count the number of 4-a.p.'s such that all but the third (or second) term of the a.p. are in a set $A$, then the same idea of the proof of Roth's Theorem can be used to prove the following lemma.

**Lemma (4.14)**

*Let $N \gg 1$, $A \subseteq [N]$ be such that $\mu_N(A) = SD(A) = \alpha > 0$, and $H = \lfloor N/8 \rfloor$. There exists an interval $x_0 + [H] \subseteq [N]$, a set $T \subseteq x_0 + [H]$ with $\mu_H(T) = 1$, and*

$$\mathcal{P}_t := \{p \subseteq [N] \mid p \text{ is a 4–a.p.}, p(1), p(2) \in A, \text{ and } p(4) = t\}$$

*such that $\mu_H(\mathcal{P}_t) = \alpha^2/3$ for each $t \in T$.*

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The reason why the number of 4-a.p.'s in $A$ is $\geq \alpha^2 H/3$ instead of $\alpha^2 H/2$ as in Theorem 3.10 is that for a 4-a.p. $p$ with $p(4) = t$ fixed, $p(4) - p(1)$ should be a multiple of 3 in order to guarantee that $p(2)$ and $p(3)$ are integers.

**Lemma (4.15)**

*Let $N \gg 1$, $B, S_\gamma \subseteq [N]$ be such that $B \subseteq S_\gamma$,*

$$\mu_N(S_\gamma) = SD(S_\gamma) = \gamma > 11/12,$$

$$\text{and } \mu_N(B) = SD_{S_\gamma}(B) = \beta > 0.$$

*There exists an interval $x_0 + [\lfloor N/24 \rfloor] \subseteq [N]$ and a set $T \subseteq x_0 + [\lfloor N/24 \rfloor]$ with $\mu_{N/24}(T) \geq 1 - 12(1-\gamma)$, and a collection of 4-a.p.'s $\{p_t \mid t \in T\}$ such that $p_t(1), p_t(2) \in B$, $p_t(3), p_t(4) \in S_\gamma$, and $p_t(3) = t$ for each $t \in T$.*

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*Proof* Let $H := [N/8]$. Notice that $\mu_H(S_\gamma \cap (x + [H])) = \gamma$ and $\mu_H(B \cap (x + [H])) = \beta$ for every $x \in [N - H]$. Let $\mathcal{Q}$ be the collection of all 4–a.p.'s in $[H]$. For each $w \in [\lfloor H/3 \rfloor, \lfloor 2H/3 \rfloor]$ let

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By (iii) of Mixing Lemma, there is an $I \in [H]$, $W_3 \subseteq [\lfloor H/3 \rfloor, \lfloor 2H/3 \rfloor]$ with $\mu_H(W_3) = 1/3$, and $W^i \subseteq [H]$ with $\mu_H(W^i) = 1$ such that

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for all $w \in W_3$ and $w' \in W^i$ for $i = 3$ or $4$. Clearly, $\mu_H(((i-1)H + (i-1)I + W^i) \cap S_\gamma) = \gamma$ for $i = 3$ or $4$.

Let $T^3 := 2H + 2I + W_3$. For each $t = 2H + 2I + w \in T^3$ let

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Notice that $\mu_H(\mathcal{P}_t) = \mu_H(B \cap (2H + 2I + E^3_w)) = \beta^2/2$ for each $t = 2H + 2I + w \in T^3$.

A 4–a.p. $p \in \mathcal{P}$ is called good if $p(i) \in S_\gamma \cap ((i-1)H + (i-1)I + [H])$ for $i = 3, 4$. Let $\mathcal{P}_g$ be the collection of all good 4–a.p.'s in $\mathcal{P}$. A 4–a.p. $p \in \mathcal{P}$ is bad if it is not good. Let $\mathcal{P}_b := \mathcal{P} \setminus \mathcal{P}_g$. Let $T^3_g := \{p(3) \mid p \in \mathcal{P}_g\}$. Then $T^3_g \subseteq S_\gamma$. We show that $\mu_H(T^3_g) \geq \frac{1}{3} - 4(1-\gamma)$.

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A 4–a.p. $p \in \mathcal{P}$ is called good if $p(i) \in S_\gamma \cap ((i-1)H + (i-1)l + [H])$ for $i = 3, 4$. Let $\mathcal{P}_g$ be the collection of all good 4–a.p.'s in $\mathcal{P}$. A 4–a.p. $p \in \mathcal{P}$ is bad if it is not good. Let $\mathcal{P}_b := \mathcal{P} \smallsetminus \mathcal{P}_g$. Let $T^3_g := \{p(3) \mid p \in \mathcal{P}_g\}$. Then $T^3_g \subseteq S_\gamma$. We show that $\mu_H(T^3_g) \geq \dfrac{1}{3} - 4(1-\gamma)$.

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So $$|\mathcal{P}_g| = |\mathcal{P}| - |\mathcal{P}_b| \geq \sum_{t \in T^3} |\mathcal{P}_t|$$

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Hence we have

$$\mu_H(T^3_g) \cdot \frac{\beta^2}{2} = st \left( \frac{1}{H} \sum_{t \in T^3_g} \frac{1}{H} |\mathcal{P}_t| \right) \geq st \left( \frac{1}{H^2} |\mathcal{P}_g| \right)$$

$$= st \left( \frac{1}{H^2} (|\mathcal{P}| - |\mathcal{P}_b|) \right) \geq st \left( \frac{1}{H^2} \sum_{t \in T^3} |\mathcal{P}_t| \right) -$$

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which implies $\mu_H(T^3_g) \geq \frac{1}{3} - 4(1-\gamma)$. Hence $\mu_{N/24}(T^3_g) \geq 1 - 12(1-\gamma)$ because $H = \lfloor N/8 \rfloor$. Now the lemma is proven if we set $x_0 := 2H + 2I + \lfloor H/3 \rfloor$, $T := T^3_g$, and choose one $p_t \in \mathcal{P}_g$ such that $P_t(3) = t$ for each $t \in T$. □

**Remark (4.16)**

*The argument for showing $\mu_{N/24}(T^3_g) > 1 - 12(1-\gamma)$ is already in the papers of Szemerédi and Tao.*

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**Theorem (4.17, E. Szemerédi, 1969)**

*If $U \subseteq \mathbb{N}_0$ and $SD(U) > 0$, then $U$ contains nontrivial 4-term arithmetic progressions.*

*Proof* Let $N \gg 1$ and $A \subseteq [N]$ be such that $\mu_N(A) = SD(A) = \alpha > 0$. Same as in the beginning of the proof of Roth's Theorem, it suffices to find a 4–a.p. in $A$. For each $n, j \in \mathbb{N}_0$ let

$$S_{j,n} := \{x \in [N-n] \mid \mu_n((x+[n]) \cap A) \geq \alpha - 1/j\}.$$

Then $\lim_{n\to\infty} \mu_{N-n}(S_{j,n}) = 1$ by Lemma 4.11. So, for all sufficiently large $n \in \mathbb{N}_0$ we have that $\gamma_{j,n} := SD(S_{j,n}) > 11/12$. Let $R_{j,n}$ be an a.p. in $[N]$ with difference $d$ and $|R_{j,n}| \gg 1$ such that $\mu_{|R_{j,n}|}(R_{j,n} \cap S_{j,n}) = \gamma_{j,n}$. For each $\tau \subseteq [n]$ let

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Let $\varphi : P_{j,n} \to [N']$ be the affine map $\varphi(x) = (x - \min P_{j,n})/d' + 1$. Applying Lemma 4.15 to $[N']$ for $S' = \varphi((S_{j,n}) \cap P_{j,n})$, and $B' = \varphi(B_{j,n} \cap P_{j,n})$, and then pulling back through $\varphi^{-1}$, we obtain $x_0 + d'\,[\lfloor |P_{j,n}|/24 \rfloor] \subseteq P_{j,n}$ and $T_{j,n} \subseteq x_0 + d'\,[\lfloor |P_{j,n}|/24 \rfloor]$ with $\mu_{N'/24}(T_{j,n}) \geq 1 - 12(1 - \gamma_{j,n})$, and there exists a collection of 4–a.p.'s $\mathcal{P}_{j,n} = \{p_t \mid t \in T_{j,n}\}$ such that $p_t(1), p_t(2) \in B_{j,n} \cap P_{j,n}$, $p_t(3), p_t(4) \in S_{j,n} \cap P_{j,n}$, and $p_t(3) = t$ for each $t \in T_{j,n}$.

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By countable saturation we can find fixed hyperfinite integer $H$ and then $J$ such that $\gamma := \gamma_{J,H} \approx 1$, $P := P_{J,H}$ with $|P| \gg 1$, $S := S_\gamma$, $B := B_{J,H} \subseteq S$, $T := T_{J,H}$, and $\mathcal{P}_{J,H} = \{p_t \mid t \in T\}$ such that $p_t(1), P_t(2) \in B$, $p_t(3), p_t(4) \in S$, and $p_t(3) = t$ for each $t \in T$.

Notice that $\mu_{N-H}(S) = 1$, $T \subseteq x_0 + d'\,[\lfloor |P|/24 \rfloor]$, $\mu_{|P|/24}(T) = 1$, $\gamma \approx 1$, $x, y \in B$ implies $((x + [H]) \cap A) - x = ((y + [H]) \cap A) - y$, and $x \in S$ implies $\mu_H((x + [H]) \cap A) = \alpha$. It may be the case that $\mu_{|P|}(B) = 0$. But the existence of the collection $\mathcal{P}_{J,H} = \{p_x \mid x \in T\}$ is guaranteed by countable saturation.

Since $\mu_{N/24}(T) = 1$, we can find an a.p. of $P' \subseteq T$ of difference $d'$ with $|P'| \gg 1$. Let $\mathcal{P}' := \{p_t \in \mathcal{P}_{J,H} \mid t \in P'\}$. Notice that for each $p_t \in \mathcal{P}'$ we have that $p_t(1), p_t(2) \in B$, $p_t(3) = t \in S$, and $p_t(4) \in S$.

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Let $\tau_0 := ((x + [H]) \cap A) - x$ for some $x \in B$. Then $\mu_H(\tau_0) = \alpha$ because $B \subseteq S$. By Lemma 4.14 with $N$ being replaced by $H$, $A$ being replaced by $\tau$, we can find $x_0 + [\lfloor H/8 \rfloor] \subseteq [H]$, $T_Q \subseteq x_0 + [\lfloor H/8 \rfloor]$ with $\mu_H(T_Q) = 1/8$,

$$\mathcal{Q}_w := \{q \subseteq [H] \mid q(1), q(2) \in \tau_0, \text{ and } p(4) = w\},$$

and $E_w = \{q(3) \mid q \in \mathcal{Q}_w\}$ such that $\mu_H(E_w) = \alpha^2/24$ for each $w \in T_Q$.

By (iii) of Mixing Lemma there is an $x' \in P'$ and $T'_Q \subseteq T_Q$ with $\mu_H(T'_Q) = 1/8$ such that $\mu_H((x' + E_w) \cap A) = \alpha\mu_H(E_w) = \alpha^3/24$ for each $w \in T'_Q$.

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Fix $p_{x'} \in \mathcal{P}'$.

Since $p_{x'}(4) \in S$, we have that $\mu_H((p_{x'}(4) + T'_Q) \cap A) = \alpha/8$.

Hence there is a $w \in T'_Q$ such that $p_{x'}(4) + w \in A$.

Let $q_w \in \mathcal{Q}_w$. Then $p_{x'}(4) + q_w(4) = p_{x'}(4) + w \in A$.

Notice that $p_{x'}(3) + q_w(3) \in (x + E_w) \cap A \subseteq A$. Notice also that $p_{x'}(1), p_{x'}(2) \in B$ imply $A \cap (p_{x'}(i) + [\lfloor H/8 \rfloor]) = p_{x'}(i) + \tau_0$ for $i = 1, 2$.

Hence $p_{x'}(i) + q_w(i) \in p_{x'}(i) + \tau_0 \subseteq A$ for $i = 1, 2$. Therefore, $p_{x'} \oplus q_w$ is a nontrivial 4–a.p. in $A$. □

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# Szemerédi's Theorem for all $k \geq 5$

Szemerédi's Theorem is an easy consequence of Lemma 4.18, denoted by $\mathbf{L}(m)$ for all $m \in [k]$.

For an integer $n \geq 2k+1$ define an interval $C_n \subseteq [n]$ by

$$C_n := \left[\left\lceil \frac{kn}{2k+1} \right\rceil, \left\lfloor \frac{(k+1)n}{2k+1} \right\rfloor\right]. \tag{16}$$

The set $C_n$ is the subinterval of $[n]$ in the middle of $[n]$ with the length $\lfloor n/(2k+1) \rfloor \pm \iota$ for $\iota = 0$ or 1. If $n \gg 1$, then $\mu_n(C_n) = 1/(2k+1)$. For notational convenience we denote

$$D := 3k^3 \text{ and } \eta_0 := 1 - \frac{1}{D}. \tag{17}$$

♦: Fix a $K \in \mathbb{N}_1 \setminus \mathbb{N}_0$. The number $K$ is the length of an interval which will play an important role in Lemma 3.15. Keeping $K$ unchanged is one of the advantages from nonstandard analysis, which is unavailable in the standard setting.

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If $p$ is a $k$-a.p. and $A$ is a set, we denote $p \oplus A$ for the sequence $\{p(l) + A \mid 1 \leq l \leq k\}$. If $p, q$ are $k$-a.p.'s and $A$ be a set, we denote $p \sqsubseteq q \oplus A$ for the statement that $p(l) \in q(l) + A$ for $1 \leq l \leq k$.

**Lemma (4.18, L($m$))**

*Given any $\alpha > 0$, $\eta > \eta_0$, any $N \in \mathbb{N}_2 \setminus \mathbb{N}_1$, and any $A \subseteq S \subseteq [N]$ and $U \subseteq [N]$ with*

$$\mu_N(U) = 1, \mu_N(S) = SD(S) = \eta, \quad \text{and } \mu_N(A) = SD_S(A) = \alpha, \tag{18}$$

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## Lemma (4.18)

**L**$_1(m)(\alpha, \eta, N, A, S, U, K)$: *There exists a k–a.p.* $\vec{x} \subseteq U$ *with* $\vec{x} \oplus [K] \subseteq [N]$ *satisfying the statement* $(\forall n \in \mathbb{N}_0)\, \xi(\vec{x}(l), \alpha, \eta, A, S, U, K, n)$ *for* $l \in 1 + [k]$, *and there exist* $T_l \subseteq C_K$ *with* $\mu_{|C_K|}(T_l) = 1$ *where* $C_K$ *is defined in (16) and* $V_l \subseteq [K]$ *with* $\mu_K(V_l) = 1$ *for every* $l \geq m$, *and collections of k–a.p.'s*

$$\mathcal{P} := \bigcup\{\mathcal{P}_{l,t} \mid t \in T_l \text{ and } l \geq m\} \text{ and}$$
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## Lemma (4.18)

$$\mathcal{Q}_{l,v} = \{q \sqsubseteq \vec{x} \oplus [K] \mid \forall l' < m\,(q(l') \in A) \text{ and } q(l) = \vec{x}(l) + v\}$$

*satisfying* $\mu_K(\mathcal{Q}_{l,v}) \leq \alpha^{m-1}$ *for all* $l \geq m$ *and* $v \in V_l$.

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## Remark (4.19)

(a) $\mathbf{L}_2(m)$ *is an internal statement in* $\mathcal{V}_2$. *Both* $\mathbf{L}_1(m)$ *and* $\mathbf{L}_2(m)$ *depend on* $K$. *Since* $K$ *is fixed throughout whole proof, it, as a parameter, may be omitted in some expressions.*

(b) *If* $H \gg 1$ *and* $T \subseteq [H]$ *with* $\mu_H(T) > 1 - \epsilon$, *then* $T$ *contains* $\lfloor 1/\epsilon \rfloor$ *consecutive integers because otherwise we have* $\mu_H(T) \leq (\lfloor 1/\epsilon \rfloor - 1)/\lfloor 1/\epsilon \rfloor$ $= 1 - 1/\lfloor 1/\epsilon \rfloor \leq 1 - 1/(1/\epsilon) = 1 - \epsilon$.

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The following lemma is a generalization of Lemma 4.15.

**Lemma (4.20)**

$\mathbf{L}_1(m)(\alpha, \eta, N, A, S, U)$ *implies* $\mathbf{L}_2(m)(\alpha, \eta, N, A, S)$ *for any* $\alpha, \eta, N, A, S, U$ *satisfying the conditions of Lemma 4.18.*

*Sketch Proof of Lemma 4.18* We prove $\mathbf{L}(m)$ by induction on $m$. By Lemmas 4.20 it suffices to prove $\mathbf{L}_1(m)$.

For $\mathbf{L}(1)$, given any $\alpha > 0$, $\eta > \eta_0$, $N \in \mathbb{N}_2 \setminus \mathbb{N}_1$, $A$, $S$, and $U$ satisfying the conditions of the lemma, by Lemma 4.11 (b) we can find a $k$–a.p. $\vec{x} \subseteq [N]$ such that

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is true for $l \in 1 + [k]$.

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Clearly, we have $\mu_K(\mathcal{P}_{l,t}) \geq 1/(k-1) > 1/k$. By some pruning we can assume that $\mu_K(\mathcal{P}_{l,t}) = 1/k$. It is trivial that $\mu_K(\mathcal{Q}_{l,v}) \leq 1$ and $q \in \mathcal{Q}_{l,v}$ iff $q(l) = \vec{x}(l) + v$ for each $q \sqsubseteq \vec{x} \oplus [K]$. This completes the proof of $\mathbf{L}_1(1)(\alpha, \eta, N, A, S, U)$. $\mathbf{L}_2(1)(\alpha, \eta, N, A, S)$ follows from Lemma 4.20.

Assume $\mathbf{L}(m-1)$ is true for some $2 \leq m \leq k$.

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We now prove $\mathbf{L}(m)$. Given any $\alpha > 0$ and $\eta > \eta_0$, fix $N \in \mathbb{N}_2 \setminus \mathbb{N}_1$, $U \subseteq [N]$, and $A \subseteq S \subseteq [N]$ satisfying the conditions of the lemma.

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The proof of Claim 1 is done in $\mathcal{V}_3$.

**Claim 2** There exists a $J \in \mathbb{N}_2 \setminus \mathbb{N}_1$ such that the $\theta(J, A, N)$ is true, i.e., $\exists W \subseteq [N] \exists \mathcal{R} \,(W$ is an a.p. $\wedge\, |W| \geq \min\{K, \lfloor 1/2D(1-d_J)\rfloor\} \wedge \mathcal{R} = \{r_w \mid w \in W\}$ is a collection of $k$–a.p.'s such that

$$\begin{aligned} \forall w \in W \,((\forall l \geq m)\,(r_w(l) \in G_{J,h_J}) \wedge r_w(m-1) = w \\ \wedge (\forall l, l' \leq m-2)\,((A \cap (r_w(l) + [h_J])) - r_w(l) \\ = (A \cap (r_w(l') + [h_J])) - r_w(l')))). \end{aligned}$$

**Claim 3** For each $s \in \mathbb{N}_0$ we can find an internal $U_s \subseteq [H]$ with $\mu_H(U_s) = 1$ such that for each $y \in U_s$ and each $l \in 1 + [k]$, $r_{w_s}(l) + y \in U$ and $(\forall n \in \mathbb{N}_0)\,\xi(r_{w_s}(l) + y, \alpha, \eta, A, S, U, K, n)$ is true.

The proof of Claim 1 is done in $\mathcal{V}_3$.

**Claim 2** There exists a $J \in \mathbb{N}_2 \setminus \mathbb{N}_1$ such that the $\theta(J, A, N)$ is true, i.e., $\exists W \subseteq [N] \exists \mathcal{R} (W$ is an a.p. $\wedge |W| \geq \min\{K, \lfloor 1/2D(1-d_J) \rfloor\} \wedge \mathcal{R} = \{r_w \mid w \in W\}$ is a collection of $k$–a.p.'s such that

$$\forall w \in W \, ((\forall l \geq m) \, (r_w(l) \in G_{J,h_J}) \wedge r_w(m-1) = w$$
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Notice that $\delta_H(\bigcap_{i=1}^{s} U_i) > 1 - 1/s$. By Overspill Principle we can find $1 \ll l \leq |W_H|$ and

$$U' := \bigcap \{U_s \mid 1 \leq s \leq l\}$$

such that $\delta_H(U') > 1 - 1/l$. Hence $\mu_H(U') = 1$. Applying the induction hypothesis for $\mathbf{L}_1(m-1)(\alpha, 1, H, \tau_H, [H], U')$, we obtain a $k$–a.p. $\vec{y} \subseteq U'$ with $\vec{y} \oplus [K] \subseteq [H]$, $T'_l \subseteq C_K \cap U'$ with $\mu_{|C_K|}(T'_l) = 1$ and $V'_l \subseteq [K]$ with $\mu_K(V'_l) = 1$ for each $l \geq m-1$, and collections of $k$–a.p.'s

$$\mathcal{P}' = \bigcup \{\mathcal{P}'_{l,t} \mid t \in T'_l \text{ and } l \geq m-1\} \text{ and}$$
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such that

(i) for each $l \geq m-1$ and $t \in T'_l$ we have $\mu_K(\mathcal{P}'_{l,t}) = \alpha^{m-2}/k$ and for each $p \in \mathcal{P}'_{l,t}$ we have $p \sqsubseteq (\vec{y} \oplus [K]) \cap U'$, $p(l') \in \tau_H$ for $l' < m-1$, $p(l) = \vec{y}(l) + t$, and

(ii) for each $l \geq m-1$ and $v \in V'_l$ we have $\mu_K(\mathcal{Q}'_{l,v}) \leq \alpha^{m-2}$, and for each $q \sqsubseteq \vec{y} \oplus [K]$ we have $q \in \mathcal{Q}'_{l,v}$ iff $q(l') \in \tau_H$ for every $l' < m-1$ and $q(l) = \vec{y}(l) + v$.

For each $l \geq m$, $t \in T_l$, and $v \in V_l$ let

$$E_{l,t} := \{p(m-1) \mid p \in \mathcal{P}'_{l,t}\} \text{ and } F_{l,v} := \{q(m-1) \mid q \in \mathcal{Q}'_{l,v}\}.$$

Then $E_{l,t}, F_{l,v} \subseteq \vec{y}(m-1) + [K]$,
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Applying Part (iii) of Mixing Lemma with $R := \{w_s + \vec{y}(m-1) \mid 1 \leq s \leq l\}$ and $H$ being replaced by $K$ we can find $s_0 \in [l]$, $T_l \subseteq T'_l$ with $\mu_{|C_K|}(T_l) = 1$ and $V_l \subseteq V'_l$ with $\mu_K(V_l) = 1$ for each $l \geq m$ such that for each $t \in T_l$ and $v \in V_l$ we have

$$\begin{aligned} &\mu_K((w_{s_0} + E_{l,t}) \cap ((w_{s_0} + \vec{y}(m-1) + [K]) \cap A)) \\ &= \alpha\mu_K(E_{l,t}) = \alpha(\alpha^{m-2}/k) = \alpha^{m-1}/k \text{ and} \end{aligned} \tag{19}$$

$$\begin{aligned} &\mu_K((w_{s_0} + F_{l,v}) \cap ((w_{s_0} + \vec{y}(m-1) + [K]) \cap A)) \\ &= \alpha\mu_K(F_{l,t}) \leq \alpha \cdot \alpha^{m-2} = \alpha^{m-1}. \end{aligned} \tag{20}$$

Let $\vec{x} := r_{w_{s_0}} \oplus \vec{y}$. Clearly, we have $\vec{x} \oplus [K] \subseteq [N]$. We also have that $\vec{x} \subseteq U$, $\mu_K((\vec{x}(l) + [K]) \cap S) = \eta$, and $\mu_K((\vec{x}(l) + [K]) \cap A) = \alpha$ because $r_{w_{s_0}} \subseteq S_H$ and $\vec{y} \subseteq U' \subseteq U_{s_0}$.

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Then $\mu_K(\mathcal{P}_{l,t}) = \alpha^{m-1}/k$ by (19). If $\bar{q} \sqsubseteq \vec{x} \oplus [K]$, then there is a $q \sqsubseteq \vec{y} \oplus [K]$ such that $\bar{q} = r_{w_{s_0}} \oplus q$. If $\bar{q}(l') \in A$ for $l' < m$ and $v \in V_l$ for some $l \geq m$ such that $\bar{q}(l) = \vec{x}(l) + v$, then $q(l') \in \tau_H$ for $l' < m-1$, $v \in V'_l$, and $q(l) = \vec{y}(l) + v$, which imply $q \in \mathcal{Q}'_{l,v}$ by induction hypothesis. Hence we have $q(m-1) \in F_{l,v}$. Clearly, $\bar{q}(m-1) = w_{s_0} + q(m-1) \in A$ implies $q(m-1) \in F_{l,v} \cap (((w_{s_0} + \vec{y}(m-1) + [K]) \cap A) - w_{s_0})$. Thus we have $\bar{q} \in \mathcal{Q}_{l,v}$. Clearly, $\mu_K(\mathcal{Q}_{l,v}) \leq \alpha^{m-1}$ by (20).

For each $l \geq m$, $t \in T_l$, and $v \in V_l$ let

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Summarizing the argument above we have that for each $r_{w_{s_0}} \oplus p \in \mathcal{P}_{l,t}$

- $r_{w_{s_0}}(l') + p(l') \in r_{w_{s_0}}(l') + \tau_H \subseteq A$ for $l' < m-1$ because $r_{w_{s_0}}(l') \in B_H$,
- $r_{w_{s_0}}(m-1) + p(m-1) = w_{s_0} + p(m-1)$
  $\in (w_{s_0} + E_{l,t}) \cap (w_{s_0} + \vec{y}(m-1) + [K]) \cap A \subseteq A$,
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For each $\bar{q} \sqsubseteq \vec{x} \oplus [K]$, $\bar{q} \in \mathcal{Q}_{l,v}$ iff there is a $q \sqsubseteq \vec{y} \oplus [K]$ with $\bar{q} = r_{w_{s_0}} \oplus q$ such that

- $r_{w_{s_0}}(l') + q(l') \in r_{w_{s_0}}(l') + \tau_H \subseteq A$ for $l' < m - 1$ because $r_{w_{s_0}}(l') \in B_H$,
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This completes the proof of $\mathbf{L}_1(m)(\alpha, \eta, N, A, S, U)$ as well as $\mathbf{L}(m)$ by Lemma 3.19. □

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The End of Day Four

Thank you for your attention.