Nonstandard Analysis and Combinatorial Number Theory

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Day Three: Easy Applications to Combinatorics

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OUTLINE:

1. Nonstandard Versions of Densities
2. By-one-get-one-free Thesis
3. Plünnecke's Inequalities

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An apparent reason why nonstandard analysis should be a useful tool for other fields of mathematics is that a limit process which involves rank 3 objects in $\mathcal{V}$ such as the limit of a sequence or a function with real values can be changed to an infinitesimal argument with rank 0 objects such as infinitesimals in ${}^*\mathcal{V}$. So, good candidates for the applications of nonstandard analysis should be something involving limit processes. This may be why the density problems receive attention from nonstandard analysts. The densities introduced in this section are Shnirel'man density, lower and upper (asymptotic) density, and lower and upper Banach density.

For two sets $A, B \subseteq \mathbb{N}$, let $A + B := \{a + b \mid a \in A \text{ and } b \in B\}$. If $A = \{a\}$ we write $a + B$ instead of $\{a\} + B$ for simplicity. If $r, r' \in {}^*\mathbb{R}$, we write $r \gtrsim r'$ for $r > r'$ or $r \approx r'$ and $r \lesssim r'$ for $r < r'$ or $r \approx r'$.

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Definition (3.1)

Let $A \subseteq \mathbb{N}$. The Shnirel'man density $\sigma(A)$, lower density $\underline{d}(A)$, upper density $\overline{d}(A)$, upper Banach density $\overline{BD}(A)$, and lower Banach density $\underline{BD}(A)$ of A are defined by

1. $\sigma(A) := \inf_{n \geq 1} \frac{|A \cap (1 + [n])|}{n}$;
2. $\underline{d}(A) := \liminf_{n \to \infty} \frac{|A \cap [n]|}{n}$;
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Remark (3.2)

1. *In the definition of* $\sigma(A)$*, we have* $1+[n]=\{1,2,\ldots,n\}$*. Hence,* 0*, in or not in* A*, does not play any role. If* $\sigma(A)>0$*, then* $1\in A$*;*
2. *If* $\underline{d}(A)=\overline{d}(A)$*, we say that the (asymptotic) density of* A *exists and is denoted by* $d(A)$*;*
3. *If* $\underline{BD}(A)=\overline{BD}(A)$*, we say that the Banach density of* A *exists and is denoted by* $BD(A)$*;*
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The following Proposition is direct consequences of the definition.

Proposition (3.3)

For any $A \subseteq \mathbb{N}$ we have

$$0 \le \min\{\sigma(A), \underline{BD}(A)\} \le \max\{\sigma(A), \underline{BD}(A)\} \le \underline{d}(A) \le \overline{d}(A) \le \overline{BD}(A) \le 1.$$

Lemma (3.4)

Let $A \subseteq \mathbb{N}$. Then, $\overline{BD}(A)$ is the largest real α in $[0,1]$ such that there exist $k_m, n_m \in \mathbb{N}$ with $n_m \to \infty$ as $m \to \infty$ such that

$$\lim_{m\to\infty} \frac{|A \cap (k_m + [n_m])|}{n_m} = \alpha.$$

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Proposition (3.5)

Let $A \subseteq \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then

1. $\underline{d}(A) \geq \alpha$ *iff* $\frac{|{}^*A \cap [N]|}{N} \gtrapprox \alpha$ *for any hyperfinite integer* N*;*
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Proof. Part 1. "$\Rightarrow$": Let N be an arbitrary hyperfinite integer. Since for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \left(n \geq n_0 \rightarrow \frac{|A \cap [n]|}{n} > \alpha - \epsilon \right).$$

By the transfer principle, it is true that

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Part 1. "$\Leftarrow$": Suppose $\underline{d}(A) < \alpha$. Let $\alpha' = (\alpha + \underline{d}(A))/2$, then there is an increasing sequence $n_1 < n_2 < \cdots$ such that $\forall i \in \mathbb{N} \left(\dfrac{|A \cap [n_i]|}{n_i} < \alpha' \right)$. By the transfer principle the sentence $\forall i \in {}^*\mathbb{N} \left(\dfrac{|{}^*A \cap [n_i]|}{n_i} < \alpha' \right)$ is true in ${}^*\mathcal{V}$.

Let N' be a hyperfinite integer and $N := n_{N'}$. Then, N is hyperfinite and $\dfrac{|{}^*A \cap [N]|}{N} \lessapprox \alpha' < \alpha$. Hence, the right side of Part 1 is false.

The proof of Part 2 is left to the reader. $\square$

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Proof. We prove Part 2. The proof of Part 1 is left to the reader.

Part 2. "$\Rightarrow$": Given $m \in \mathbb{N}$, there exist $k_m \in \mathbb{N}$ and $n_m > m$ such that

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1. $\underline{BD}(A) \geq \alpha$ *iff* $\dfrac{|{}^*A \cap (k + [N])|}{N} \gtrapprox \alpha$ *for any* $k \in {}^*\mathbb{N}$ *and any hyperfinite integer* N*;*
2. $\overline{BD}(A) \geq \alpha$ *iff* $\dfrac{|{}^*A \cap (k + [N])|}{N} \gtrapprox \alpha$ *for some* $k \in {}^*\mathbb{N}$ *and some hyperfinite integer* N*.*

Proof: We prove Part 2. The proof of Part 1 is left to the reader.

Part 2. "$\Rightarrow$": Given $m \in \mathbb{N}$, there exist $k_m \in \mathbb{N}$ and $n_m > m$ such that

$$\frac{|A \cap (k_m + [n_m])|}{n_m} > \alpha - \frac{1}{m}.$$

By the transfer principle, we have that for any $m \in {}^*\mathbb{N}$ there exist $k_m \in {}^*\mathbb{N}$ and $n_m > m$ such that

$$\frac{|{}^*A \cap (k_m + [n_m])|}{n_m} > \alpha - \frac{1}{m}.$$

Now let m be a hyperfinite integer, $k := k_m$, and $N := n_m > m$. Then,

$$\frac{|{}^*A \cap (k + [N])|}{N} \gtrapprox \alpha.$$

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Part 2. "$\Leftarrow$": Assume that $\overline{BD}(A) < \alpha$. Let $\alpha' = (\alpha + \overline{BD}(A))/2$. Then, there exists an $n_0 \in \mathbb{N}$ such that the following sentence is true in $\mathcal{V}$:

$$\forall k, n \in \mathbb{N} \left(n \geq n_0 \to \frac{|A \cap (k + [n])|}{n} \leq \alpha' \right).$$

By the transfer principle, the following is true in ${}^*\mathcal{V}$:

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Shnirel'man density and lower density are most used densities by number theorists. For example, Shnirel'man proved that if a set A has positive Shnirel'man density, then there is a fixed k such that every positive integer is the sum of at most k numbers in A. If P is the set of all prime numbers, then $A := (\{0,1\} \cup P) + (\{0,1\} \cup P)$ has positive Shnirel'man density, therefore, every positive integer is the sum of at most $2k$ prime numbers. This is the first nontrivial result towards the solution of Goldbach conjecture.

The buy-one-get-one-free thesis is the following statement:

There is a parallel result involving upper Banach density for every existing result involving Shnirel'man density or lower density.

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Theorem (3.7)

If $A \subseteq \mathbb{N}$ and $\overline{BD}(A) = \alpha$, then there is an $k \in {}^*\mathbb{N}$ and a hyperfinite integer N such that for μ_Ω-almost all $n \in k + [N]$ where μ_Ω is the Loeb measure on $\Omega := k + [N]$, we have $\underline{d}(({}^*A - n) \cap \mathbb{N}) = \alpha$.

On the other hand, if $A \subseteq \mathbb{N}$ and there is a positive integer $n \in {}^*\mathbb{N}$ such that $\underline{d}(({}^*A - n) \cap \mathbb{N}) \geq \alpha$, then $\overline{BD}(A) \geq \alpha$.

Theorem (3.8)

If $A \subseteq \mathbb{N}$ and $\overline{BD}(A) = \alpha$, then there is an $n \in {}^*\mathbb{N}$ such that

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To present short proofs of Theorem 3.7 and Theorem 3.8 we borrow the following Birkhoff's Ergodic Theorem.

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Let (Ω, Σ, μ) be a probability space and T be a measure-preserving transformation from Ω to Ω. For every $f \in L_1(\Omega)$, there exists a T-invariant $\bar{f} \in L_1(\Omega)$ such that for μ-almost all $x \in \Omega$,

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By Birkhoff Ergodic Theorem there is a T-invariant $\bar{f} \in L_1(\Omega)$ such that there is a $X \subseteq \Omega$ with $\mu_\Omega(X) = 1$ such that for all $n \in X$ we have

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Since

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Proof of Theorem 3.8: By Theorem 3.7 we can find $k \in {}^*\mathbb{N}$ such that $\underline{d}(({}^*A - k) \cap \mathbb{N}) = \alpha$.

For each $m \in \mathbb{N}$ let

$$n_m := \max\{n \in \mathbb{N} \mid |({}^*A - k) \cap [n_m] \leq \alpha - 1/m\}.$$

Note that n_m exists because otherwise we would have $\underline{d}(({}^*A - k) \cap \mathbb{N}) \leq \alpha - 1/m$. Note that $|({}^*A - k - n_m) \cap [n]|/n > \alpha - 1/m$ for any $n \in 1 + [m]$.

By Proposition 2.8 we can find a hyperfinite integer N such that $|({}^*A - k - n_N) \cap [n]|/n > \alpha - 1/N$ for any $n \in 1 + [N]$. This implies that $\sigma(({}^*A - k - n_N) \cap \mathbb{N}) \geq \alpha$.

Since $\sigma(({}^*A - k - n_N) \cap \mathbb{N}) > \alpha$ implies $\underline{d}(({}^*A - k - n_N) \cap \mathbb{N}) > \alpha$ which is impossible by Theorem 3.7 we conclude that $\sigma(({}^*A - k - n_N) \cap \mathbb{N}) = \alpha$. $\square$.

Theorem (3.10, Mann's Theorem)

Let $A, B \subseteq \mathbb{N}$ and $0 \in A \cap B$. Then

$$\sigma(A+B) \geq \min\{\sigma(A)+\sigma(B), 1\}.$$

Theorem (3.11, Upper Banach Density Version)

Let $A, B \subseteq \mathbb{N}$. Then

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Definition (3.12)

Let $B \subseteq \mathbb{N}$*. For a positive integer* $h \in \mathbb{N}$*, let*

$$hB := \{b_1 + b_2 + \cdots + b_h \mid b_i \in B \text{ for } i = 1, 2, \ldots, h\}.$$

(1) *The set* B *is a* basis *if* $hB = \mathbb{N}$ *for some* $h \in \mathbb{N}$*. The least such* h *is called the* order *of* B*. Clearly, a basis must contain* 0*;*

(2) *Suppose* B *is a basis of order* h*. For each* $m \geq 1$ *let* $h(m) := \min\{h' \in \mathbb{N} \mid m \in h'B\}$*. Then, the number*

$$h^* := \sup_{n \geq 1} \frac{1}{n} \sum_{m=1}^{n} h(m)$$

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(3) *The set B is an asymptotic basis if $\mathbb{N} \setminus h_a B$ is finite for some $h_a \in \mathbb{N}$. The least such h_a is called the asymptotic order of B;*

(4) *Suppose B is an asymptotic basis of order $h_a \in \mathbb{N}$ and $\mathbb{N} \setminus [n_0] \subseteq h_a B$ for some minimal $n_0 \in \mathbb{N}$. For each $m \geq n_0$ let $h(m) := \min\{h' \in \mathbb{N} \mid m \in h' B\}$. Then, the number*

$$h_a^* := \limsup_{n \to \infty} \frac{1}{n} \sum_{m=n_0}^{n_0+n-1} h(m)$$

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Definition (3.12)

(5) *The set B is a piecewise basis if there exists some $h_p \in \mathbb{N}$ such that one can find a sequence $k_n + [m_n]$ with $m_n \to \infty$ as $n \to \infty$ satisfying*

$$k_n + ([m_n]) \subseteq h_p((B - k_n) \cap \mathbb{N}) + k_n$$

for every $n \in \mathbb{N}$. The least such h_p is called the piecewise order of B;

(6) *The set B is a piecewise asymptotic basis if there is an $h_{pa} \in \mathbb{N}$ such that one can find a sequence $k_n + [m_n]$ with $m_n \to \infty$ as $n \to \infty$ and a number $n_0 \in \mathbb{N}$ satisfying*

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Definition (3.12)

(7) *Suppose that B is a piecewise asymptotic basis of piecewise asymptotic order h_{pa}. Let $\mathcal{I}$ be the sequence $k_n + [m_n]$ and $n_0 \in \mathbb{N}$ such that $k_n + ([m_n] \setminus [n_0]) \subseteq h_{pa}((B - k_n) \cap \mathbb{N}) + k_n$ for every $n \in \mathbb{N}$. For each $m \in k_n + ([m_n] \setminus [n_0])$ let* $h(m) := \min\{h' \in \mathbb{N} \mid m \in h'((B - k_n) \cap \mathbb{N}) + k_n$. *Let*

$$h_n^* := \sup_{k_n+n_0 \leq m < k_n+m_n} \frac{1}{m_n - n_0} \sum_{i=k_n+n_0}^{k_n+m_n-1} h(m) \text{ and}$$

$$h_{\mathcal{I}}^* := \limsup_{n\to\infty} h_n^*.$$

Then, the number

$$h_{pa}^* := \inf\{h_{\mathcal{I}}^* \mid \text{ for all suitable } \mathcal{I}\}$$

is called a piecewise asymptotic average order of B.

Theorem (3.13, Rohrback's Theorem)

If B is an asymptotic basis of asymptotic average order h_a^*, then for any $A \subseteq \mathbb{N}$ we have

$$\underline{d}(A+B) \geq \underline{d}(A) + \frac{1}{2h_a^*}\underline{d}(A)(1-\underline{d}(A)).$$

Theorem (3.14, Upper Banach Density Version)

If B is a piecewise asymptotic basis of piecewise asymptotic average order h_{pa}^*, then for any $A \subseteq \mathbb{N}$ we have

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Rohrbach's Theorem is a generalization of Erdős' Theorem that if B is a basis of order h, then for any $A \subseteq \mathbb{N}$ it is true that

$$\sigma(A+B) \geq \sigma(A) + \frac{1}{2h}\sigma(A)(1-\sigma(A)).$$

Erdős' theorem is for the study of so-called essential component problems. A set B is called essential component if $\sigma(A+B) > \sigma(A)$ for any $A \subseteq \mathbb{N}$ with $0 < \sigma(A) < 1$. Hence, a basis must be an essential component.

There is another generalization of Erdős' theorem, which is much more significant than Rohrbach's Theorem does. The following generalization of Erdős' Theorem used a completely different idea from Erdős'.

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Theorem (3.15, Plünnecke's Theorem)

Let B be a basis of order h. Then, for any $A \subseteq \mathbb{N}$ we have

$$\sigma(A+B) \geq \sigma(A)^{1-\frac{1}{h}}.$$

It is not too hard to show that $\sigma(A)^{1-\frac{1}{h}} \geq \sigma(A) + \frac{1}{h}\sigma(A)(1-\sigma(A))$.

The key component used in the proof of Plünnecke's Theorem is a version of Plünnecke's Inequality based on graph theoretic argument. The following lemma is a translation of an inequality from the language of graph theory to the language of additive number theory.

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Lemma (3.16, Plünnecke's Inequality)

Let $A, B \subseteq \mathbb{N}$ and $h, n \geq 1$ be such that $A \cap [n] \neq \emptyset$. For each $1 \leq i \leq h$ define

$$D_{A,B,n,i} = \min\left\{\frac{|(A' + iB) \cap [n]|}{|A' \cap [n]|} : \emptyset \neq A' \subseteq A \cap [n]\right\}.$$

$$\textit{Then, } D_{A,B,n,1} \geq (D_{A,B,n,2})^{1/2} \geq \cdots \geq (D_{A,B,n,h})^{1/h}.$$

Many interesting subsets of $\mathbb{N}$ are not bases but asymptotic bases. For example,

$P := \{p \in \mathbb{N} \mid p \text{ is a prime number}\}$,
$C_k := \{n^k \mid n \in \mathbb{N}\}$ for $k \geq 1$,
$P^2 := \{a^2b^3 \mid a, b \in \mathbb{N} \text{ and } a, b \geq 1\}$, etc.

are asymptotic bases. Therefore, it is interesting to see whether Plünnecke's Theorem can be generalized to some versions involving other densities.

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1. *The set $B \subseteq \mathbb{N}$ is a lower asymptotic basis of order $h \in \mathbb{N}$ if $\underline{d}(hB) = 1$;*
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Note that P is an asymptotic basis of order 4 by Vinogradov's Theorem, or 3 if Goldbach conjecture is true. It is also known that P is a lower asymptotic basis of order 3. P^2 is an asymptotic basis of order 3 by a result of Heath-Brown. C_2 is a basis of order 4 and C_3 is an asymptotic basis of order at most 7. Note also that P, C_k, P^2 are all have upper density 0.

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Note that P is an asymptotic basis of order 4 by Vinogradov's Theorem, or 3 if Goldbach conjecture is true. It is also known that P is a lower asymptotic basis of order 3. P^2 is an asymptotic basis of order 3 by a result of Heath-Brown. C_2 is a basis of order 4 and C_3 is an asymptotic basis of order at most 7. Note also that P, C_k, P^2 are all have upper density 0.

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1. *The set* $B \subseteq \mathbb{N}$ *is a* lower asymptotic basis *of order* $h \in \mathbb{N}$ *if* $\underline{d}(hB) = 1$;
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Theorem (3.18)

Let $A, B \subseteq \mathbb{N}$ and B be a lower asymptotic basis of order h. Then

$$\underline{d}(A+B) \geqslant \underline{d}(A)^{1-\frac{1}{h}}.$$

Corollary (3.19)

For any $A \subseteq \mathbb{N}$ we have

1. $\underline{d}(A+P) \geq \underline{d}(A)^{2/3}$;
2. $\underline{d}(A+C_2) \geq \underline{d}(A)^{3/4}$;
3. $\underline{d}(A+C_3) \geq \underline{d}(A)^{6/7}$;
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There are $A, B \subseteq \mathbb{N}$ *with* $\overline{d}(A) = \frac{1}{2}$, $\overline{d}(2B) = 1$, *and*

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Let $A, B \subseteq \mathbb{N}$ *and* B *be a upper Banach basis of order* h. *Then*

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Note that Theorem 3.18 and Theorem 3.20 show that lower density and upper density are asymmetrical on generalizing Plünnecke's Theorem. Theorem 3.21 and Theorem 3.22 look like following the same pattern but they show also that upper Banach density and lower Banach density are mildly asymmetrical. Both of the theorems require B be upper Banach basis.

We will prove Theorem 3.18 and Theorem 3.21. The arguments used in the proof of Theorem 3.15 deal with finite intervals of integers and are purely combinatorial. It becomes messy when the limit processes for $\underline{d}$ or $\overline{BD}$ are involved. Using nonstandard analysis, we can transfer the limit processes to combinatorial arguments on intervals of hyperfinite length, which simplify the proofs.

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Proof of Theorem 3.18: Let A and B be in Theorem 3.18 such that $\underline{d}(A) = \alpha$ and $\underline{d}(hB) = 1$. Without loss of generality, we can assume $0 < \alpha < 1$. Let N be any hyperfinite integer. We want to show that

$$\frac{|{}^*(A+B) \cap [N]|}{N} = \frac{|({}^*A + {}^*B) \cap [N]|}{N} \gtrsim \alpha^{1-\frac{1}{h}},$$

which implies Theorem 3.18 by Proposition 3.5. Choose hyperfinite integers $N' < K < N$ such that $(N-K)/N \approx 0$ and $(K-N')/(N-N') \approx 0$ (for example $K = N - \left\lfloor \sqrt{N} \right\rfloor$ and $N' = K - \left\lfloor \sqrt[4]{N} \right\rfloor$ satisfy the requirements). Let $C_0 = {}^*A \cap [K]$. Then $(|C_0 \cap [N]|)/N \gtrsim \alpha$. Next we want to trim C_0 so that the density of the trimmed set in each interval $\{x, x+1, \ldots, N-1\}$ for every $x \leq K$ would not be too large.

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We define C_k inductively for $k = 0, 1, \ldots, N' - 1$ so that

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{N'-1},\ \frac{|C_{N'-1} \cap [N]|}{N} \approx \alpha, \text{ and}$$

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It is easy to see that $C_0, C_1, \ldots, C_{N'-1}$ has the desired properties. Let $A_0 = C_{N'-1}$ and nonempty $A' \subseteq A_0$ be such that

$$D_{A_0, {}^*B, N, h} = \frac{|(A' + h^*B) \cap [N]|)}{|A' \cap [N]|}.$$

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Let $z = \min A'$.

Then $z < K$ because $A_0 \subseteq [K]$. Hence $N - z$ is hyperfinite, which implies $\frac{|(h^*B) \cap [N-z]|}{N-z} \approx 1$. By Lemma 3.16 we have

$$\frac{|(A_0 + {}^*B) \cap [N]|}{|A_0 \cap [N]|}$$

$$\geq D_{A_0,{}^*B,N,1} \geq (D_{A_0,{}^*B,N,h})^{1/h} = \left(\frac{|(A' + h^*B) \cap [N]|}{|A' \cap [N]|}\right)^{1/h}$$

$$\gtrapprox \left(\frac{|(z + h^*B) \cap [N]|}{|A' \cap [N]|)}\right)^{1/h}$$

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$$\frac{|{}^*(A+B)\cap[N]|}{N} \geq \frac{|(A_0+{}^*B)\cap[N]|}{N} \gtrapprox \frac{|A_0\cap[N]|}{N}\cdot\frac{1}{\alpha^{1/h}} \gtrapprox \alpha^{1-\frac{1}{h}}.$$

Since N is an arbitrary hyperfinite integer, Theorem 3.18 is proven with the help of Proposition 3.5. □

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Proof of Theorem 3.21: Let A and B be in Theorem 3.21 with $\overline{BD}(A) = \alpha$ and $\overline{BD}(hB) = 1$ for some $h \in \mathbb{N}$.

Theorem 3.21 is trivially true if $\overline{BD}(A) = 0$ or $\overline{BD}(A) = 1$. So, we can assume that $0 < \alpha = \overline{BD}(A) < 1$.

Let $n \in {}^*\mathbb{N}$ and K be a hyperfinite integer such that $n + [K] \subseteq (h\,{}^*B)$.

Choose N large enough so that $(n + K)/N \approx 0$ and $|{}^*A \cap (m + [N])|/N \approx \alpha$ for some $m \in {}^*\mathbb{N}$.

It suffices to show that

$$\frac{|({}^*A \cap (m + [N]) + {}^*B) \cap (m + [N])|}{N} \gtrapprox \alpha^{1-\frac{1}{h}}$$

by Proposition 3.6.

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Choose N large enough so that $(n + K)/N \approx 0$ and $|{}^*A \cap (m + [N])|/N \approx \alpha$ for some $m \in {}^*\mathbb{N}$.

It suffices to show that

$$\frac{|({}^*A \cap (m + [N]) + {}^*B) \cap (m + [N])|}{N} \gtrapprox \alpha^{1-\frac{1}{h}}$$

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Let $A_0 = ({}^*A \cap (m + [N - n - K]) - m$. By the choice of N and A_0 we have

$$\frac{|A_0 \cap [N]|}{N} \approx \alpha \text{ and } \frac{|(A_0 + {}^*B) \cap [N]|}{N} \lessapprox \frac{|({}^*A + {}^*B) \cap (m + [N])|}{N}.$$

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$$\frac{|(A_0 + {}^*B) \cap [N]|}{N} \gtrapprox \alpha^{1-\frac{1}{h}}.$$

Let $A' \subseteq A_0$ be nonempty such that

$$D_{A_0, {}^*B, N, h} = \frac{|(A' + h\,{}^*B) \cap [N]|}{|A' \cap [N]|}.$$

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Claim:
$$\frac{|(A' + h\,^*B) \cap [N]|}{|A' \cap [N]|} = D_{A_0, ^*B, H, h} \gtrapprox \frac{1}{\alpha}.$$

Proof of Claim: Let $H = \lfloor K/2 \rfloor$ and let $I_i = iH + [H]$ for $i = 0, 1, \ldots \lfloor N/H \rfloor - 1$, and let
$$I_{\lfloor N/H \rfloor} = \lfloor N/H \rfloor \cdot H + [N - \lfloor N/H \rfloor \cdot H].$$

Denote
$$\mathcal{I} := \{I_i \mid i \in [\lfloor N/H \rfloor + 1] \text{ and } I_i \cap A' \neq \emptyset\}.$$

Then
$$|(A' + h\,^*B) \cap [N]| \geq |\mathcal{I}| \cdot H$$

because $H \leq K/2$, every element in A' is less than or equal to $N - n - K$, and $H + n + I_i \subseteq (A' + h\,^*B) \cap [N]$ if $A' \cap I_i \neq \emptyset$ for every $i = 0, 1, \ldots, \lfloor H/N \rfloor$.

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Given a positive standard real ϵ, we have

$$|A' \cap [N]| \leqslant |\mathcal{I}| \cdot (\alpha + \epsilon) H$$

because $|A' \cap I_i|/|I_i| \lessapprox \alpha$ when $|I_i|$ is hyperfinite by Proposition 3.6. Because ϵ is an arbitrary standard positive real number, we have that

$$\frac{|(A' + h{}^*B) \cap [N]|}{|A' \cap [N]|} \gtrapprox \frac{|\mathcal{I}| \cdot H}{|\mathcal{I}| \cdot \alpha H} = \frac{1}{\alpha}.$$

This completes the proof of the claim.

We continue to prove Theorem 3.21. Combine the arguments above and Theorem 3.16 we now have

$$\frac{|(A_0 + {}^*B) \cap [N]|}{|A_0 \cap [N]|} \gtrapprox D_{A_0, {}^*B, N, 1} \geq (D_{A_0, {}^*B, N, h})^{1/h}$$

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Hence

$$\frac{|{}^*(A+B)\cap[N]|}{N} \gtrapprox \frac{|(A_0 + {}^*B)\cap[N]|}{N}$$
$$\gtrapprox \frac{|A_0\cap[N]|}{N}\cdot\frac{1}{\alpha^{1/h}} \approx \alpha^{1-\frac{1}{h}},$$

which implies Theorem 3.21 by Proposition 3.6. □

Hence

$$\frac{|^*(A+B)\cap[N]|}{N} \gtrapprox \frac{|(A_0+{}^*B)\cap[N]|}{N}$$
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The End of Day Three

Thank you for your attention.